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On Leray's problem in an infinitely long pipe with the Navier-slip boundary condition

Zijin Li[1], Xinghong Pan[2,*] & Jiaqi Yang[3]

[1]*School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing* 210044*, China;*
[2]*School of Mathematics and Key Laboratory of Ministry of Industry and Information Technology, Nanjing University of Aeronautics and Astronautics, Nanjing* 211106*, China;*
[3]*School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an* 710129*, China*

Email: zijinli@nuist.edu.cn, xinghong_87@nuaa.edu.cn, yjqmath@nwpu.edu.cn

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Abstract The original Leray's problem concerns the well-posedness of weak solutions to the steady incompressible Navier-Stokes equations in a distorted pipe, which approach the Poiseuille flow subject to the no-slip boundary condition at spatial infinity. In this paper, the same problem with the Navier-slip boundary condition instead of the no-slip boundary condition, is addressed. Due to the complexity of the boundary condition, some new ideas, presented as follows, are introduced to handle the extra difficulties caused by boundary terms. First, the Poiseuille flow in the semi-infinite straight pipe with the Navier-slip boundary condition will be introduced, which will serve as the asymptotic profile of the solution to the generalized Leray's problem at spatial infinity. Second, a solenoidal vector function defined in the whole pipe, satisfying the Navier-slip boundary condition, having the designated flux and equalling the Poiseuille flow at a large distance, will be carefully constructed. This plays an important role in reformulating our problem. Third, the energy estimates depend on a combined L^2-estimate of the gradient and the stress tensor of the velocity.

Keywords stationary Navier-Stokes system, Navier-slip boundary condition, Leray's problem

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1 Introduction

The 3D stationary Navier-Stokes (NS) equations which describe the motion of stationary viscous incompressible fluids are as follows:

$$\begin{cases} \boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p-\Delta\boldsymbol{u}=0, \\ \nabla\cdot\boldsymbol{u}=0, \end{cases} \quad \text{in } \mathcal{D}\subset\mathbb{R}^3. \tag{1.1}$$

* Corresponding author

Here, $\boldsymbol{u}(x) \in \mathbb{R}^3$ and $p(x) \in \mathbb{R}$ represent the velocity and the scalar pressure, respectively. In this paper, we consider the smooth domain $\mathcal{D}$ to be an infinitely long pipe with two straight "outlets", the left and the right, and a compact distortion, a bubble or a bulge, in the middle. We define it as follows:

$$\mathcal{D} = \mathcal{D}_L \cup \mathcal{D}_M \cup \mathcal{D}_R, \tag{1.2}$$

where $\mathcal{D}_L$ and $\mathcal{D}_R$ are semi-infinite smooth straight pipes with their cross sections Σ_L and Σ_R being perpendicular to the x_3-axis, i.e.,

$$\mathcal{D}_L = \Sigma_L \times (-\infty, -Z/2] \quad \text{and} \quad \mathcal{D}_R = \Sigma_R \times [Z/2, +\infty).$$

Here, $\Sigma_L, \Sigma_R \subset \mathbb{R}^2$ are smooth bounded domains. The distortion part $\mathcal{D}_M \subset \mathbb{R}^2 \times (-Z, Z)$ is a compact smooth domain in $\mathbb{R}^3$ (see Figure 1).

Moreover, technically we assume that there exists an infinitely long smooth straight pipe getting through $\mathcal{D}$, which means that there exists a $\Sigma' \subset\subset \Sigma_L \cap \Sigma_R$ such that $\Sigma' \times \mathbb{R} \subset \mathcal{D}$. This will be applied in the construction of the profile vector in Subsection 3.2.

In the current paper, the Navier-Stokes equations (1.1) will be equipped with the following boundary condition, i.e., **the Navier-slip boundary condition**:

$$\begin{cases} 2(\mathbb{S}\boldsymbol{u} \cdot \boldsymbol{n})_{\tan} + \alpha \boldsymbol{u}_{\tan} = 0, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \end{cases} \quad \text{on } \partial\mathcal{D}. \tag{1.3}$$

Here, $\mathbb{S}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^{\mathrm{T}} \boldsymbol{u})$ is the stress tensor, where $\nabla^{\mathrm{T}} \boldsymbol{u}$ stands for the transpose of the Jacobian matrix $\nabla \boldsymbol{u}$, and $\boldsymbol{n}$ is the unit outer normal vector of $\partial\mathcal{D}$. For a vector field $\boldsymbol{v}$, we denote by $\boldsymbol{v}_{\tan}$ its tangential part, i.e., $\boldsymbol{v}_{\tan} := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$. $\alpha > 0$ stands for the friction constant which may depend on various elements such as the property of the boundary and the viscosity of the fluid. When $\alpha \to 0_+$, the boundary condition (1.3) turns to be the total Navier-slip boundary condition, while when $\alpha \to \infty$, the boundary condition (1.3) degenerates into the no-slip boundary condition $u \equiv 0$ on the boundary. In this paper, we assume $0 < \alpha < +\infty$.

Throughout this paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on $a, b, c, \ldots$, which may be different from line to line. For a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we define $x_h := (x_1, x_2)$. For a two-dimensional scalar function f or a vector-valued function $\boldsymbol{f} := (f_1, f_2)$, we define

$$\nabla_h f = (\partial_{x_1} f, \partial_{x_2} f), \quad \Delta_h f = \partial_{x_1}^2 f + \partial_{x_2}^2 f, \quad \mathrm{div}_h \boldsymbol{f} := \partial_{x_1} f_1 + \partial_{x_2} f_2.$$

Meanwhile, for any $\zeta > 1$, we define

$$\mathcal{D}_\zeta := \{x \in \mathcal{D} : -\zeta < x_3 < \zeta\},$$

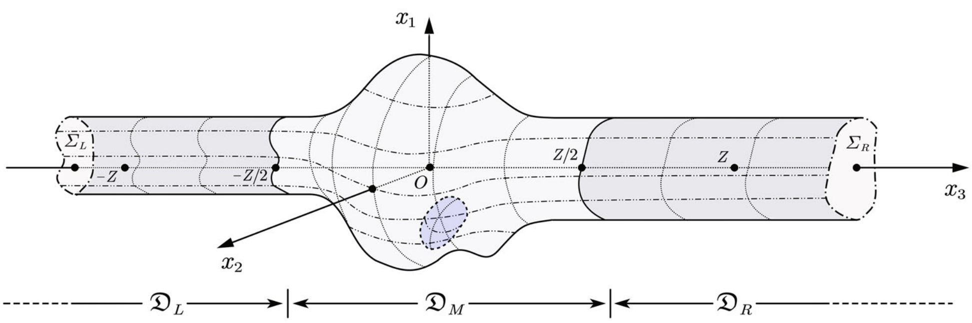

Figure 1 (Color online) Infinite pipe $\mathcal{D}$, with a bubble and an obstacle in the middle

the truncated pipe with the length of 2ζ. Meanwhile, the notations $\Omega_\zeta^{\pm}$ are denoted by

$$\Omega_\zeta^+ := (\mathcal{D}_\zeta - \mathcal{D}_{\zeta-1}) \cap \{x \in \mathcal{D} :\, x_3 > 0\}, \quad \Omega_\zeta^- := (\mathcal{D}_\zeta - \mathcal{D}_{\zeta-1}) \cap \{x \in \mathcal{D} :\, x_3 < 0\},$$

respectively. We also apply $A \lesssim B$ to stating $A \leqslant CB$. Moreover, $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$.

For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$, L^p denotes the usual Lebesgue space with the norm

$$\|f\|_{L^p(D)} := \begin{cases} \left(\displaystyle\int_D |f(x)|^p dx \right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{esssup}_{x\in D} |f(x)|, & p = \infty, \end{cases}$$

while $W^{k,p}$ denotes the usual Sobolev space with its norm

$$\|f\|_{W^{k,p}(D)} := \sum_{0 \leqslant |L| \leqslant k} \|\nabla^L f\|_{L^p(D)},$$

where $L = (l_1, l_2, l_3)$ is a multi-index. We also simply denote $W^{k,p}$ by H^k provided that $p = 2$. Finally, $\bar{D}$ denotes the closure of a domain D. A function $g \in W^{k,p}_{\mathrm{loc}}(D)$ or $g \in W^{k,p}_{\mathrm{loc}}(\bar{D})$ means $g \in W^{k,p}(\tilde{D})$ for any $\tilde{D}$ compactly contained in D or $\bar{D}$.

For the 3D vector-valued function, we define

$$\begin{aligned} &\mathcal{H}(\mathcal{D}) := \{\boldsymbol{\varphi} \in H^1(\mathcal{D}; \mathbb{R}^3) : \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\mathcal{D}} = 0\}, \\ &\mathcal{H}_\sigma(\mathcal{D}) := \{\boldsymbol{\varphi} \in H^1(\mathcal{D};\, \mathbb{R}^3) : \nabla \cdot \boldsymbol{\varphi} = 0,\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\mathcal{D}} = 0\} \end{aligned}$$

and

$$\mathcal{H}_{\sigma,\mathrm{loc}}(\overline{\mathcal{D}}) := \{\boldsymbol{\varphi} \in H^1_{\mathrm{loc}}(\overline{\mathcal{D}};\, \mathbb{R}^3) : \nabla \cdot \boldsymbol{\varphi} = 0,\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\mathcal{D}} = 0\}.$$

We also define

$$\boldsymbol{X} := \{\boldsymbol{\varphi} \in C_c^\infty(\overline{\mathcal{D}}; \mathbb{R}^3) : \nabla \cdot \boldsymbol{\varphi} = 0,\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\mathcal{D}} = 0\}.$$

Clearly, $\boldsymbol{X}$ is dense in $\mathcal{H}_\sigma$ in the $H^1(\mathcal{D})$-norm. For matrices $\boldsymbol{\Gamma} = (\gamma_{ij})_{1\leqslant i,j\leqslant 3}$ and $\boldsymbol{K} = (\kappa_{ij})_{1\leqslant i,j\leqslant 3}$, we define

$$\boldsymbol{\Gamma} : \boldsymbol{K} = \sum_{i,j=1}^{3} \gamma_{ij}\kappa_{ij}.$$

Next, we state the main problem of the paper.

1.1 Leray's problem with the Navier-slip boundary condition

For a given flux Φ which is supposed to be nonnegative without loss of generality, if we consider the Poiseuille flow, $\boldsymbol{g}_\Phi^i$, of (1.1) with the boundary condition (1.3) in $\mathcal{D}_i$ (i denotes L or R), then it satisfies

$$\begin{cases} \boldsymbol{g}_\Phi^i = g_\Phi^i(x_h)\boldsymbol{e}_3, \\ -\Delta_h g_\Phi^i(x_h) = C_i \quad \text{in } \Sigma_i, \\ \dfrac{\partial g_\Phi^i}{\partial \bar{\boldsymbol{n}}} = -\alpha g_\Phi^i \quad \text{on } \partial\Sigma_i, \\ \displaystyle\int_{\Sigma_i} g_\Phi^i(x_h)dx_h = \Phi, \end{cases}$$

where the constant C_i is uniquely related to Φ, while $\bar{\boldsymbol{n}}$ is the unit outer normal vector on $\partial\Sigma_i$. We can see that $\boldsymbol{g}_\Phi^i$ is a solution of (1.1) with the Navier-slip boundary (1.3) in $\Sigma_i \times \mathbb{R}$.

The main objective of this paper is to study the solvability of the following generalized Leray's problem: for a given flux Φ, find a pair $(\boldsymbol{u}, p)$ such that

$$\begin{cases} \boldsymbol{u} \cdot \nabla\boldsymbol{u} + \nabla p - \Delta\boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \mathcal{D}, \\ 2(\mathbb{S}\boldsymbol{u} \cdot \boldsymbol{n})_{\tan} + \alpha\boldsymbol{u}_{\tan} = 0, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\mathcal{D} \end{cases} \tag{1.4}$$

with

$$\int_{\Sigma_i} u_3(x_h, x_3)dx_h = \Phi \quad \text{for } |x_3| > Z/2 \tag{1.5}$$

and

$$\boldsymbol{u} \to \boldsymbol{g}_\Phi^i \quad \text{as } |x| \to \infty \text{ in } \mathcal{D}_i. \tag{1.6}$$

To prove the existence of the above generalized Leray's problem, we first introduce a weak formulation. Multiplying $(1.4)_1$ with $\boldsymbol{\varphi} \in \boldsymbol{X}$ and integrating by parts, by the boundary condition $(1.4)_2$, we can obtain

$$2\int_{\mathcal{D}} \mathbb{S}\boldsymbol{u} : \mathbb{S}\boldsymbol{\varphi} dx + \alpha \int_{\partial\mathcal{D}} \boldsymbol{u}_{\tan} \cdot \boldsymbol{\varphi}_{\tan} dS + \int_{\mathcal{D}} \boldsymbol{u} \cdot \nabla\boldsymbol{\varphi} \cdot \boldsymbol{u} dx = 0 \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{X}. \tag{1.7}$$

Now we define the weak solution of the generalized Leray's problem.

Definition 1.1. A vector $\boldsymbol{u} : \mathcal{D} \to \mathbb{R}^3$ is a called a *weak* solution of the generalized Leray's problem (1.4)–(1.6) if and only if

(i) $\boldsymbol{u} \in \mathcal{H}_{\sigma,\mathrm{loc}}(\overline{\mathcal{D}})$;

(ii) $\boldsymbol{u}$ satisfies (1.7);

(iii) $\boldsymbol{u}$ satisfies (1.5) in the trace sense;

(iv) $\boldsymbol{u} - \boldsymbol{g}_\Phi^i \in H^1(\mathcal{D}_i)$ for $i = L, R$.

Remark 1.2. The weak solution also satisfies a generalized version of (1.6). Actually, it follows from the trace inequality (see [8, Theorem II.4.1]) that for any $x_3 > Z$,

$$\int_{\Sigma_R} |\boldsymbol{u}(x_h, x_3) - \boldsymbol{g}_\Phi^R(x_h)|^2 dx_h \leqslant C\|\boldsymbol{u} - \boldsymbol{g}_\Phi^R\|^2_{H^1(\Sigma_R \times (x_3, +\infty))},$$

where the constant C is independent of x_3. This implies that

$$\int_{\Sigma_R} |\boldsymbol{u}(x_h, x_3) - \boldsymbol{g}_\Phi^R(x)|^2 dx_h \to 0 \quad \text{as } x_3 \to \infty.$$

The case of $x_3 < -Z$ is similar.

The following result shows that for each weak solution, we can associate a corresponding pressure field with it (see the proof in Subsection 3.3.2 below).

Lemma 1.3. *Let $\boldsymbol{u}$ be a weak solution to the generalized Leray's problem defined above. Then there exists a scalar function $p \in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$ such that*

$$\int_{\mathcal{D}} \nabla\boldsymbol{u} : \nabla\boldsymbol{\psi} dx + \int_{\mathcal{D}} \boldsymbol{u} \cdot \nabla\boldsymbol{u} \cdot \boldsymbol{\psi} dx = \int_{\mathcal{D}} p\nabla \cdot \boldsymbol{\psi} dx$$

holds for any $\boldsymbol{\psi} \in C_c^\infty(\mathcal{D}; \mathbb{R}^3)$.

1.2 Main results

Now we are ready to state the main theorems of this paper. The first one is the existence of weak solutions, the second one addresses the uniqueness of the weak solution, and the third one concerns the regularity and decay estimates of the weak solution.

Theorem 1.4. *Assume that $\mathcal{D}$ is the aforementioned smoothness domain in* (1.2). *Then there exists a positive constant Φ_0 depending only on α and $\mathcal{D}$ such that for any $\Phi \leqslant \Phi_0$, the generalized Leray's problem* (1.4)–(1.6) *has a weak solution $(\boldsymbol{u}, p) \in \mathcal{H}_{\sigma,\mathrm{loc}(\overline{\mathcal{D}})} \times L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$ satisfying*

$$\sum_{i=L,R} \|\boldsymbol{u} - \boldsymbol{g}_\Phi^i\|_{H^1(\mathcal{D}_i)} \leqslant C_{\alpha,\mathcal{D}}\Phi, \tag{1.8}$$

where $C_{\alpha,\mathcal{D}}$ depends only on α and $\mathcal{D}$.

Theorem 1.4 only states the existence of a weak solution, which has the finite energy property after the background Poiseuille flows are subtracted. However, the uniqueness of the solution is not shown. Actually, the estimate (1.8) is enough for us to deduce the following uniqueness theorem, which allows the energy of the weak solution in $\mathcal{D}_\zeta$ to satisfy 3/2-order growth with respect to ζ. The achievement of this relaxation is due to the application of a key lemma in [15], which will be presented in Section 2.

Theorem 1.5. *Let $(\boldsymbol{u}, p)$ be a weak solution to* (1.4)–(1.6)*. Suppose that for any $\zeta > Z$,*

$$\|\nabla \boldsymbol{u}\|_{L^2(\mathcal{D}_\zeta)} = o(\zeta^{3/2}). \tag{1.9}$$

Then the weak solution is unique[1] *provided that the flux Φ is sufficiently small.*

The following theorem gives the smoothness and the asymptotic behavior of $(\boldsymbol{u}, p)$, which decays exponentially to the Poiseuille flow $\boldsymbol{g}_\Phi^i$ at each outlet $\mathcal{D}_i$ as x_3 tends to infinity.

Theorem 1.6. *Let $\boldsymbol{u}$ be the weak solution in Theorem* 1.4 *and p be the corresponding pressure. Then $(\boldsymbol{u}, p) \in C^\infty(\overline{\mathcal{D}})$ such that for any $m = |\beta| \geqslant 0$,*

$$\sum_{i=L,R} \|\nabla^\beta(\boldsymbol{u} - \boldsymbol{g}_\Phi^i)\|_{L^2(\mathcal{D}_i)} + \|\nabla^\beta \boldsymbol{u}\|_{L^2(\mathcal{D}_M)} \leqslant C_{m,\alpha,\mathcal{D}}\Phi. \tag{1.10}$$

Meanwhile, the following pointwise decay estimates hold:

$$\begin{aligned} &|\nabla^\beta(\boldsymbol{u} - \boldsymbol{g}_\Phi^L)(x)| \leqslant C_{m,\alpha,\mathcal{D}}\Phi \exp\{-\sigma_{m,\alpha,\mathcal{D}}|x_3|\} \quad \textit{for all } x_3 < -Z - 1, \\ &|\nabla^\beta(\boldsymbol{u} - \boldsymbol{g}_\Phi^R)(x)| \leqslant C_{m,\alpha,\mathcal{D}}\Phi \exp\{-\sigma_{m,\alpha,\mathcal{D}}|x_3|\} \quad \textit{for all } x_3 > Z + 1. \end{aligned} \tag{1.11}$$

Here, $C_{m,\alpha,\mathcal{D}}$ and $\sigma_{m,\alpha,\mathcal{D}}$ are positive constants depending on m, α and $\mathcal{D}$.

Remark 1.7. For the pressure p which is generated in Lemma 1.3, there exist two constants $C_{P,L}$, $C_{P,R} > 0$ (see (3.3)) and a smooth cut-off function η with

$$\eta(x_3) = \begin{cases} 1 & \text{for } x_3 > Z, \\ 0 & \text{for } x_3 < Z/2 \end{cases}$$

(which is given in (3.6)) such that for any $m = |\beta| \geqslant 0$,

$$\left\| \nabla^\beta \nabla \left(p + \frac{\Phi \int_{-\infty}^{x_3} \eta(s) ds}{C_{P,R}} - \frac{\Phi \int_{-\infty}^{-x_3} \eta(s) ds}{C_{P,L}} \right) \right\|_{L^2(\mathcal{D})} \leqslant C_{m,\alpha,\mathcal{D}}\Phi.$$

Meanwhile, the following pointwise decay estimate holds: for all $|x_3| > Z + 1$,

$$\left| \nabla^\beta \nabla \left(p + \frac{\Phi \int_{-\infty}^{x_3} \eta(s) ds}{C_{P,R}} - \frac{\Phi \int_{-\infty}^{-x_3} \eta(s) ds}{C_{P,L}} \right)(x) \right| \leqslant C_{m,\alpha,\mathcal{D}}\Phi \exp\{-\sigma_{m,\alpha,\mathcal{D}}|x_3|\},$$

where $C_{m,\alpha,\mathcal{D}}$ and $\sigma_{m,\alpha,\mathcal{D}}$ are positive constants depending on m, α and $\mathcal{D}$. The subtracted term

$$p_{\boldsymbol{g}} := -\frac{\Phi \int_{-\infty}^{x_3} \eta(s) ds}{C_{P,R}} + \frac{\Phi \int_{-\infty}^{-x_3} \eta(s) ds}{C_{P,L}}$$

is set to balance the pressure of the Poiseuille flows.

1.3 Main difficulties, strategies and outline of the proof

Difficulties. Compared with the no-slip boundary condition, the main difficulties of the problem with the Navier-slip boundary condition lie in

[1] The pressure p is unique up to subtracting an arbitrary constant.

(i) the absence of the Korn-type inequality (the L^2-norm equivalence between $\nabla \boldsymbol{v}$ and $\mathbb{S}\boldsymbol{v}$) on $\mathcal{D}$ with a noncompact boundary;

(ii) the construction of a smooth solenoidal vector field $\boldsymbol{a}$ satisfying the Navier-slip boundary condition and equalling the Poiseuille flow at a large distance for a given flux;

(iii) achieving Poincaré-type inequalities under the Navier-slip boundary condition;

(iv) derivation of the global H^2-estimate of the H^1-weak solution.

Strategies. To overcome the difficulties listed above, our main strategies are as follows:

(i) During the proof of both existence and uniqueness, an H^1-estimate of the solution is required. Owing to the boundary condition, we only have the L^2-estimate of the stress tensor $\mathbb{S}v$. However, the Korn-type inequality is not applicable to our domain considered here with a noncompact boundary. Fortunately, the energy estimate of the stress tensor $\mathbb{S}v$ will produce a boundary integration with a good sign, which can be used to control the bad terms coming from the energy estimate of the gradient of the velocity. At last, by combining the uniform energy estimates of the stress tensor with the L^2-estimate of the gradient of the velocity, we can achieve the H^1-estimate of $\boldsymbol{v}$.

(ii) Our main idea in constructing $\boldsymbol{a}$ is to smoothly connect Poiseuille flows in $\boldsymbol{g}_\Phi^L$ and $\boldsymbol{g}_\Phi^R$ with a compactly supported divergence-free vector $(0, 0, h(x_h))$ in $\mathcal{D}_M$. In the intermediate parts, we glue them by solving a 2D divergence equation in the cross section with the 2D Navier-slip boundary condition.

(iii) For the no-slip boundary condition, the Poincaré inequality can be applied directly. However, in the case of the Navier-slip boundary condition, the Poincaré inequality is not obvious in both straight pipes $\mathcal{D}_L$, $\mathcal{D}_R$ and the truncated pipe $\mathcal{D}_\zeta$. To handle the case in $\mathcal{D}_L$ or $\mathcal{D}_R$, we divide a vector-valued function into the x_h-direction part and the x_3-direction part. The first part follows from a 2D Payne's identity (2.5) and the impermeable boundary condition, while the second part is achieved by subtracting the constant flux so that v_3 has zero mean value in any cross section of $\mathcal{D}_L$ or $\mathcal{D}_R$ (see Lemma 2.5). Based on the result of the straight pipe, we derive the Poincaré inequality in $\mathcal{D}_\zeta$ by the trace theorem and a 3D Payne's identity (2.11) (see Lemma 2.6).

(iv) Our idea of obtaining the global H^2-estimate is to decompose $\mathcal{D}$ into a series of bounded smooth domains $\tilde{\mathfrak{D}}_k$ which only have three shapes so that the related estimate constant in $\tilde{\mathfrak{D}}_k$ could be uniform with k. In each $\tilde{\mathfrak{D}}_k$, we establish the H^2-estimate of the solution via the known conclusions for the linear Stokes system with the Navier-slip boundary condition in [21]. Then we achieve the global H^2-estimate by summarizing those estimates in $\tilde{\mathfrak{D}}_k$.

Outline of the proof. The existence of the solution will be given in Section 3. First, the Poiseuille flows in $\mathcal{D}_i$ $(i = L, R)$ with their fluxes being Φ and satisfying the Navier-slip boundary condition will be constructed. Then a smooth divergence-free vector field in $\mathcal{D}$ subject to the Navier-slip boundary condition and equalling the Poiseuille flows at the far left and the far right will be introduced. In this way, we can reduce the existence problem to a related one in which the solution approaches zero at spatial infinity. Then this problem can be handled by the standard Galerkin method.

The proof of the uniqueness is derived in Section 4. The main idea is applying Lemma 2.7, which was originally announced in [15] as far as the authors know. If $(\boldsymbol{u}, p)$ and $(\tilde{\boldsymbol{u}}, \tilde{p})$ are two distinct solutions, we define the energy integral in terms of $\boldsymbol{w} := \tilde{\boldsymbol{u}} - \boldsymbol{u}$ as follows:

$$Y(K) := \int_{K-1}^{K} \int_{\mathcal{D}_\zeta} |\nabla \boldsymbol{w}|^2 dx d\zeta.$$

An ordinary differential inequality of $Y(K)$, which satisfies the assumption in Lemma 2.7, will be derived. The derivation of this inequality involves a series of estimates, two terms of which are especially different from the previous literature, i.e., $\int_{\Omega_\zeta} p v_3 dx$ and $\int_{\mathcal{D}_\zeta} \boldsymbol{v} \cdot \Delta \boldsymbol{v} dx$. The estimate of the first term involves an application of the partial Poincaré inequality in Lemma 2.4 and a divergence-gradient operator estimate in Lemma 4.1. The estimate of the second term is derived by combining the L^2-norms of both the stress tensor and the gradient of the velocity. At last, the vanishing of $Y(K)$ will be proved, which indicates the uniqueness of the solution.

Proofs of the smoothness and exponential decay of the solution to the Poiseuille flow at a large spatial distance are given in Section 5. By the "decomposing-summarizing" technique in the strategy (iv),

$W^{1,3}$ and then H^2 global estimates will be derived. By the bootstrapping argument, higher-order global estimates will follow. For the exponential decay estimates, here goes the idea: by defining the following energy for $\zeta > Z+1$:

$$\mathcal{G}(\zeta) := \int_{\Sigma_L\times(-\infty,-\zeta)} |\nabla(\boldsymbol{u}-\boldsymbol{g}_\Phi^L)|^2dx + \int_{\Sigma_R\times(\zeta,+\infty)} |\nabla(\boldsymbol{u}-\boldsymbol{g}_\Phi^R)|^2dx,$$

then we can derive a first-order ordinary differential inequality, which will result in the exponential decay of $\mathcal{G}(\zeta)$. Finally, higher-order estimates of the solution in $\mathcal{D}-\mathcal{D}_\zeta$, the Sobolev embedding, and the exponential decay of $\mathcal{G}(\zeta)$ will validate the pointwise decay of the solution in (1.11).

1.4 Related works

Before ending Section 1, we review some works related to the solvability of Leray's problem in (1.4)–(1.6). The original Leray's problem concerns the existence, uniqueness, regularity, and asymptotic behavior of the system (1.4)–(1.6) with the no-slip boundary condition (corresponding to $\alpha=+\infty$ in $(1.4)_2$). See the description in [14, p. 77] and [13, p. 551]. Amick [2, 3] contributed the first remarkable work on the solvability of Leray's problem with a small flux and the no-slip boundary condition, which reduced the solvability problem to the resolution of a variational problem. However, the uniqueness is left open. A detailed analysis of the existence, uniqueness, and asymptotic behavior of small-flux solutions is given by Ladyženskaya and Solonnikov [15]. More details on well-posedness, decay, and far-field asymptotic analysis of solutions for Leray's problem with the no-slip boundary condition and related topics can be found in [1, 11, 20] and the references therein. Readers can trace to [8, Chapter XIII] for a systematic review and study of Leray's problem with the no-slip boundary condition. Recently, Yang and Yin [27] studied the well-posedness of weak solutions to the steady non-Newtonian fluids in pipe-like domains. Wang and Xie [23, 24] studied the existence, uniqueness, and uniform structural stability of Poiseuille flows for the 3D axially symmetric inhomogeneous Navier-Stokes equations in the 3D pipe.

Compared with the no-slip boundary condition, Leray's problem with the Navier-slip boundary condition, which also has different physical interpretations and mathematical properties, seems to be much more complicated. Konieczny [12] and Mucha [18, 19] studied the solvability of the steady Navier-Stokes equations with the perfect Navier-slip condition ($\alpha=0$), where they employed a constant vector field as its asymptotic profile at the spatial infinity. Only the existence, regularity, and asymptotic behavior of weak solutions were addressed there. The uniqueness was left open, and the asymptotic behavior at far fields was not given there. The problem raised there could be recognized as Leray's problem with the complete Navier-slip boundary condition in a two-dimensional strip, where the asymptotic profile is a constant vector. Our problem raised in (1.4)–(1.6) is a perfect extension of the original Leray's problem with the no-slip boundary condition. The background Poiseuille flows considered here tend to Leray's Poiseuille flows with the no-slip boundary condition as $\alpha\to+\infty$. As far as the authors know, there is little literature concerning the solvability of the generalized Leray's problem (1.4)–(1.6), which settles the well-posedness issue on the steady Navier-Stokes equations subject to the Navier-slip boundary in an unbounded domain with an unbounded boundary, while for the well-posedness of the solutions to the steady Navier-Stokes equations with the Navier-slip boundary in bounded domains, there have already been many works that we can refer to (see [4, 9, 21] and the references therein). Recently, Wang and Xie [25] gave the uniqueness and uniform structural stability of Poiseuille flows in an infinitely long pipe with the Navier-slip boundary condition for the inhomogeneous axially symmetric Navier-Stokes equations. Their primary strategy is a delicate decomposition in the 2D plane for the slip coefficient and the frequency corresponding to the Fourier variable in the axial direction and energy estimates are performed on the stream function, which are essentially different from ours as shown in Subsection 1.3. Li et al. [16] gave the characterization of bounded smooth solutions for the axially symmetric Navier-Stokes equations with the perfect Navier-slip boundary condition (corresponding to $\alpha=0$ in $(1.3)_1$) in the infinitely long cylinder, mainly with the aid of the Moser iteration technique and an energy estimate solely for the stress tensor, which is different from the compound energy estimates in this paper.

The rest of this paper is organized as follows. Section 2 contains the preliminary work of the proof, in which the Navier-slip boundary condition will be written under the "natural" moving frame of $\partial\mathcal{D}$, and some useful lemmas will be presented. Section 3 is devoted to the proof of existence results. In Section 4, we finish the proof of the uniqueness of the solution. Finally, we focus on the higher-order regularity and exponential decay properties of the solution in Section 5.

At last, we emphasize that the domain considered in this paper is a 3D distorted pipe, while if the domain is a two-dimensional distorted strip, similar results as stated in Theorems 1.4–1.6 will also be obtained. More precisely, if we consider the Navier-Stokes equation (1.1) with the Navier-slip boundary condition (1.3) in the strip $[0,1]\times\mathbb{R}$, the following two-dimensional Poiseuille flow will be obtained by a direct calculation:

$$\boldsymbol{g}_\Phi = \left(0, \frac{6\alpha\Phi}{6+\alpha}(-x_1^2+x_1)+\frac{6\Phi}{6+\alpha}\right). \tag{1.12}$$

After a compact perturbation of the domain $\mathbb{R}\times[0,1]$, the existence, uniqueness, and regularity of the solutions, which approach $\boldsymbol{g}_\Phi$ in (1.12) at spatial infinity will be presented in our forthcoming paper, where the flux at the cross section Φ can be relatively large.

2 Preliminaries

2.1 Reformulation of the boundary condition in the local orthogonal curvilinear coordinates

First, we rewrite the boundary condition (1.3) in the locally moving coordinate framework.

Regarding the smoothness of the pipe $\mathcal{D}$, for any given point on $\partial\mathcal{D}$, we define $(\gamma_1,\gamma_2,\gamma_3)$ to be a system of orthogonal curvilinear coordinates in $U\subset\mathbb{R}^3$, where U is a neighborhood of the aforementioned point. The surface $\gamma_3=0$ represents a portion of the surface $\mathcal{D}$, and the surfaces $\gamma_3=$ constant are parallel to this portion with γ_3 increasing towards the outside of $\mathcal{D}$. On each surface $\gamma_3=$ constant, two families of curves, the γ_1-curve and the γ_2-curve, are lines of the curvature of the surface. Their unit tangent vectors $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ and the normal vector $\boldsymbol{n}$ form an orthogonal basis at each point of the neighborhood U with the Lamé coefficients $H_1,H_2,H_3>0$ such that

$$\begin{cases} \partial_{\gamma_i}x = H_i\boldsymbol{\tau_i} \quad \text{for } i=1,2, \\ \partial_{\gamma_3}x = H_3\boldsymbol{n} \end{cases}$$

(see Figure 2). Under these (local) curvilinear coordinates, one can write

$$\boldsymbol{u} = u_{\tau_1}\boldsymbol{\tau_1}+u_{\tau_2}\boldsymbol{\tau_2}+u_n\boldsymbol{n}.$$

Then (1.3) enjoys the following simplified expression:

$$\begin{cases} \partial_{\boldsymbol{n}}u_{\tau_1} = (\kappa_1(x)-\alpha)u_{\tau_1}, \\ \partial_{\boldsymbol{n}}u_{\tau_2} = (\kappa_2(x)-\alpha)u_{\tau_2}, \quad \text{on } U\cap\partial\mathcal{D} \\ u_n = 0, \end{cases} \tag{2.1}$$

(see [26, Proposition 2.1 and Corollary 2.2]). Here, for $i=1,2$,

$$\kappa_i(x) = -\frac{\boldsymbol{n}}{H_i}\cdot\frac{\partial\boldsymbol{\tau}_i}{\partial\gamma_i} \tag{2.2}$$

are the principal curvatures of $\partial\mathcal{D}$ corresponding to the γ_i-curves, respectively.

Note that $\partial\mathcal{D}$ can be divided into two parts. The part $\partial\mathcal{D}\cap\partial\mathcal{D}_M$ is a compact manifold, and thus one can deduce the uniform boundedness of κ_i ($i=1,2$) there. The other part $\partial\mathcal{D}\cap\partial(\mathcal{D}_L\cup\mathcal{D}_R)$ is a combination of two semi-infinite smooth straight pipes, whose curvature depends only on the scalar curvatures of smooth Jordan curves $\partial\Sigma_L$ and $\partial\Sigma_R$, which is clearly uniformly bounded. More details on the locally natural moving frame on $\partial\mathcal{D}\cap(\partial\mathcal{D}_R\cup\partial\mathcal{D}_L)$ is shown in the following remark.

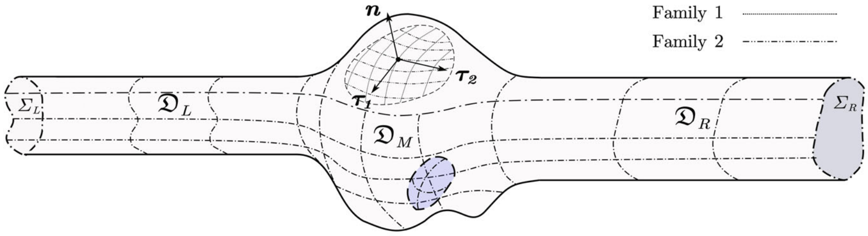

Figure 2 (Color online) The local orthogonal curvilinear coordinates on $\partial\mathcal{D}$

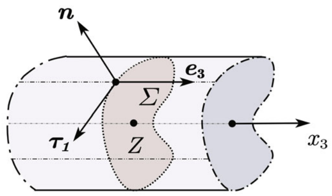

Figure 3 (Color online) The orthogonal curvilinear coordinates on $\partial\mathcal{D}\cap(\partial\mathcal{D}_R\cup\partial\mathcal{D}_L)$

Remark 2.1. In the "straight" part of the pipe (i.e., $\mathcal{D}_L\cup\mathcal{D}_R$), one can always choose $\boldsymbol{\tau_2}=\boldsymbol{e_3}$, which is a constant vector. Meanwhile, $\boldsymbol{\tau_1}$ and $\boldsymbol{n}$ are the unit tangent vector and the unit outer normal vector of the cross section Σ, respectively (see Figure 3).

In this case, we find $\boldsymbol{\tau_1}$ and $\boldsymbol{n}$, which are perpendicular to the x_3-axis and depend only on (x_1,x_2). From (2.2), the principal curvature κ_2 is identically zero. By writing

$$\boldsymbol{u}=u_{\tau_1}\boldsymbol{\tau_1}+u_3\boldsymbol{e_3}+u_n\boldsymbol{n},$$

we see that (2.1) is simplified to

$$\begin{cases}\partial_{\boldsymbol{n}}u_{\tau_1}=(\kappa_1(x)-\alpha)u_{\tau_1},\\ \partial_{\boldsymbol{n}}u_3=-\alpha u_3, & \text{on } \partial\mathcal{D}\cap(\partial\mathcal{D}_R\cup\partial\mathcal{D}_L).\\ u_n=0,\end{cases} \tag{2.3}$$

Since $\partial\Sigma_i$ ($i=L$ or R) is smooth and compact, the principal curvature κ_1 must be uniformly bounded on $\partial\mathcal{D}\cap(\partial\mathcal{D}_R\cup\partial\mathcal{D}_L)$.

Thus one deduces the following result at the end of this subsection.

Proposition 2.2. *The principal curvature κ_i ($i=1$ or $i=2$) is uniformly bounded on $\partial\mathcal{D}$.*

2.2 Useful lemmas

In this subsection, we give some useful lemmas which will be frequently used throughout the rest of this paper. Lemmas 2.3–2.6 concern the Poincaré inequalities of the solution $\boldsymbol{u}$ in a part of the straight pipe $\Sigma\times\mathbb{R}$ and the truncated finite pipe $\mathcal{D}_\zeta$ with only the impermeable condition $(1.3)_2$. Lemma 2.7 is introduced to show the uniqueness of the solution. Lemmas 2.8 and 2.9 are regularity results of linear Stokes systems on a bounded domain, which will be applied in the bootstrapping argument in Section 5.

In the standard Euclidean coordinate frameworks $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$, let $\boldsymbol{u}=u_1\boldsymbol{e_1}+u_2\boldsymbol{e_2}+u_3\boldsymbol{e_3}$. $\Sigma\subset\mathbb{R}^2$ is a smooth bounded domain in the x_h directions and $I\subset\mathbb{R}$ is an (infinite or finite) interval in the x_3 direction. We set $\boldsymbol{n}=(n_1,n_2,0)$ to be the unit outer normal vector on $\partial\Sigma\times I$, where $\bar{\boldsymbol{n}}=(n_1,n_2)$ is the unit outer normal vector on $\partial\Sigma$.

Lemma 2.3. *Let $\Sigma\subset\mathbb{R}^2$ be a compact domain with a C^1 boundary, and $\boldsymbol{f}=f_1\boldsymbol{e_1}+f_2\boldsymbol{e_2}$ be a two-dimensional vector function with components in $H^1(\Sigma)$, and $\boldsymbol{f}\cdot\bar{\boldsymbol{n}}=0$ on $\partial\Sigma$, where $\bar{\boldsymbol{n}}$ is the unit outer normal vector of $\partial\Sigma$. Then the following Poincaré inequality holds*:

$$\|\boldsymbol{f}\|_{L^2(\Sigma)}\leqslant C_\Sigma\|\nabla_h\boldsymbol{f}\|_{L^2(\Sigma)}, \tag{2.4}$$

where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ is the gradient operator on the x_1 and x_2 directions and $C_\Sigma = Cd_\Sigma$, where C is a constant and $d_\Sigma := \max_{x_h, y_h \in \Sigma}\{|x_h - y_h|\}$ is the diameter of Σ.

Proof. See the hint in [8, Exercise II.5.6, p. 71]. Here, we give the proof for completeness. First, we choose a fixed point $x_0 = (x_{0,1}, x_{0,2}) \in \Sigma$, and then it is easy to check the following equality:

$$\sum_{i,j=1}^{2} [\partial_{x_i}(f_i(x_j - x_{0,j})f_j) - \partial_{x_i} f_i(x_j - x_{0,j})f_j - |\boldsymbol{f}|^2 - f_i(x_j - x_{0,j})\partial_{x_i} f_j] = 0. \tag{2.5}$$

Integrating the above equality on Σ, one deduces

$$\begin{aligned}\int_\Sigma |\boldsymbol{f}|^2 dx_h = & \underbrace{\sum_{i,j=1}^{2} \int_\Sigma \partial_i(f_i(x_j - x_{0,j})f_j) dx_h}_{I_1} - \sum_{i,j=1}^{2} \int_\Sigma \partial_i f_i(x_j - x_{0,j}) f_j dx_h \\ & - \sum_{i,j=1}^{2} \int_\Sigma f_i(x_j - x_{0,j})\partial_i f_j dx_h. \end{aligned} \tag{2.6}$$

Using the divergence theorem and the boundary condition $\boldsymbol{f} \cdot \bar{\boldsymbol{n}} = 0$, we can obtain

$$I_1 = \sum_{j=1}^{2} \int_{\partial\Sigma} \bar{\boldsymbol{n}} \cdot \boldsymbol{f}(x_j - x_{0,j}) f_j dS = 0.$$

Thus by (2.6) and the Cauchy-Schwarz inequality, we arrive at

$$\int_\Sigma |\boldsymbol{f}|^2 dx_h \leqslant \frac{1}{2} \int_\Sigma |\boldsymbol{f}|^2 dx_h + C d_\Sigma^2 \int_\Sigma |\nabla_h \boldsymbol{f}|^2 dx_h,$$

which indicates (2.4). □

If we choose $\boldsymbol{f} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2}$ as in Lemma 2.3, we deduce the following lemma.

Lemma 2.4 (Partial Poincaré inequality in a straight pipe). *Let $\boldsymbol{u} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3}$ be an H^1 vector field in $\Sigma \times I$. If $\boldsymbol{u}$ satisfies the boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, then the following Poincaré inequality holds*:

$$\|u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2}\|_{L^2(\Sigma\times I)} \leqslant C\|\nabla_h(u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2})\|_{L^2(\Sigma\times I)}, \tag{2.7}$$

where C is a positive constant.

Proof. For any $x = (x_h, x_3) \in \Sigma \times I$, by the impermeable condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, one sees that

$$(u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2})(x_h, x_3) \cdot \bar{\boldsymbol{n}} = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{for any } x_h \in \partial\Sigma.$$

Then Lemma 2.3 indicates

$$\|(u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2})(\cdot, x_3)\|^2_{L^2(\Sigma)} \leqslant C^2\|\nabla_h(u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2})(\cdot, x_3)\|^2_{L^2(\Sigma)}.$$

Integrating with x_3 over I on both sides, respectively, one concludes (2.7). □

We notice that the Poincaré inequality could not hold for u_3, due to the existence of the parallel flow (see, e.g., (3.1) below). Nevertheless, the mean value of u_3 through the cross section Σ is conserved for $x_3 \in I$ if $\boldsymbol{u}$ is a divergence-free vector field. The reason is that denoting the flux flowing across Σ by

$$\Phi(x_3) = \int_\Sigma \boldsymbol{u}(x_h, x_3) \cdot \boldsymbol{e_3} dx_h,$$

and then applying the divergence-free property of $\boldsymbol{u}$ and the impermeable condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, we have

$$\begin{aligned}\frac{d}{dx_3}\Phi(x_3) &= \int_\Sigma \partial_{x_3} u_3(x_h, x_3) dx_h \\ &= -\int_\Sigma (\partial_{x_1} u_1 + \partial_{x_2} u_2)(x_h, x_3) dx_h \\ &= -\int_{\partial\Sigma} (\boldsymbol{u} \cdot \boldsymbol{n})(x_h, x_3) dS(x_h) = 0.\end{aligned}$$

This implies the constancy of the flux Φ. Then we have the following lemma.

Lemma 2.5. *Let $\boldsymbol{u} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3}$ be an H^1 vector field in $\Sigma \times I$, which is divergence-free and satisfies the boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. Set $g(x_h) \in H^1(\Sigma)$ satisfying*

$$\int_\Sigma g(x_h)dx_h = \Phi,$$

and define $\boldsymbol{v} := \boldsymbol{u} - g(x_h)\boldsymbol{e_3}$. Then we have

$$\|\boldsymbol{v}\|_{L^2(\Sigma\times I)} \leqslant C\|\nabla_h \boldsymbol{v}\|_{L^2(\Sigma\times I)}, \tag{2.8}$$

where C is a positive constant.

Proof. Notice that v_3 has a vanishing mean value on the cross section Σ, and therefore it enjoys the following 2D Poincaré inequality:

$$\int_\Sigma |v_3(x_h,x_3)|^2dx_h = \int_\Sigma \left|v_3(x_h,x_3) - \frac{1}{|\Sigma|}\int_\Sigma v_3(x_h,x_3)dx_h\right|^2 dx_h \leqslant C^2\int_\Sigma |\nabla_h v_3(x_h,x_3)|^2dx_h. \tag{2.9}$$

This indicates the 3D Poincaré inequality

$$\|v_3\|_{L^2(\Sigma\times I)} \leqslant C\|\nabla_h v_3\|_{L^2(\Sigma\times I)}$$

if one integrates (2.9) with x_3 on the interval I. Combining the result in Lemma 2.4, we can obtain (2.8). □

Based on Lemma 2.5, one has the following Poincaré-type inequality in the truncated distorted pipe $\mathcal{D}_\zeta = \{x \in \mathcal{D} : -\zeta \leqslant x_3 \leqslant \zeta\}$.

Lemma 2.6. *Given $\zeta \geqslant Z$, let $\boldsymbol{w} = (w_1,w_2,w_3) \in H^1(\mathcal{D}_\zeta)$ with zero flux in $\mathcal{D}_\zeta$, i.e.,*

$$\int_{\Sigma_R} \boldsymbol{w}(x_h, Z/2)\cdot\boldsymbol{e_3}dx_h = 0.$$

If we suppose $\boldsymbol{w}\cdot\boldsymbol{n} \equiv 0$ on $\partial\mathcal{D}\cap\partial\mathcal{D}_\zeta$, where $\boldsymbol{n}$ is the unit outer normal vector on $\partial\mathcal{D}$, then the following Poincaré inequality holds:

$$\|\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)} \leqslant C_{\mathcal{D}}\|\nabla\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)}. \tag{2.10}$$

Here, $C_{\mathcal{D}} > 0$ is a constant which is uniform with ζ.

Proof. Integrating the following identity on $\mathcal{D}_M = \{x \in \mathcal{D} : -Z/2 \leqslant x_3 \leqslant Z/2\}$:

$$\sum_{i,j=1}^3 [\partial_{x_i}(w_ix_jw_j) - \partial_{x_i}w_ix_jw_j - |\boldsymbol{w}|^2 - w_ix_j\partial_{x_i}w_j] = 0, \tag{2.11}$$

and using an approach similar to our estimation of the terms on the right-hand side of (2.6), one derives

$$\begin{aligned}\int_{\mathcal{D}_M}|\boldsymbol{w}|^2dx \leqslant{}& \frac12\int_{\mathcal{D}_M}|\boldsymbol{w}|^2dx + C_{\mathcal{D}}\int_{\mathcal{D}_M}|\nabla\boldsymbol{w}|^2dx\\ &+\left|\int_{\Sigma_L}(w_3(x\cdot\boldsymbol{w}))(x_h,-Z/2)dS\right| + \left|\int_{\Sigma_R}(w_3(x\cdot\boldsymbol{w}))(x_h,Z/2)dS\right|,\end{aligned}$$

which indicates

$$\int_{\mathcal{D}_M}|\boldsymbol{w}|^2dx \leqslant C_{\mathcal{D}}\bigg(\int_{\mathcal{D}_M}|\nabla\boldsymbol{w}|^2dx + \int_{\Sigma_L}|\boldsymbol{w}(x_h,-Z/2)|^2dS + \int_{\Sigma_R}|\boldsymbol{w}(x_h,Z/2)|^2dS\bigg). \tag{2.12}$$

Meanwhile, using the trace theorem in $\Sigma_L\times[-Z,-Z/2]$ and Lemma 2.5, one derives

$$\int_{\Sigma_L}|\boldsymbol{w}(x_h,-Z/2)|^2dS \leqslant C_{\mathcal{D}}\bigg(\int_{\Sigma_L\times[-Z,-Z/2]}|\boldsymbol{w}|^2dx + \int_{\Sigma_L\times[-Z,-Z/2]}|\nabla\boldsymbol{w}|^2dx\bigg)$$

$$\leqslant C_{\mathcal{D}} \int_{\Sigma_L \times [-Z, -Z/2]} |\nabla \boldsymbol{w}|^2 dx. \tag{2.13}$$

Similarly, one derives that

$$\int_{\Sigma_R} |\boldsymbol{w}(x_h, Z/2)|^2 dS \leqslant C_{\mathcal{D}} \int_{\Sigma_R \times [Z/2, Z]} |\nabla \boldsymbol{w}|^2 dx. \tag{2.14}$$

Substituting (2.13) and (2.14) into the right-hand side of (2.12), one deduces

$$\int_{\mathcal{D}_M} |\boldsymbol{w}|^2 dx \leqslant C_{\mathcal{D}} \int_{D_\zeta} |\nabla \boldsymbol{w}|^2 dx.$$

This finishes the estimate in $\mathcal{D}_M$. Noting that the remaining parts of the domain D_ζ are a union of two straight truncated pipes, we see that the estimates in $\mathcal{D}_\zeta - \mathcal{D}_M$ are direct conclusions of Lemma 2.5. We conclude the proof of (2.10). □

The following asymptotic estimate of a function that satisfies an ordinary differential inequality will be useful in our further proof. To the best of the authors' knowledge, it was originally derived by Ladyženskaya and Solonnikov [15].

Lemma 2.7. *Let $Y(\zeta) \not\equiv 0$ be a nondecreasing nonnegative differentiable function satisfying*

$$Y(\zeta) \leqslant \Psi(Y'(\zeta)), \quad \forall\, \zeta > 0. \tag{2.15}$$

Here, $\Psi : [0, \infty) \to [0, \infty)$ is a monotonically increasing function with $\Psi(0) = 0$, and there exist $C, \tau_1 > 0$ and $m > 1$ such that

$$\Psi(\tau) \leqslant C\tau^m, \quad \forall\, \tau > \tau_1. \tag{2.16}$$

Then

$$\liminf_{\zeta \to +\infty} \zeta^{-\frac{m}{m-1}} Y(\zeta) > 0. \tag{2.17}$$

Proof. Since Y is not identically zero, there exists a $\zeta_0 > 0$ such that $Y(\zeta_0) =: Y_0 > 0$. Using the monotonicity of Ψ, one knows that

$$Y'(\zeta_0) \geqslant \Psi^{-1}(Y_0) =: \eta_0 > 0.$$

Therefore, we have

$$Y(\zeta) \geqslant Y_0 + \eta_0(\zeta - \zeta_0) \quad \text{for } \zeta \geqslant \zeta_0.$$

From (2.15), we see that

$$Y'(\zeta) \geqslant \Psi^{-1}(Y(\zeta)) \geqslant \Psi^{-1}(Y_0 + \eta_0(\zeta - \zeta_0)) \quad \text{for } \zeta \geqslant \zeta_0.$$

When $\zeta \to +\infty$, we can deduce that $\Psi^{-1}(Y_0 + \eta_0(\zeta - \zeta_0)) \to +\infty$, otherwise if $\Psi^{-1}(Y_0 + \eta_0(\zeta - \zeta_0)) \to A < +\infty$,

$$Y_0 + \eta_0(\zeta - \zeta_0) = \Psi(\Psi^{-1}(Y_0 + \eta_0(\zeta - \zeta_0))) \leqslant \Psi(A) < +\infty,$$

which is invalid as $\zeta \to +\infty$.

So we see that there exists a $\zeta_1 \geqslant \zeta_0$ such that $Y'(\zeta) \geqslant \tau_1$ for any $\zeta \geqslant \zeta_1$. Then from (2.15) and (2.16), we have that for $\zeta \geqslant \zeta_1$,

$$Y(\zeta) \leqslant C(Y'(\zeta))^m.$$

Integrating the above inequality on $[\zeta_1, \infty)$, one concludes the proof of (2.17). □

At the end of this subsection, we introduce the following results which focus on the $W^{1,3}$-weak solution and the H^m-strong solution of the linear Stokes equations on *bounded domains* with the Navier-slip boundary condition.

Lemma 2.8 (See [21, Corollary 5.7]). *Let Ω be a bounded smooth domain, $\boldsymbol{f} \in L^{\frac{3}{2}}(\Omega), \boldsymbol{F} \in L^3(\Omega)$ and $\boldsymbol{h} \in W^{-\frac{1}{3},3}(\partial\Omega)$. Then the Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla P = \operatorname{div} \boldsymbol{F} + \boldsymbol{f}, & \nabla \cdot \boldsymbol{v} = 0, \quad \text{in } \Omega, \\ 2(\mathbb{S}\boldsymbol{v} \cdot \boldsymbol{n})_{\tan} + \alpha \boldsymbol{v}_{\tan} = \boldsymbol{h}, & \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega \end{cases}$$

has a unique solution $(\boldsymbol{v}, P) \in W^{1,3}(\Omega) \times L^3(\Omega)$, which satisfies the estimate

$$\|\boldsymbol{v}\|_{W^{1,3}(\Omega)} + \|P\|_{L^3(\Omega)} \leqslant C_{\alpha,\Omega}(\|\boldsymbol{f}\|_{L^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{F}\|_{L^3(\Omega)} + \|\boldsymbol{h}\|_{W^{-\frac{1}{3},3}(\partial\Omega)}). \tag{2.18}$$

Lemma 2.9 (See [21, Theorem 4.5] and [9, Theorem 2.5.10]). *Let Ω be a bounded smooth domain, $m \in \mathbb{N}$, $\boldsymbol{f} \in H^m(\Omega)$ and $\boldsymbol{h} \in H^{m+\frac{1}{2}}(\partial\Omega)$. Then the solution of the Stokes problem*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla P = \boldsymbol{f}, & \nabla \cdot \boldsymbol{v} = 0, \quad \text{in } \Omega, \\ 2(\mathbb{S}\boldsymbol{v} \cdot \boldsymbol{n})_{\tan} + \alpha \boldsymbol{v}_{\tan} = \boldsymbol{h}, & \boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega \end{cases}$$

satisfies $(\boldsymbol{v}, P) \in H^{m+2}(\Omega) \times H^{m+1}(\Omega)$. Also, it enjoys the following estimate:

$$\|\boldsymbol{v}\|_{H^{m+2}(\Omega)} + \|P\|_{H^{m+1}(\Omega)} \leqslant C_{\alpha,\Omega}(\|\boldsymbol{f}\|_{H^m(\Omega)} + \|\boldsymbol{h}\|_{H^{m+\frac{1}{2}}(\partial\Omega)}). \tag{2.19}$$

3 Existence

3.1 On Poiseuille flows in pipes $\mathcal{D}_L$ and $\mathcal{D}_R$

In this subsection, we introduce Poiseuille flows in pipes $\mathcal{D}_L$ and $\mathcal{D}_R$, which are solutions of the system (1.4) in $\Sigma_i \times \mathbb{R}$ ($i = L$ or $i = R$). We drop the index i for convenience in this subsection. To find a Poiseuille flow $\boldsymbol{g}_\Phi$ in $\Sigma \times \mathbb{R}$ with a given flux Φ, one needs to find a function $g_\Phi : \Sigma \to \mathbb{R}$ such that

$$\begin{cases} \boldsymbol{g}_\Phi = g_\Phi \boldsymbol{e_3}, \\ -\Delta_h g_\Phi(x_h) = \text{constant} \quad \text{in } \Sigma, \\ \dfrac{\partial g_\Phi}{\partial \bar{\boldsymbol{n}}} = -\alpha g_\Phi \quad \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma g_\Phi(x_h) dx_h = \Phi. \end{cases} \tag{3.1}$$

Here and below, we assume $\Phi \geqslant 0$ without loss of generality (see Figure 4).

Remark 3.1. If Σ is the unit disk in $\mathbb{R}^2$, one has the following exact formula of $\boldsymbol{g}_\Phi$:

$$\boldsymbol{g}_\Phi(x) = \frac{2(\alpha+2)\Phi}{(\alpha+4)\pi}\left(1 - \frac{\alpha}{\alpha+2}|x_h|^2\right)\boldsymbol{e_3} \quad \text{with its pressure } p_\Phi(x) = -\frac{8\alpha\Phi}{(\alpha+4)\pi}x_3,$$

which could be considered as a generalization of the Hagen-Poiseuille flow under the no-slip boundary condition ($\alpha \to +\infty$).

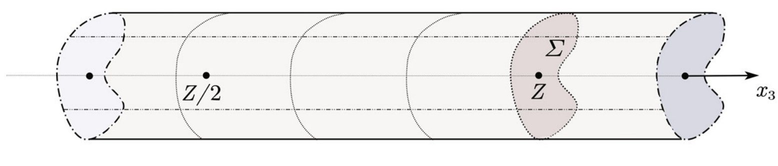

Figure 4 (Color online) A straight infinite pipe $\Sigma \times \mathbb{R}$

The existence and uniqueness of g_Φ in (3.1) could be derived by routine methods of elliptic equations in a bounded smooth domain with the Robin boundary condition. Here, we omit the details. We only derive an H^m-estimate of g_Φ in terms of Φ by the scaling technique.

By the linearity of the problem (3.1), one considers the classical Poisson equation subject to the Robin boundary condition

$$\begin{cases} -\Delta_h\varphi(x_h)=1 & \text{in } \Sigma,\\ \dfrac{\partial\varphi}{\partial\bar{\boldsymbol{n}}}+\alpha\varphi=0 & \text{on } \partial\Sigma. \end{cases} \tag{3.2}$$

The existence and uniqueness of the problem (3.2) are classical. We refer the reader to [10] for details. Moreover, the following estimate of φ can be derived by classical results:

$$\|\varphi\|_{H^m(\Sigma)}\leqslant C_{\alpha,m,\Sigma}.$$

Multiplying $(3.2)_1$ by φ and using integration by parts, we also note that

$$\int_\Sigma \varphi dx_h=-\int_\Sigma \varphi\Delta_h\varphi dx=\int_\Sigma|\nabla_h\varphi|^2dx+\int_{\partial\Sigma}\alpha|\varphi|^2dS_h=:C_P>0. \tag{3.3}$$

Thus one concludes that

$$g_\Phi(x_h)=\frac{\Phi}{C_P}\varphi(x_h)$$

satisfies the problem (3.1). Then

$$\|g_\Phi\|_{H^m(\Sigma)}=\frac{\Phi}{C_P}\|\varphi(x_h)\|_{H^m(\Sigma)}\leqslant C_{\alpha,m,\Sigma}\Phi,\quad \forall\, m\in\mathbb{N}, \tag{3.4}$$

where $C_{\alpha,m,\Sigma}>0$ is a constant independent of Φ. Later, for $i=L$ or $i=R$, we denote by $\boldsymbol{g}_\Phi^i=g_\Phi^i\boldsymbol{e_3}$ the Poiseuille flows in $\Sigma_i\times\mathbb{R}$.

3.2 Construction of the profile vector

In this subsection, we focus on the construction of a smooth divergence-free vector $\boldsymbol{a}$, which satisfies the Navier-slip boundary condition (1.3). Meanwhile, the vector $\boldsymbol{a}$ equals $\boldsymbol{g}_\Phi^L$ in the far left of $\mathcal{D}$, and it is identical to $\boldsymbol{g}_\Phi^R$ in the far right of $\mathcal{D}$. Here is the result.

Proposition 3.2. *There exists a smooth vector field $\boldsymbol{a}(x)$ which enjoys the following properties*:

(i) $\boldsymbol{a}\in C^\infty(\overline{\mathcal{D}})$, *and* $\nabla\cdot\boldsymbol{a}=0$ *in* $\mathcal{D}$.

(ii) $2(\mathbb{S}\boldsymbol{a}\cdot\boldsymbol{n})_{\tan}+\alpha\boldsymbol{a}_{\tan}=0$, *and* $\boldsymbol{a}\cdot\boldsymbol{n}=0$ *on* $\partial\mathcal{D}$.

(iii) $\boldsymbol{a}=\boldsymbol{g}_\Phi^L$ *in* $\mathcal{D}\cap\{x\in\mathbb{R}^3:x_3\leqslant -Z\}$, *and* $\boldsymbol{a}=\boldsymbol{g}_\Phi^R$ *in* $\mathcal{D}\cap\{x\in\mathbb{R}^3:x_3\geqslant Z\}$.

(iv) $\|\boldsymbol{a}\|_{H^m(\mathcal{D}_M)}\leqslant C_{\alpha,m,\mathcal{D}}\Phi$ *for any* $m\in\mathbb{N}$.

Proof. Recalling the assumption of the domain $\mathcal{D}$, one notices that there exists a smooth domain $\Sigma'\subset\mathbb{R}^2$ such that $\Sigma'\times\mathbb{R}\subset\subset\mathcal{D}$ (*which means that* $\Sigma'\times\mathbb{R}\subset\mathcal{D}$ *and* $\mathrm{dist}(\Sigma'\times\mathbb{R},\partial\mathcal{D})\geqslant\varepsilon_0>0$). Let $h=h(x_h)$ be a smooth function supported on Σ', which satisfies

$$\int_{\Sigma'}h(x_h)dx_h=\Phi.$$

By a scaling, we can assume that

$$\|h\|_{H^m(\Sigma')}\leqslant C_m\Phi,\quad \forall\, m\in\mathbb{N}. \tag{3.5}$$

Let $\eta=\eta(x_3)$ be a smooth cut-off function such that

$$\eta(x_3)=\begin{cases} 1 & \text{for } x_3>Z,\\ 0 & \text{for } x_3<Z/2. \end{cases} \tag{3.6}$$

Now we define the vector $\boldsymbol{a}$ by

$$\boldsymbol{a} = \begin{pmatrix} A_1^R(x_h)\eta'(x_3) - A_1^L(x_h)\eta'(-x_3) \\ A_2^R(x_h)\eta'(x_3) - A_2^L(x_h)\eta'(-x_3) \\ h(x_h) + (g_\Phi^R - h(x_h))\eta(x_3) + (g_\Phi^L - h(x_h))\eta(-x_3) \end{pmatrix}^{\mathrm{T}}, \tag{3.7}$$

where the 2D vector function $\boldsymbol{A}^i := (A_1^i, A_2^i)$ ($i = L$ or $i = R$) satisfies the following partial differential equation in Σ_i:

$$\mathrm{div}_h \boldsymbol{A}^i(x_h) = h(x_h) - g_\Phi^i(x_h) \quad \text{in } \Sigma_i \tag{3.8}$$

subject to the two-dimensional Navier-slip boundary condition

$$\begin{cases} 2(\mathbb{S}\boldsymbol{A}^i \cdot \bar{\boldsymbol{n}})_{\tan} + \alpha \boldsymbol{A}^i_{\tan} = 0, \\ \boldsymbol{A}^i \cdot \bar{\boldsymbol{n}} = 0, \end{cases} \quad \text{on } \partial\Sigma_i, \tag{3.9}$$

where $\bar{\boldsymbol{n}} = (n_1(x_h), n_2(x_h))$ is the unit outward normal vector on $\partial\Sigma_i$.

Now let us verify the validity of the above construction.

First, combining (3.7) and (3.8), we see that direct computation shows the divergence-free property of $\boldsymbol{a}$. The smoothness of $\boldsymbol{a}$ follows from the smoothness of h, η, g_Φ^L, and g_Φ^R, which are provided in their definitions, together with the smoothness of $\boldsymbol{A}^L$ and $\boldsymbol{A}^R$ which will be derived below.

Second, concerning the validity of the boundary condition (ii) in Proposition 3.2, we first see that

$$\boldsymbol{a} = \begin{cases} g_\Phi^L(x_h)\boldsymbol{e_3} & \text{in } \mathcal{D} \cap \{x : x_3 \leqslant -Z\}, \\ g_\Phi^R(x_h)\boldsymbol{e_3} & \text{in } \mathcal{D} \cap \{x : x_3 \geqslant Z\}, \\ h(x_h)\boldsymbol{e_3} & \text{in } \mathcal{D} \cap \{x : |x_3| \leqslant Z/2\}. \end{cases}$$

Due to the fact that $g_\Phi^i \boldsymbol{e_3}$ ($i = L, R$) is the Poiseuille flow in (3.1) which satisfies the same Navier-slip boundary condition, and the auxiliary function $h(x_h)$ is compactly supported in each *cross section* of $\mathcal{D}$, we see that $\boldsymbol{a}$ satisfies the Navier-slip boundary condition on $\partial\mathcal{D} \cap \{x : |x_3| \leqslant Z/2 \text{ or } |x_3| \geqslant Z\}$. For the remaining part $\partial\mathcal{D} \cap \{x : Z/2 \leqslant |x_3| \leqslant Z\}$, the unit outer normal vector enjoys the following form:

$$\boldsymbol{n} = (\bar{\boldsymbol{n}}, 0) = (n_1(x_h), n_2(x_h), 0) \quad \text{on } \partial\mathcal{D} \cap \{x : Z/2 \leqslant |x_3| \leqslant Z\},$$

which is independent of the x_3 variable. Recalling (2.3), we see that the Navier-slip boundary condition

$$\begin{cases} 2(\mathbb{S}\boldsymbol{a} \cdot \boldsymbol{n})_{\tan} + \alpha \boldsymbol{a}_{\tan} = 0, \\ \boldsymbol{a} \cdot \boldsymbol{n} = 0, \end{cases} \quad \text{on } \partial\mathcal{D} \cap \{x : Z/2 \leqslant |x_3| \leqslant Z\}$$

enjoys the following form in the orthogonal curvilinear coordinates on the boundary:

$$\begin{cases} \partial_{\boldsymbol{n}} a_{\tau_1} = (\kappa_1(x) - \alpha) a_{\tau_1}, \\ \partial_{\boldsymbol{n}} a_3 = -\alpha a_3, \\ a_n = 0, \end{cases} \quad \text{on } \partial\mathcal{D} \cap \{x : Z/2 \leqslant |x_3| \leqslant Z\}. \tag{3.10}$$

Therefore, noting that the cut-off function η depends only on the x_3 variable, we see that $(3.10)_1$ and $(3.10)_3$ are guaranteed by $(3.9)_1$ and $(3.9)_2$, respectively. Moreover, by direct calculations,

$$\begin{aligned} \partial_{\boldsymbol{n}} a_3 &= \eta(x_3)\partial_{\boldsymbol{n}} g_\Phi^R + \eta(-x_3)\partial_{\boldsymbol{n}} g_\Phi^L \\ &= -\alpha\eta(x_3) g_\Phi^R - \alpha\eta(-x_3) g_\Phi^L \\ &= -\alpha a_3 \quad \text{on } \partial\mathcal{D} \cap \{x : Z/2 \leqslant |x_3| \leqslant Z\}, \end{aligned}$$

and one proves $(3.10)_2$. Thus we finish the proof of Proposition 3.2(ii).

Third, the property (iii) in this proposition follows directly from the definition of $\boldsymbol{a}$ in (3.7).

Finally, we derive the H^m-estimate on $\boldsymbol{a}$ in $\mathcal{D}_M$. Using (3.7), we see that

$$\begin{aligned}\|\boldsymbol{a}\|_{H^m(\mathcal{D}_M)} &\lesssim \sum_{i=L,R}\|\boldsymbol{A}^i\|_{H^m(\Sigma_i)} + \|h\|_{H^m(\Sigma')} + \sum_{i=L,R}\|g^i_\Phi\|_{H^m(\Sigma_i)}\\ &\lesssim_{\alpha,m,\mathcal{D}} \Phi + \sum_{i=L,R}\|\boldsymbol{A}^i\|_{H^m(\Sigma_i)},\end{aligned} \tag{3.11}$$

where the last inequality follows from the estimates in (3.4) and (3.5). Now we only need to show the H^m-estimate of 2D vectors $\boldsymbol{A}^i$ $(i = L, R)$, by solving the boundary value problem (3.8)–(3.9), which is derived in the following lemma.

Lemma 3.3. *Problem* (3.8)–(3.9) *has a smooth solution* $\boldsymbol{A}^i \in C^\infty(\overline{\Sigma})$ *satisfying*

$$\|\boldsymbol{A}^i\|_{H^m(\Sigma_i)} \leqslant C_{\alpha,m,\Sigma_i}\Phi, \quad \forall\, m \in \mathbb{N}. \tag{3.12}$$

Proof. For the simplicity of notation, we omit the index L or R in the following proof if no ambiguity is caused. Using the Helmholtz-Weyl decomposition, we can split $\boldsymbol{A}$ into

$$\boldsymbol{A} =: \nabla_h\phi + \boldsymbol{G}. \tag{3.13}$$

Here, $\phi = \phi(x_h)$ is a scalar function, which satisfies

$$\begin{cases}\Delta_h\phi = h(x_h) - g_\Phi(x_h) & \text{in } \Sigma,\\ \dfrac{\partial\phi}{\partial\bar{\boldsymbol{n}}} = 0 & \text{on } \partial\Sigma,\\ \displaystyle\int_\Sigma \phi dx_h = 0.\end{cases} \tag{3.14}$$

By the definition of the auxiliary function $h(x_h)$, one sees that $h - g_\Phi$ satisfies the following compatibility condition:

$$\int_\Sigma (h(x_h) - g_\Phi(x_h))dx_h = 0.$$

Thus, the classical theory of Poisson equations indicates the solvability and regularity $\phi \in C^\infty(\overline{\Sigma})$ of the problem (3.14). In addition, ϕ satisfies

$$\|\phi\|_{H^{m+2}(\Sigma)} \leqslant C_{m,\Sigma}\|h - g_\Phi\|_{H^m(\Sigma)} \leqslant C_{m,\Sigma}\Phi, \quad \forall\, m \in \mathbb{N}. \tag{3.15}$$

It remains to construct the smooth vector $\boldsymbol{G}$ in (3.13). Notice that $\boldsymbol{G}$ should satisfy

$$\begin{cases}\mathrm{div}_h\boldsymbol{G} = 0 & \text{in } \Sigma,\\ 2(\mathbb{S}\boldsymbol{G}\cdot\bar{\boldsymbol{n}})_{\tan} + \alpha\boldsymbol{G}_{\tan} = 2(\mathbb{S}(\nabla\phi)\cdot\bar{\boldsymbol{n}})_{\tan} + \alpha(\nabla\phi)_{\tan} & \text{on } \partial\Sigma,\\ \boldsymbol{G}\cdot\bar{\boldsymbol{n}} = 0 & \text{on } \partial\Sigma.\end{cases} \tag{3.16}$$

There is too much space for us to construct a solution G satisfying (3.16) such that $\|G\|_{H^m(\Sigma)} \leqslant C_{\alpha,m,\Sigma}\Phi$. For example, we can choose (G, π) to be the pair of solutions to the following linear Stokes equations with the Navier-slip boundary condition:

$$\begin{cases}-\Delta_h\boldsymbol{G} + \nabla\pi = 0, \quad \mathrm{div}_h\boldsymbol{G} = 0, & \text{in } \Sigma,\\ 2(\mathbb{S}\boldsymbol{G}\cdot\bar{\boldsymbol{n}})_{\tan} + \alpha\boldsymbol{G}_{\tan} = 2(\mathbb{S}(\nabla\phi)\cdot\bar{\boldsymbol{n}})_{\tan} + \alpha(\nabla\phi)_{\tan} & \text{on } \partial\Sigma,\\ \boldsymbol{G}\cdot\bar{\boldsymbol{n}} = 0 & \text{on } \partial\Sigma.\end{cases}$$

From [21, Theorem 4.5] or [9, Theorem 2.5.10][2)], we have the following estimate of $\boldsymbol{G}$:

$$\|\boldsymbol{G}\|_{H^{m+2}(\Sigma)} \leqslant C_{\alpha,m,\Sigma}\|2(\mathbb{S}(\nabla\phi)\cdot\bar{\boldsymbol{n}})_{\tan} + \alpha(\nabla\phi)_{\tan}\|_{H^{m+1/2}(\partial\Sigma)}$$

2) Strictly speaking, the theorems in [9, 21] are derived for 3D linear Stokes systems. However, their methods are also valid for related 2D problems (see the introduction part of [9]).

$$\leqslant C_{\alpha,m,\Sigma}\|\phi\|_{H^{m+3}(\Sigma)} \leqslant C_{\alpha,m,\Sigma}\Phi, \tag{3.17}$$

where on the last line, we have used the trace theorem and (3.15). Then (3.12) is proved by combining (3.15) and (3.17). □

Remark 3.4. Combining the estimates (3.4), (3.11) and (3.17) above, we see that the following global $W^{1,\infty}$-estimate of $\boldsymbol{a}$ is a direct conclusion of the Sobolev embedding:

$$\|\boldsymbol{a}\|_{W^{1,\infty}(\mathcal{D})} \leqslant C_{\alpha,\mathcal{D}}\Phi. \tag{3.18}$$

This completes the proof of Proposition 3.2 □

3.3 The proof of the existence

In this subsection, we study the solvability of the generalized Leray's problem subject to the Navier-slip boundary condition. Considering the asymptotic behavior of the prescribed weak solution in Definition 1.1, we write

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{a}, \tag{3.19}$$

where $\boldsymbol{a}$ is constructed in the previous subsection. Therefore, the generalized Leray's problem (1.4)–(1.6) has the following equivalent form in the viewpoint of $\boldsymbol{v}$.

Problem 3.5 (Modified problem)**.** *Find $(\boldsymbol{v}, p)$ such that*

$$\begin{cases} \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \boldsymbol{a}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \nabla p - \Delta\boldsymbol{v} = \Delta\boldsymbol{a} - \boldsymbol{a}\cdot\nabla\boldsymbol{a}, \\ \nabla\cdot\boldsymbol{v} = 0, \end{cases} \quad \text{in } \mathcal{D} \tag{3.20}$$

subject to the Navier-slip boundary condition

$$\begin{cases} 2(\mathbb{S}\boldsymbol{v}\cdot\boldsymbol{n})_{\tan} + \alpha\boldsymbol{v}_{\tan} = 0, \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, \end{cases} \quad \text{on } \mathcal{D} \tag{3.21}$$

with the asymptotic behavior as

$$\boldsymbol{v}(x) \to \boldsymbol{0} \quad \text{as } |x_3| \to \infty. \tag{3.22}$$

Substituting the expression (3.19) into the weak formulation (1.7), we arrive at the following weak formulation of $\boldsymbol{v}$.

Definition 3.6. Let $\boldsymbol{a}$ be a smooth vector satisfying the properties stated in Proposition 3.2. We say that $\boldsymbol{v} \in \mathcal{H}_\sigma(\mathcal{D})$ is a weak solution of Problem 3.5, if

$$\begin{aligned} &2\int_{\mathcal{D}} \mathbb{S}\boldsymbol{v} : \mathbb{S}\boldsymbol{\varphi} dx + \alpha\int_{\partial\mathcal{D}} \boldsymbol{v}_{\tan}\cdot\boldsymbol{\varphi}_{\tan} dS + \int_{\mathcal{D}} \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi} dx + \int_{\mathcal{D}} \boldsymbol{v}\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{\varphi} dx + \int_{\mathcal{D}} \boldsymbol{a}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi} dx \\ &\quad = \int_{\mathcal{D}} (\Delta\boldsymbol{a} - \boldsymbol{a}\cdot\nabla\boldsymbol{a})\cdot\boldsymbol{\varphi} dx \end{aligned} \tag{3.23}$$

holds for any vector-valued function $\boldsymbol{\varphi} \in \mathcal{H}_\sigma(\mathcal{D})$.

To establish the existence of the weak solution defined in Definition 3.6, we first introduce the following Brouwer's fixed point theorem. It could be found in [17] (see also [8, Lemma IX.3.1]).

Lemma 3.7. *Let P be a continuous operator which maps $\mathbb{R}^N$ into itself such that for some $\rho > 0$,*

$$P(\xi)\cdot\xi \geqslant 0 \quad \text{for all } \xi \in \mathbb{R}^n \text{ with } |\xi| = \rho.$$

Then there exists a $\xi_0 \in \mathbb{R}^N$ with $|\xi_0| \leqslant \rho$ such that $P(\xi_0) = 0$.

Now, we go to the existence theorem.

Theorem 3.8. *There is a constant $\Phi_0 > 0$ depending on α and the curvature of $\partial\mathcal{D}$ such that if $\Phi \leqslant \Phi_0$, then Problem* 3.5 *admits at least one weak solution*

$$(\boldsymbol{v}, p) \in \mathcal{H}_\sigma(\mathcal{D}) \times L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$$

with

$$\|\boldsymbol{v}\|_{H^1(\mathcal{D})} \leqslant C_{\alpha,\mathcal{D}}\Phi. \tag{3.24}$$

Remark 3.9. The weak solution satisfies a generalized version of (3.22). Actually, it follows from the trace inequality (see [8, Theorem II.4.1]) that

$$\int_{\Sigma_R} |\boldsymbol{v}(x_h, x_3)|^2 dx_h \leqslant C \int_{z>x_3} \int_{\Sigma_R} (|\boldsymbol{v}|^2 + |\nabla \boldsymbol{v}|^2)(x_h, z) dx_h dz,$$

where the constant C is independent of the x_3 variable. This implies

$$\int_{\Sigma_R} |\boldsymbol{v}(x_h, x_3)|^2 dx_h \to 0 \quad \text{as } x_3 \to +\infty.$$

The case $x_3 \to -\infty$ is similar.

Now we are ready to provide the proof of Theorem 3.8.

3.3.1 *Constructing the velocity field by the Galerkin method*

Using the Galerkin method, we first construct an approximate solution and then pass to the limit by compactness arguments. Recall

$$\boldsymbol{X} := C^\infty_{\sigma,c}(\overline{\mathcal{D}}; \mathbb{R}^3) = \{\boldsymbol{\varphi} \in C^\infty_c(\overline{\mathcal{D}}; \mathbb{R}^3) : \nabla \cdot \boldsymbol{\varphi} = 0, \, \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\mathcal{D}} = 0\},$$

and $\{\boldsymbol{\varphi}_k\}_{k=1}^\infty \subset \boldsymbol{X}$ is an unit orthonormal basis of $\mathcal{H}_\sigma(\mathcal{D})$, i.e.,

$$\langle \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j \rangle_{H^1(\mathcal{D})} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad \forall\, i, j \in \mathbb{N}.$$

Now we construct an approximation of $\boldsymbol{v}$ of the form

$$\boldsymbol{v}_N(x) = \sum_{i=1}^N c_i^N \boldsymbol{\varphi}_i(x).$$

To determine $\boldsymbol{v}_N$, one tests the weak formulation (3.23) by $\boldsymbol{\varphi}_i$ with $i = 1, 2, \ldots, N$. This indicates

$$\begin{aligned}
2\sum_{i=1}^N c_i^N \int_{\mathcal{D}} \mathbb{S}\boldsymbol{\varphi}_i : \mathbb{S}\boldsymbol{\varphi}_j dx + \alpha \sum_{i=1}^N c_i^N \int_{\partial\mathcal{D}} (\boldsymbol{\varphi}_i)_{\tan} (\boldsymbol{\varphi}_j)_{\tan} dS + \sum_{i,k=1}^N c_i^N c_k^N \int_{\mathcal{D}} \boldsymbol{\varphi}_i \cdot \nabla \boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_j dx \\
+ \sum_{i=1}^N \int_{\mathcal{D}} \boldsymbol{\varphi}_i \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{\varphi}_j dx + \sum_{i=1}^N c_i^N \int_{\mathcal{D}} \boldsymbol{a} \cdot \nabla \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j dx \\
= \int_{\mathcal{D}} (\Delta \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{a}) \cdot \boldsymbol{\varphi}_j dx, \quad \forall\, j = 1, 2, \ldots, N.
\end{aligned}$$

As we see, this is a system of nonlinear algebraic equations of the N-dimensional vector

$$\boldsymbol{c}^N := (c_1^N, c_2^N, \ldots, c_N^N).$$

We solve the above system by Lemma 3.7 (Brouwer's fixed point theorem). To this end, we define $P:\mathbb{R}^N\to\mathbb{R}^N$ such that

$$
\begin{aligned}
(P(\boldsymbol{c}^N))_j = {}& 2\sum_{i=1}^N c_i^N\int_{\mathcal{D}}\mathbb{S}\boldsymbol{\varphi}_i:\mathbb{S}\boldsymbol{\varphi}_j dx+\alpha\sum_{i=1}^N c_i^N\int_{\partial\mathcal{D}}(\boldsymbol{\varphi}_i)_{\tan}\cdot(\boldsymbol{\varphi}_j)_{\tan}dS\\
&+\sum_{i,k=1}^N c_i^N c_k^N\int_{\mathcal{D}}\boldsymbol{\varphi}_i\cdot\nabla\boldsymbol{\varphi}_k\cdot\boldsymbol{\varphi}_j dx\\
&+\sum_{i=1}^N\int_{\mathcal{D}}\boldsymbol{\varphi}_i\cdot\nabla\boldsymbol{a}\cdot\boldsymbol{\varphi}_j dx+\sum_{i=1}^N c_i^N\int_{\mathcal{D}}\boldsymbol{a}\cdot\nabla\boldsymbol{\varphi}_i\cdot\boldsymbol{\varphi}_j dx\\
&-\int_{\mathcal{D}}(\Delta\boldsymbol{a}-\boldsymbol{a}\cdot\nabla\boldsymbol{a})\cdot\boldsymbol{\varphi}_j dx,\quad \forall\, j=1,2,\dots,N.
\end{aligned}
$$

Clearly, one observes that P is continuous. Then we can obtain that

$$
\begin{aligned}
P(\boldsymbol{c}^N)\cdot\boldsymbol{c}^N = {}& \underbrace{2\int_{\mathcal{D}}|\mathbb{S}\boldsymbol{v}_N|^2dx+\alpha\int_{\partial\mathcal{D}}|(\boldsymbol{v}_N)_{\tan}|^2dS}_{I_1}\\
&+\underbrace{\int_{\mathcal{D}}((\boldsymbol{v}_N+\boldsymbol{a})\cdot\nabla(\boldsymbol{v}_N+\boldsymbol{a}))\cdot\boldsymbol{v}_N dx}_{I_2}-\underbrace{\int_{\mathcal{D}}\boldsymbol{v}_N\cdot\Delta\boldsymbol{a}dx}_{I_3}.
\end{aligned}
$$

First, we estimate the term I_1. We show that there exists a constant $C_{\alpha,\mathcal{D}}$ depending on α and $\mathcal{D}$ such that

$$
I_1\geqslant C_{\alpha,\mathcal{D}}\int_{\mathcal{D}}|\nabla\boldsymbol{v}_N|^2dx. \tag{3.25}
$$

By the definition of the stress tensor and integration by parts, one notices that

$$
\begin{aligned}
\int_{\mathcal{D}}|\mathbb{S}\boldsymbol{v}_N|^2dx &= \frac12\int_{\mathcal{D}}|\nabla\boldsymbol{v}_N|^2dx+\frac12\sum_{i,j=1}^3\int_{\mathcal{D}}\partial_{x_i}(\boldsymbol{v}_N)_j\partial_{x_j}(\boldsymbol{v}_N)_i dx\\
&= \frac12\int_{\mathcal{D}}|\nabla\boldsymbol{v}_N|^2dx+\underbrace{\frac12\sum_{i,j=1}^3\int_{\partial\mathcal{D}}(\boldsymbol{v}_N)_j\partial_{x_j}(\boldsymbol{v}_N)_i n_i dS}_{I_{11}}\\
&\quad -\underbrace{\frac12\int_{\mathcal{D}}\boldsymbol{v}_N\cdot\nabla\mathrm{div}\,(\boldsymbol{v}_N)dx}_{I_{12}}.
\end{aligned}\tag{3.26}
$$

Here, the term I_{12} vanishes due to the fact that $\boldsymbol{v}_N$ is divergence-free. Noting that $\boldsymbol{v}_N\cdot\boldsymbol{n}\equiv 0$ on $\partial\mathcal{D}$, we have

$$
I_{11}=\int_{\partial\mathcal{D}}\boldsymbol{v}_N\cdot(\nabla(\boldsymbol{v}_N\cdot\boldsymbol{n})-\boldsymbol{v}_N\cdot\nabla\boldsymbol{n})dS=-\sum_{i,j=1}^3\int_{\partial\mathcal{D}}(\boldsymbol{v}_N)_j\partial_{x_j}n_i(\boldsymbol{v}_N)_j dS.
$$

Thus, I_{11} can be bounded by

$$
|I_{11}|\leqslant C_{\mathcal{D}}\int_{\partial\mathcal{D}}|(\boldsymbol{v}_N)_{\tan}|^2dS,
$$

where $C_{\mathcal{D}}>0$ is a universal constant depending only on $\partial\mathcal{D}$. Inserting the above calculations for I_{11} and I_{12} in (3.26), one arrives at

$$
I_1\geqslant 2\int_{\mathcal{D}}|\mathbb{S}\boldsymbol{v}_N|^2dx\geqslant\int_{\mathcal{D}}|\nabla\boldsymbol{v}_N|^2dx-C_\kappa\int_{\partial\mathcal{D}}|(\boldsymbol{v}_N)_{\tan}|^2dS,
$$

i.e.,

$$
\frac{\alpha}{C_{\mathcal{D}}}I_1\geqslant\frac{\alpha}{C_{\mathcal{D}}}\int_{\mathcal{D}}|\nabla\boldsymbol{v}_N|^2dx-\alpha\int_{\partial\mathcal{D}}|(\boldsymbol{v}_N)_{\tan}|^2dS.
$$

Hence, we deduce that

$$I_1 + \frac{\alpha}{C_{\mathcal{D}}} I_1 \geqslant \frac{\alpha}{C_{\mathcal{D}}} \int_{\mathcal{D}} |\nabla \boldsymbol{v}_N|^2 dx.$$

This indicates (3.25).

Next, we turn to the estimate of I_2. Using integration by parts, together with the divergence-free property of $\boldsymbol{v}_N$ and $\boldsymbol{a}$, one knows that

$$I_2 = \underbrace{\int_{\mathcal{D}} \boldsymbol{v}_N \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{v}_N dx}_{I_{21}} + \underbrace{\int_{\mathcal{D}} \boldsymbol{a} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{v}_N dx}_{I_{22}}.$$

Noting that $\boldsymbol{v}_N \cdot \boldsymbol{n} = 0$ on $\partial\mathcal{D}$, and using integration by parts, one finds

$$I_{21} = \sum_{k,l=1}^{3} \int_{\mathcal{D}} (\boldsymbol{v}_N)_k \partial_{x_k} a_l (\boldsymbol{v}_N)_j dx = - \sum_{k,l=1}^{3} \int_{\mathcal{D}} (\boldsymbol{v}_N)_k a_l \partial_{x_k} (\boldsymbol{v}_N)_l dx.$$

Using Hölder's inequality, we have

$$I_{21} \leqslant C \|\boldsymbol{a}\|_{L^\infty(\mathcal{D})} \|\boldsymbol{v}_N\|^2_{H^1(\mathcal{D})}.$$

For the term I_{22}, one notices that $\boldsymbol{a}$ equals the Poiseuille flow $\boldsymbol{g}^L_\Phi$ or $\boldsymbol{g}^R_\Phi$ in $\mathcal{D} - \mathcal{D}_Z$, and thus $\boldsymbol{a} \cdot \nabla \boldsymbol{a} \equiv 0$ in $\mathcal{D} - \mathcal{D}_Z$. This indicates

$$|I_{22}| = \left| \int_{\mathcal{D}_Z} \boldsymbol{a} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{v}_N dx \right| \leqslant \|\boldsymbol{a}\|_{L^3(\mathcal{D}_Z)} \|\nabla \boldsymbol{a}\|_{L^2(\mathcal{D}_Z)} \|\boldsymbol{v}_N\|_{L^2(\mathcal{D})} \leqslant C_{\mathcal{D}} \|\boldsymbol{a}\|^2_{H^1(\mathcal{D}_Z)} \|\boldsymbol{v}_N\|_{H^1(\mathcal{D})}.$$

Finally, it remains to estimate I_3. Similar to I_{22}, we also claim that

$$|I_3| = \left| \int_{\mathcal{D}_Z} \boldsymbol{v}_N \cdot \Delta \boldsymbol{a} dx \right| \leqslant C_Z \|\boldsymbol{a}\|_{H^2(\mathcal{D}_Z)} \|\boldsymbol{v}_N\|_{H^1(\mathcal{D})}.$$

Here goes the proof of the claim: by the construction of the Poiseuille flow $\boldsymbol{g}^L_\Phi$, one knows

$$\int_{\Sigma_L \times (-\infty, -Z)} \boldsymbol{v}_N \cdot \Delta \boldsymbol{a} dx = C \int_{-Z}^{-\infty} \int_{\Sigma_L} (\boldsymbol{v}_N)_3 dx = 0.$$

Actually, we can show that

$$\int_{\Sigma_L} (\boldsymbol{v}_N)_3 (x_h, x_3) dx_h$$

is independent of x_3 by using div $\boldsymbol{v}_N = 0$ and $\boldsymbol{v}_N \cdot \boldsymbol{n} = 0$. Then using the compact support of $\boldsymbol{v}_N$, we can get the above equality.

Substituting the above estimates for I_1–I_3, and applying the Poincaré inequalities in Lemmas 2.5 and 2.6, one derives

$$P(\boldsymbol{c}^N) \cdot \boldsymbol{c}^N \geqslant \|\boldsymbol{v}_N\|_{H^1(\mathcal{D})} ((C_{\alpha,\mathcal{D}} - C_\alpha \Phi) \|\boldsymbol{v}_N\|_{H^1(\mathcal{D})} - \tilde{C}_{\alpha,\mathcal{D}} (\Phi + \Phi^2)),$$

which guarantees

$$P(\boldsymbol{c}^N) \cdot \boldsymbol{c}^N \geqslant 0,$$

provided that

$$\Phi \leqslant \Phi_0 := C_\alpha^{-1} C_{\alpha,\mathcal{D}}$$

and

$$|\boldsymbol{c}^N| = \|\boldsymbol{v}_N\|_{H^1(\mathcal{D})} \geqslant \frac{\tilde{C}_{\alpha,\mathcal{D}} (\Phi + \Phi^2)}{C_{\alpha,\mathcal{D}} - C_\alpha \Phi} =: \rho.$$

Using Lemma 3.7, we see that there exists $\boldsymbol{v}_N^* \in \operatorname{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_N\}$ such that

$$\|\boldsymbol{v}_N^*\|_{H^1(\mathcal{D})} \leqslant \frac{\tilde{C}_{\alpha,\mathcal{D}}(\Phi + \Phi^2)}{C_{\alpha,\mathcal{D}} - C_\alpha \Phi} \tag{3.27}$$

and

$$\begin{aligned}
&2\int_{\mathcal{D}} \mathbb{S}\boldsymbol{v}_N^* : \mathbb{S}\boldsymbol{\phi}_N dx + \alpha \int_{\partial\mathcal{D}} (\boldsymbol{v}_N^*)_{\tan} \cdot (\boldsymbol{\phi}_N)_{\tan} dS \\
&\quad + \int_{\mathcal{D}} \boldsymbol{v}_N^* \cdot \nabla \boldsymbol{v}_N^* \cdot \boldsymbol{\phi}_N dx + \int_{\mathcal{D}} \boldsymbol{v}_N^* \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{\phi}_N dx + \int_{\mathcal{D}} \boldsymbol{a} \cdot \nabla \boldsymbol{v}_N^* \cdot \boldsymbol{\phi}_N dx \\
&\quad = \int_{\mathcal{D}} (\Delta \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{a}) \cdot \boldsymbol{\phi}_N dx, \quad \forall\, \boldsymbol{\phi}_N \in \operatorname{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_N\}.
\end{aligned} \tag{3.28}$$

The above bound (3.27) and the Rellich-Kondrachov embedding theorem imply the existence of a field $\boldsymbol{v} \in \mathcal{H}_\sigma(\mathcal{D})$ and a subsequence, which we always denote by $\boldsymbol{v}_N^*$, such that

$$\boldsymbol{v}_N^* \to \boldsymbol{v} \quad \text{weakly in } \mathcal{H}_\sigma(\mathcal{D})$$

and

$$\boldsymbol{v}_N^* \to \boldsymbol{v} \quad \text{strongly in } L^2(\mathcal{D}') \text{ for all bounded } \mathcal{D}' \subset \mathcal{D}.$$

Therefore, we can pass to the limit in (3.28) and obtain

$$\begin{aligned}
&2\int_{\mathcal{D}} \mathbb{S}\boldsymbol{v} : \mathbb{S}\boldsymbol{\varphi} dx + \alpha \int_{\partial\mathcal{D}} \boldsymbol{v}_{\tan} \cdot \boldsymbol{\varphi}_{\tan} dS + \int_{\mathcal{D}} \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} dx + \int_{\mathcal{D}} \boldsymbol{v} \cdot \nabla \boldsymbol{a} \cdot \boldsymbol{\varphi} dx + \int_{\mathcal{D}} \boldsymbol{a} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{\varphi} dx \\
&\quad = \int_{\mathcal{D}} (\Delta \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \boldsymbol{a}) \cdot \boldsymbol{\varphi} dx \quad \text{for any } \boldsymbol{\varphi} \in \mathcal{H}_\sigma(\mathcal{D}).
\end{aligned} \tag{3.29}$$

Finally, the H^1-estimate of $\boldsymbol{v}$:

$$\|\boldsymbol{v}\|_{H^1(\mathcal{D})} \leqslant C_{\alpha,\mathcal{D}} \Phi$$

follows from (3.27) and the Fatou lemma for weakly convergent sequences. This completes the construction of $\boldsymbol{v}$.

3.3.2 *Creating the pressure field*

While processing the Galerkin method in the previous subsection, we did nothing with the pressure. This is because all the test functions are divergence-free. To find the pressure, we introduce the following lemma, which is a special case of [7, Theorem 17] by de Rham (see also [22, Proposition 1.1]).

Lemma 3.10. *For a given open set $\Omega \subset \mathbb{R}^3$, let $\boldsymbol{\mathcal{F}}$ be a distribution in $(C_c^\infty(\Omega; \mathbb{R}^3))'$ which satisfies*

$$\langle \boldsymbol{\mathcal{F}}, \boldsymbol{\phi} \rangle = 0 \quad \textit{for all } \boldsymbol{\phi} \in \{\boldsymbol{g} \in C_c^\infty(\Omega; \mathbb{R}^3) : \operatorname{div} \boldsymbol{g} = 0\}.$$

Then there exists a distribution $q \in (C_c^\infty(\Omega; \mathbb{R}))'$ such that

$$\boldsymbol{\mathcal{F}} = \nabla q.$$

Let $\boldsymbol{v}$ be a weak solution of (3.23) constructed in the previous subsection. Using (3.29), one finds that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{a}$ satisfies

$$\int_{\mathcal{D}} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\phi} dx + \int_{\mathcal{D}} \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\phi} dx = 0 \quad \text{for all } \boldsymbol{\phi} \in \{\boldsymbol{g} \in C_c^\infty(\mathcal{D}; \mathbb{R}^3) : \operatorname{div} \boldsymbol{g} = 0\}.$$

Thus by Lemma 3.10, there exists a $p \in (C_c^\infty(\mathcal{D}; \mathbb{R}))'$ such that

$$\Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla p \tag{3.30}$$

in the sense of distribution.

To derive the regularity of the pressure, one first notices that (3.30) is equivalent to

$$\operatorname{div}(\nabla \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{a} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{a}) + \Delta \boldsymbol{a} + \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}}\right)\Phi \boldsymbol{e_3} - \boldsymbol{a} \cdot \nabla \boldsymbol{a} = \nabla \Pi \tag{3.31}$$

with

$$\Pi = p + \frac{\Phi \int_{-\infty}^{x_3} \eta(s)ds}{C_{P,R}} - \frac{\Phi \int_{-\infty}^{-x_3} \eta(s)ds}{C_{P,L}}. \tag{3.32}$$

Here, $C_{P,i}$ $(i = L, R)$ are Poiseuille constants defined in (3.3), where we have dropped indexes L and R there. In addition, η is the cut-off function which is given in (3.6). By the definition of $\boldsymbol{a}$ in (3.7), both

$$\Delta \boldsymbol{a} + \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}}\right)\Phi \boldsymbol{e_3}$$

and $\boldsymbol{a} \cdot \nabla \boldsymbol{a}$ are smooth and have compact support. Meanwhile, since $\boldsymbol{v} \in H^1(\mathcal{D})$ and $\boldsymbol{a}$ is uniformly bounded, one deduces

$$\nabla \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{a} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{a} \in L^2(\mathcal{D})$$

directly by the Sobolev embedding and Hölder's inequality. Therefore, one concludes that the left-hand side of (3.31) belongs to $H^{-1}(\mathcal{D})$. Then the following lemma implies $\Pi \in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$, which leads to $p \in L^2_{\mathrm{loc}}(\overline{\mathcal{D}})$ by (3.32).

Lemma 3.11 (See [22, Proposition 1.2])**.** *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^3$. If a distribution q has all its first derivatives $\partial_{x_i} q$ $(1 \leqslant i \leqslant 3)$ in $H^{-1}(\Omega)$, then $q \in L^2(\Omega)$ and*

$$\|q - \bar{q}_\Omega\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla q\|_{H^{-1}(\Omega)}, \tag{3.33}$$

where $\bar{q}_\Omega = \frac{1}{|\Omega|}\int_\Omega q dx$. Moreover, if Ω is any Lipschitz open set in $\mathbb{R}^3$, then $q \in L^2_{\mathrm{loc}}(\overline{\Omega})$.

This completes the proof of Theorem 3.8.

4 Uniqueness of the weak solution

Recall the solution $(\boldsymbol{u}, p)$ we constructed in the last section with its flux being Φ. In this section, we show it is unique if Φ is sufficiently small.

4.1 Estimate of the pressure

The following lemma shows the existence of the solution to the problem $\nabla \cdot \boldsymbol{V} = f$ in a truncated pipe.

Lemma 4.1. *Let $D = \Sigma \times [0, 1]$, $f \in L^2(D)$ with*

$$\int_D f dx = 0.$$

Then there exists a vector-valued function $\boldsymbol{V} : D \to \mathbb{R}^3$ belonging to $H^1_0(D)$ such that

$$\nabla \cdot \boldsymbol{V} = f \quad \text{and} \quad \|\nabla \boldsymbol{V}\|_{L^2(D)} \leqslant C\|f\|_{L^2(D)}. \tag{4.1}$$

Here, $C > 0$ is a constant.

See [5, 6] and [8, Chapter III] for the detailed proof of this lemma.

Below, the proposition shows that an L^2-estimate related to the pressure in the truncated pipe Ω_Z^+ or Ω_Z^- could be bounded by the L^2-norm of $\nabla \boldsymbol{u}$.

Proposition 4.2. *Let $(\tilde{\boldsymbol{u}}, \tilde{p})$ be an alternative weak solution of* (1.1) *in the pipe $\mathcal{D}$ subject to the Navier-slip boundary condition* (1.3)*. If the total flux satisfies*

$$\int_{\mathcal{D}\cap\{x_3=z\}} \tilde{\boldsymbol{u}}(x_h, z) \cdot \boldsymbol{e_3} dx_h = \Phi = \int_{\mathcal{D}\cap\{x_3=z\}} \boldsymbol{u}(x_h, z) \cdot \boldsymbol{e_3} dx_h \quad \text{for any } |z| \geqslant Z,$$

then the following estimate of $\boldsymbol{w} := \tilde{\boldsymbol{u}} - \boldsymbol{u}$ *and the pressure holds*:

$$\left|\int_{\Omega_K^{\pm}} (\tilde{p} - p) w_3 dx\right| \leqslant C_{\mathcal{D}}(\|\boldsymbol{u}\|_{L^4(\Omega_K^{\pm})}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^{\pm})} + \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^{\pm})} + \|\nabla \boldsymbol{w}\|^3_{L^2(\Omega_K^{\pm})}), \quad \forall\, K \geqslant Z+1,$$

where $C_{\mathcal{D}} > 0$ *is a constant independent of* K.

Proof. During the proof, we cancel the upper index "$\pm$" of the domain for simplicity. Noticing

$$\int_{\mathcal{D}\cap\{x_3=z\}} w_3(x_h, z) dx_h \equiv 0, \quad \forall\, |z| \geqslant Z,$$

we deduce that

$$\int_{\Omega_K} w_3 dx = 0, \quad \forall\, K \geqslant Z+1.$$

Using Lemma 4.1, one derives the existence of a vector field V satisfying (4.1) with $f = w_3$. Applying the equation $(1.1)_1$, one arrives

$$\int_{\Omega_K} (\tilde{p} - p) w_3 dx = -\int_{\Omega_K} \nabla(\tilde{p} - p) \cdot \boldsymbol{V} dx = \int_{\Omega_K} (\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} - \Delta\boldsymbol{w}) \cdot \boldsymbol{V} dx.$$

Using integration by parts, one deduces

$$\int_{\Omega_K} (\tilde{p} - p) w_3 dx = \sum_{i,j=1}^{3} \int_{\Omega_K} (\partial_i w_j - w_i w_j - u_i w_j - u_j w_i) \partial_i V_j dx.$$

By applying Hölder's inequality and (4.1) in Lemma 4.1, one deduces that

$$\left|\int_{\Omega_K} (\tilde{p} - p) w_3 dx\right| \leqslant C(\|\nabla \boldsymbol{w}\|_{L^2(\Omega_K)} + \|\boldsymbol{w}\|^2_{L^4(\Omega_K)} + \|\boldsymbol{u}\|_{L^4(\Omega_K)}\|\boldsymbol{w}\|_{L^4(\Omega_K)})\|w_3\|_{L^2(\Omega_K)}. \tag{4.2}$$

Since v_3 has a zero mean value on each cross section Σ, and $(\boldsymbol{w} - w_3\boldsymbol{e_3})$ satisfies

$$(\boldsymbol{w} - w_3\boldsymbol{e_3}) \cdot \boldsymbol{n} = 0 \quad \text{for any } x \in \partial\mathcal{D} \cap \partial\Omega_K,$$

the vector $\boldsymbol{w}$ enjoys the Poincaré inequality

$$\|\boldsymbol{w}\|_{L^2(\Omega_K)} \leqslant C_{\mathcal{D}}\|\nabla_h \boldsymbol{w}\|_{L^2(\Omega_K)}. \tag{4.3}$$

Substituting (4.3) into (4.2) and also noting the Gagliardo-Nirenberg inequality

$$\|\boldsymbol{w}\|^2_{L^4(\Omega_K)} \leqslant C_{\mathcal{D}}(\|\boldsymbol{w}\|^{1/2}_{L^2(\Omega_K)}\|\nabla\boldsymbol{w}\|^{3/2}_{L^2(\Omega_K)} + \|\boldsymbol{w}\|^2_{L^2(\Omega_K)}),$$

one concludes

$$\left|\int_{\Omega_K} (\tilde{p} - p) w_3 dx\right| \leqslant C_{\mathcal{D}}(\|\boldsymbol{u}\|_{L^4(\Omega_K)}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K)} + \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K)} + \|\nabla \boldsymbol{w}\|^3_{L^2(\Omega_K)}).$$

This completes the proof. □

4.2 Main estimates

Subtracting the equation of $\boldsymbol{u}$ from the equation of $\tilde{\boldsymbol{u}}$, one finds

$$\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} + \nabla(\tilde{p} - p) - \Delta\boldsymbol{w} = 0. \tag{4.4}$$

Multiplying $\boldsymbol{w}$ on both sides of (4.4), and integrating on $\mathcal{D}_\zeta$, one derives

$$\int_{\mathcal{D}_\zeta} \boldsymbol{w}\cdot\Delta\boldsymbol{w} dx = \int_{\mathcal{D}_\zeta} \boldsymbol{w}(\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} + \nabla(\tilde{p} - p)) dx. \tag{4.5}$$

Using the divergence-free property and the Navier-slip boundary condition of $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$, one deduces

$$\begin{aligned}
&\int_{\mathcal{D}_\zeta} \boldsymbol{w}\cdot\Delta\boldsymbol{w}dx\\
&=\int_{\mathcal{D}_\zeta} w_i\partial_{x_j}(\partial_{x_j}w_i+\partial_{x_i}w_j)dx\\
&=-\sum_{i,j=1}^{3}\int_{\mathcal{D}_\zeta}\partial_{x_j}w_i(\partial_{x_j}w_i+\partial_{x_i}w_j)dx+\sum_{i,j=1}^{3}\int_{\partial\mathcal{D}_\zeta} w_in_j(\partial_{x_j}w_i+\partial_{x_i}w_j)dx\\
&=-2\int_{\mathcal{D}_\zeta}|\mathbb{S}\boldsymbol{w}|^2dx-\alpha\int_{\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}}(|w_{\tau_1}|^2+|w_{\tau_2}|^2)dS\\
&\quad+\sum_{i=1}^{3}\int_{\Sigma_R\times\{x_3=\zeta\}} w_i(\partial_{x_3}w_i+\partial_{x_i}w_3)dx_h-\sum_{i=1}^{3}\int_{\Sigma_L\times\{x_3=-\zeta\}} w_i(\partial_{x_3}w_i+\partial_{x_i}w_3)dx_h. \qquad (4.6)
\end{aligned}$$

Here, $\boldsymbol{n}=(n_1,n_2,n_3)$ is the unit outer normal vector on $\partial\mathcal{D}$.

On the other hand, using integration by parts, we may alternatively derive

$$\int_{\mathcal{D}_\zeta}\boldsymbol{w}\cdot\Delta\boldsymbol{w}dx=-\int_{\mathcal{D}_\zeta}|\nabla\boldsymbol{w}|^2dx+\underbrace{\frac{1}{2}\int_{\partial\mathcal{D}_\zeta}\nabla|\boldsymbol{w}|^2\cdot\boldsymbol{n}dS}_{T_1}, \qquad (4.7)$$

where

$$\begin{aligned}
T_1=&\underbrace{\frac{1}{2}\int_{\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_M}\nabla|\boldsymbol{w}|^2\cdot\boldsymbol{n}dS}_{T_{11}}+\underbrace{\frac{1}{2}\int_{\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_L}\nabla|\boldsymbol{w}|^2\cdot\boldsymbol{n}dS}_{T_{12}}+\underbrace{\frac{1}{2}\int_{\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_R}\nabla|\boldsymbol{w}|^2\cdot\boldsymbol{n}dS}_{T_{13}}\\
&+\frac{1}{2}\left(\int_{\mathcal{D}\cap\{x_3=\zeta\}}\partial_{x_3}|\boldsymbol{w}|^2(x_h,\zeta)dx_h-\int_{\mathcal{D}\cap\{x_3=-\zeta\}}\partial_{x_3}|\boldsymbol{w}|^2(x_h,-\zeta)dx_h\right).
\end{aligned}$$

To bound the term T_{11}, we apply the local orthogonal curvilinear coordinates on $\partial\mathcal{D}$. Thus, we split $\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_M$ into finitely many pieces, i.e.,

$$\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_M=\bigcup_{i=1}^{N}V_i,$$

and in each piece V_i, there exists an orthogonal curvilinear frame $\{\boldsymbol{\tau_1^i},\boldsymbol{\tau_2^i},\boldsymbol{n}^i\}$ such that

$$\nabla|\boldsymbol{w}|^2=\partial_{\boldsymbol{\tau_1^i}}|\boldsymbol{w}|^2\boldsymbol{\tau_1^i}+\partial_{\boldsymbol{\tau_2^i}}|\boldsymbol{w}|^2\boldsymbol{\tau_2^i}+\partial_{\boldsymbol{n}^i}|\boldsymbol{w}|^2\boldsymbol{n}^i \quad \text{on } V_i.$$

Using (2.1), one derives

$$\begin{aligned}
|T_{11}|&\leqslant\sum_{i=1}^{N}\int_{V_i}|w_{\tau_1^i}(\alpha-\kappa_1^i(x))w_{\tau_1^i}|dS+\sum_{i=1}^{N}\int_{V_i}|w_{\tau_2^i}(\alpha-\kappa_2^i(x))w_{\tau_2^i}|dS\\
&\leqslant C_{\alpha,\mathcal{D}}\int_{\partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_M}|\boldsymbol{w}_{\tan}|^2dS. \qquad (4.8)
\end{aligned}$$

Here, $C_{\alpha,\mathcal{D}}>0$ is a constant depending only on the friction ratio α and the domain $\mathcal{D}$. The existence of this constant $C_{\alpha,\mathcal{D}}$ follows from the boundedness of the principal curvature of $\partial\mathcal{D}$ (see Proposition 2.2 for details).

Noting that $\mathcal{D}_L$ is a straight pipe, one can find the global natural coordinates $\{\boldsymbol{\tau_1},\boldsymbol{e_3},\boldsymbol{n}\}$ of $\partial\mathcal{D}\cap\partial\mathcal{D}_L$, where $\boldsymbol{\tau_1}$ and $\boldsymbol{n}$ are the unit tangent vector and the unit outer normal vector of $\partial\Sigma_L$ in the x_1Ox_2 plane, while $\boldsymbol{e_3}$ is the Euclidean coordinate vector in the x_3-direction. In this case, one writes

$$\nabla|\boldsymbol{w}|^2=\partial_{\boldsymbol{\tau_1}}|\boldsymbol{w}|^2\boldsymbol{\tau_1}+\partial_{x_3}|\boldsymbol{w}|^2\boldsymbol{e_3}+\partial_{\boldsymbol{n}}|\boldsymbol{w}|^2\boldsymbol{n} \quad \text{on } \partial\mathcal{D}_\zeta\cap\partial\mathcal{D}_L.$$

Using (2.3), one finds that T_{12} satisfies

$$|T_{12}| \leqslant \left| \int_{\partial\mathcal{D}_\zeta \cap \partial\mathcal{D}_L} (w_\tau(\alpha - \kappa_1(x))w_\tau + \alpha|w_3|^2) dS \right| \leqslant C_{\alpha,\mathcal{D}} \int_{\partial\mathcal{D}_\zeta \cap \partial\mathcal{D}_L} |\boldsymbol{w}_{\tan}|^2 dS. \tag{4.9}$$

Similar to (4.9), one derives

$$|T_{13}| \leqslant C_{\alpha,\mathcal{D}} \int_{\partial\mathcal{D}_\zeta \cap \partial\mathcal{D}_R} |\boldsymbol{w}_{\tan}|^2 dS. \tag{4.10}$$

Substituting (4.8)–(4.10) into (4.7), one concludes that

$$\int_{\mathcal{D}_\zeta} \boldsymbol{w} \cdot \Delta \boldsymbol{w} dx \leqslant -\int_{\mathcal{D}_\zeta} |\nabla \boldsymbol{w}|^2 dx + C_{\alpha,\mathcal{D}} \int_{\partial\mathcal{D}_\zeta \cap \partial\mathcal{D}} |\boldsymbol{w}_{\tan}|^2 dS + C \int_{\mathcal{D}\cap\{x_3=\pm\zeta\}} |\boldsymbol{w}||\nabla \boldsymbol{w}| dx_h. \tag{4.11}$$

Now we focus on the right-hand side of (4.5). Applying integration by parts, one derives

$$\begin{aligned}\int_{\mathcal{D}_\zeta} \boldsymbol{w}(\boldsymbol{w}\cdot\nabla\boldsymbol{w} + \nabla(\tilde{p}-p))dx &= \int_{\mathcal{D}\cap\{x_3=\zeta\}} w_3 \left(\frac{1}{2}|\boldsymbol{w}|^2 + (\tilde{p}-p)\right) dx \\ &\quad - \int_{\mathcal{D}\cap\{x_3=-\zeta\}} w_3 \left(\frac{1}{2}|\boldsymbol{w}|^2 + (\tilde{p}-p)\right) dx. \end{aligned}\tag{4.12}$$

Applying Hölder's inequality, and noting that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{a}$, where $\boldsymbol{a}$ is the profile vector defined in Subsection 3.2, while $\boldsymbol{v}$ is the H^1-weak solution constructed in Subsection 3.3, one has

$$\begin{aligned}\left| \int_{\mathcal{D}_\zeta} (\boldsymbol{w}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w} + \boldsymbol{u}\cdot\nabla\boldsymbol{w}\cdot\boldsymbol{w}) dx \right| &\leqslant \|\nabla\boldsymbol{v}\|_{L^2(\mathcal{D}_\zeta)} \|\boldsymbol{w}\|^2_{L^4(\mathcal{D}_\zeta)} + \|\boldsymbol{v}\|_{L^4(\mathcal{D}_\zeta)} \|\nabla\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)} \|\boldsymbol{w}\|_{L^4(\mathcal{D}_\zeta)} \\ &\quad + \|\nabla\boldsymbol{a}\|_{L^\infty(\mathcal{D}_\zeta)} \|\boldsymbol{w}\|^2_{L^2(\mathcal{D}_\zeta)} + \|\boldsymbol{a}\|_{L^\infty(\mathcal{D}_\zeta)} \|\nabla\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)} \|\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)} \\ &\leqslant C_{\mathcal{D}}(\|\boldsymbol{v}\|_{H^1(\mathcal{D}_\zeta)} + \|\boldsymbol{a}\|_{W^{1,\infty}(\mathcal{D}_\zeta)}) \int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx \\ &\leqslant C_{\alpha,\mathcal{D}} \Phi \int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx. \end{aligned}\tag{4.13}$$

Here in the second inequality, we have applied the Gagliardo-Nirenberg inequality and the Poincaré inequality (2.10) in Lemma 2.6, which indicate

$$\|\boldsymbol{w}\|_{L^4(\mathcal{D}_\zeta)} \leqslant C_{\mathcal{D}}(\|\boldsymbol{w}\|^{1/4}_{L^2(\mathcal{D}_\zeta)} \|\nabla\boldsymbol{w}\|^{3/4}_{L^2(\mathcal{D}_\zeta)} + \|\boldsymbol{w}\|_{L^2(\mathcal{D}_\zeta)}) \leqslant C_{\mathcal{D}} \left(\int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx \right)^{1/2}.$$

Meanwhile, the third inequality in (4.13) is guaranteed by (3.24) and (3.18).

Therefore, by calculating

$$(4.6) \times C_{\alpha,\mathcal{D}} + (4.11) \times \alpha,$$

we derive

$$\int_{\mathcal{D}_\zeta} \boldsymbol{w}\cdot\Delta\boldsymbol{w} dx \leqslant -2C_{\alpha,\mathcal{D}} \int_{\mathcal{D}_\zeta} |\mathbb{S}\boldsymbol{w}|^2 dx - \alpha \int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx + C \int_{\mathcal{D}\cap\{x_3=\pm\zeta\}} |\boldsymbol{w}||\nabla\boldsymbol{w}| dx_h. \tag{4.14}$$

Substituting (4.12)–(4.14) into (4.5), one arrives

$$\begin{aligned}&(1 - C_{\alpha,\mathcal{D}}\Phi) \int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx \\ &\quad\leqslant \frac{C_{\alpha,\mathcal{D}}}{\alpha} \left(\int_{\mathcal{D}\cap\{x_3=\pm\zeta\}} |\boldsymbol{w}|(|\nabla\boldsymbol{w}| + |\boldsymbol{w}|^2) dx_h - \int_{\mathcal{D}\cap\{x_3=\zeta\}} w_3(\tilde{p}-p) dx_h + \int_{\mathcal{D}\cap\{x_3=-\zeta\}} w_3(\tilde{p}-p) dx_h \right).\end{aligned}$$

Now one concludes that if $\Phi \ll 1$ being small enough such that $C_{\alpha,\mathcal{D}}\Phi < \frac{1}{2}$,

$$\begin{aligned}\int_{\mathcal{D}_\zeta} |\nabla\boldsymbol{w}|^2 dx &\leqslant \frac{C_\alpha}{\alpha(1 - C_{\alpha,\mathcal{D}}\Phi)} \left(\int_{\mathcal{D}\cap\{x_3=\pm\zeta\}} |\boldsymbol{w}|(|\nabla\boldsymbol{w}| + |\boldsymbol{w}|^2) dx_h \right. \\ &\quad \left. - \int_{\mathcal{D}\cap\{x_3=\zeta\}} w_3(\tilde{p}-p) dx_h + \int_{\mathcal{D}\cap\{x_3=-\zeta\}} w_3(\tilde{p}-p) dx_h \right).\end{aligned}$$

Therefore, one derives the following estimate by integrating with ζ on $[K-1,K]$, where $K \geqslant Z+1$:

$$\int_{K-1}^{K}\int_{\mathcal{D}_\zeta}|\nabla \boldsymbol{w}|^2 dx d\zeta \leqslant C_{\alpha,\mathcal{D}}\bigg(\int_{\Omega_K^+\cup\Omega_K^-}|\boldsymbol{w}|(|\nabla \boldsymbol{w}|+|\boldsymbol{w}|^2)dx + \bigg|\int_{\Omega_K^+\cup\Omega_K^-} w_3(\tilde{p}-p)dx\bigg|\bigg). \tag{4.15}$$

Here, $C_{\alpha,\mathcal{D}}>0$ is a constant. Now we only handle integrations on Ω_K^+ since the case of Ω_K^- is similar. Using the Cauchy-Schwarz inequality and the Poincaré inequality in Lemma 2.5, one has

$$\int_{\Omega_K^+}|\boldsymbol{w}||\nabla \boldsymbol{w}|dx \leqslant \|\boldsymbol{w}\|_{L^2(\Omega_K^+)}\|\nabla \boldsymbol{w}\|_{L^2(\Omega_K^+)} \leqslant C\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^+)}. \tag{4.16}$$

Moreover, by Hölder's inequality and the Gagliardo-Nirenberg inequality, one writes

$$\int_{\Omega_K^+}|\boldsymbol{w}|^3 dx \leqslant C_{\mathcal{D}}(\|\boldsymbol{w}\|^{3/2}_{L^2(\Omega_K^+)}\|\nabla \boldsymbol{w}\|^{3/2}_{L^2(\Omega_K^+)} + \|\boldsymbol{w}\|^3_{L^2(\Omega_K^+)}).$$

It follows from the Poincaré inequality that

$$\int_{\Omega_K^+}|\boldsymbol{w}|^3 dx \leqslant C\|\nabla \boldsymbol{w}\|^3_{L^2(\Omega_K^+)}.$$

Recalling Proposition 4.2, one arrives at

$$\bigg|\int_{\Omega_K^+} w_3(\tilde{p}-p)dx\bigg| \leqslant C(\|\boldsymbol{u}\|_{L^4(\Omega_K^+)}\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^+)} + \|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^+)} + \|\nabla \boldsymbol{w}\|^3_{L^2(\Omega_K^+)}). \tag{4.17}$$

Substituting (4.16) and (4.17), together with their related inequality on the domain Ω_K^-, into (4.15), one concludes

$$\int_{K-1}^{K}\int_{\mathcal{D}_\zeta}|\nabla \boldsymbol{w}|^2 dx d\zeta \leqslant C_{\alpha,\mathcal{D}}(\|\nabla \boldsymbol{w}\|^2_{L^2(\Omega_K^+\cup\Omega_K^-)} + \|\nabla \boldsymbol{w}\|^3_{L^2(\Omega_K^+\cup\Omega_K^-)}). \tag{4.18}$$

4.3 End of the proof

Finally, by defining

$$Y(K) := \int_{K-1}^{K}\int_{\mathcal{D}_\zeta}|\nabla \boldsymbol{w}|^2 dx d\zeta,$$

we see that (4.18) indicates

$$Y(K) \leqslant C_{\alpha,\mathcal{D}}(Y'(K) + (Y'(K))^{3/2}), \quad \forall\, K \geqslant Z+1.$$

By Lemma 2.7, we derive

$$\liminf_{\zeta\to\infty} K^{-3}Y(K) > 0,$$

i.e., there exists a $C_0>0$ such that

$$\int_{K-1}^{K}\int_{\mathcal{D}_\zeta}|\nabla \boldsymbol{w}|^2 dx d\zeta \geqslant C_0 K^3.$$

However, this leads to a paradox to the condition (1.9). Thus, $Y(K)\equiv 0$ for all $K \geqslant Z+1$, which proves $\boldsymbol{u}\equiv\tilde{\boldsymbol{u}}$.

5 Regularity and decay estimates of the weak solution

In this section, we show that the weak solution, which is proved to be unique in Section 4, is smooth, and it decays exponentially to Poiseuille flows $\boldsymbol{g}_\Phi^R$ and $\boldsymbol{g}_\Phi^L$ as $x_3\to\pm\infty$, respectively. Recall

$$\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{a}, \quad \Pi = p + \frac{\Phi\int_{-\infty}^{x_3}\eta(s)ds}{C_{P,R}} - \frac{\Phi\int_{-\infty}^{-x_3}\eta(s)ds}{C_{P,L}}. \tag{5.1}$$

The route of the proof is as follows:

(i) the global $W^{1,3}$- and H^2-estimates of $\boldsymbol{v}$, together with the global L^2-estimate of $\nabla\Pi$;

(ii) the higher-order regularity of $(\boldsymbol{v},\Pi)$;

(iii) the H^1-exponential decay estimate of $\boldsymbol{v}$;

(iv) the exponential decay for higher-order norms of $(\boldsymbol{v},\Pi)$.

5.1 The global $\boldsymbol{H^2}$-estimate of the solution

In this subsection, we show that the weak solution constructed in Section 3 is strong. Our strategy is treating the Navier-Stokes system (3.20) as the following linear Stokes equations:

$$\begin{cases} -\Delta\boldsymbol{v}+\nabla\Pi=\mathrm{div}\boldsymbol{F}+\boldsymbol{f}, & \nabla\cdot\boldsymbol{v}=0, \quad \text{in } \mathcal{D},\\ 2(\mathbb{S}\boldsymbol{v}\cdot\boldsymbol{n})_{\tan}+\alpha\boldsymbol{v}_{\tan}=0, & \boldsymbol{v}\cdot\boldsymbol{n}=0, \quad \text{on } \partial\mathcal{D}, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} &\boldsymbol{F}:=-\boldsymbol{v}\otimes\boldsymbol{v}-\boldsymbol{a}\otimes\boldsymbol{v}-\boldsymbol{v}\otimes\boldsymbol{a},\\ &\boldsymbol{f}:=\Delta\boldsymbol{a}+\Phi\bigg(\frac{\eta(x_3)}{C_{P,R}}+\frac{\eta(-x_3)}{C_{P,L}}\bigg)\boldsymbol{e_3}-\boldsymbol{a}\cdot\nabla\boldsymbol{a}, \end{aligned}$$

and then applying the bootstrapping method. Noting that $\boldsymbol{v}\in H^1(\mathcal{D})$, we see that the Sobolev embedding indicates $\boldsymbol{v}\in L^2(\mathcal{D})\cap L^6(\mathcal{D})$, which indicates $\boldsymbol{v}\otimes\boldsymbol{v}\in L^3(\mathcal{D})$. Moreover, noting that $\boldsymbol{a}$ is smooth and uniformly bounded in $\mathcal{D}$, and both

$$\Delta\boldsymbol{a}+\Phi\bigg(\frac{\eta(x_3)}{C_{P,R}}+\frac{\eta(-x_3)}{C_{P,L}}\bigg)\boldsymbol{e_3}$$

and $\boldsymbol{a}\cdot\nabla\boldsymbol{a}$ have compact support, one concludes

$$\boldsymbol{F}\in L^3(\mathcal{D}),\quad \boldsymbol{f}\in C_c^\infty(\overline{\mathcal{D}}). \tag{5.3}$$

Here is the main result of this subsection.

Proposition 5.1. *Let $(\boldsymbol{u},p)$ be the weak solution to* (1.4)–(1.6)*, and $(\boldsymbol{v},\Pi)$ be defined as in* (5.1)*. Then*

$$(\boldsymbol{v},\nabla\Pi)\in H^2(\mathcal{D})\times L^2(\mathcal{D}), \tag{5.4}$$

which satisfies

$$\|\boldsymbol{v}\|_{H^2(\mathcal{D})}+\|\nabla\Pi\|_{L^2(\mathcal{D})}\leqslant C_{\alpha,\mathcal{D}}\Phi. \tag{5.5}$$

Proof. The proof consists of two parts. First, we show $\boldsymbol{v}\in W^{1,3}(\mathcal{D})$ by applying the regularity results in (5.3). This leads to

$$\mathrm{div}\,\boldsymbol{F}\in L^2(\mathcal{D}). \tag{5.6}$$

Then based on (5.6), we can obtain (5.4).

Now we split the problem (5.2) into a sequence of problems on bounded domains so that Lemmas 2.8 and 2.9 are valid for each one of them. To do this, we define

$$\mathcal{D}=\bigcup_{k\in\mathbb{Z}}\mathfrak{D}_k,\quad \text{where } \mathfrak{D}_k:=\mathcal{D}\cap\bigg\{x\in\mathbb{R}^3:\bigg(\frac{3k}{2}-1\bigg)Z\leqslant x_3\leqslant\bigg(\frac{3k}{2}+1\bigg)Z\bigg\},$$

and the related cut-off function

$$\psi_k=\psi\bigg(x_3-\frac{3kZ}{2}\bigg),$$

which satisfies

$$\begin{cases} \mathrm{supp}\,\psi\subset[-9Z/10,9Z/10], & \\ \psi\equiv 1 & \text{in } [-4Z/5,4Z/5],\\ 0\leqslant\psi\leqslant 1 & \text{in } [-Z,Z],\\ |\psi^{(m)}|\leqslant C/Z^m\leqslant C & \text{for } m=1,2. \end{cases}$$

Remark 5.2. According to the splitting and construction above, one notices that the "bubble part" in $\mathcal{D}$ is totally contained in $\mathfrak{D}_0$, and ψ_k', for each $k \in \mathbb{Z}$, is totally supported away from the "bubble part" of $\mathcal{D}$. Moreover, any point in $\mathcal{D}$ is contained in at most two neighboring $\mathfrak{D}_k$, while the union of sets

$$\mathfrak{D}_k' := \{x_3 \in \mathfrak{D}_k : \psi_k(x_3) = 1\}, \quad k \in \mathbb{Z}$$

covers $\mathcal{D}$ (see Figure 5).

Multiply the linearized equation $(5.2)_1$ with ψ_k, and then we convert the problem (5.2) to a related problem in the domain $\mathfrak{D}_k$ with $k \in \mathbb{Z}$, i.e.,

$$\begin{cases} -\Delta(\psi_k \boldsymbol{v}) + \nabla(\psi_k(\Pi - \overline{\Pi}_{\mathfrak{D}_k})) \\ \quad = \mathrm{div}\,(\psi_k \boldsymbol{F}) + \psi_k \boldsymbol{f} + (\Pi - \overline{\Pi}_{\mathfrak{D}_k})\psi_k' \boldsymbol{e_3} - 2\psi_k' \partial_{x_3} \boldsymbol{v} - \psi_k'' \boldsymbol{v} - \psi_k' \boldsymbol{F} \cdot \boldsymbol{e_3} & \text{in } \mathfrak{D}_k, \\ \nabla \cdot (\psi_k \boldsymbol{v}) = v_3 \psi_k' & \text{in } \mathfrak{D}_k, \\ 2(\mathbb{S}(\psi_k \boldsymbol{v}) \cdot \boldsymbol{n})_{\tan} + \alpha(\psi_k \boldsymbol{v})_{\tan} = 0, \quad (\psi_k \boldsymbol{v}) \cdot \boldsymbol{n} = 0, & \text{on } \partial\mathfrak{D}_k \cap \partial\mathcal{D}, \end{cases} \tag{5.7}$$

Here, the first two lines in (5.7) follow from direct calculations, and thus we only give some explanation for the boundary condition in $(5.7)_3$. According to the construction of the cut-off function ψ_k, one knows that for any $k \in \mathbb{Z}$, ψ_k' is supported away from the "bubble part" of $\mathcal{D}$. This indicates that the Navier-slip boundary condition $(5.2)_2$ enjoys the following form in the orthogonal curvilinear coordinates on the boundary:

$$\begin{cases} \partial_{\boldsymbol{n}} v_{\tau_1} = (\kappa_1(x) - \alpha) v_{\tau_1}, \\ \partial_{\boldsymbol{n}} v_3 = -\alpha v_3, & \text{on } \partial\mathcal{D} \cap (\partial\mathcal{D}_R \cup \partial\mathcal{D}_L) \\ v_n = 0, \end{cases} \tag{5.8}$$

(see Remark 2.1 for details). Noting that the normal vector $\boldsymbol{n}$ depends only on x_h in the "straight part" of $\mathcal{D}$, while ψ_k depends only on x_3, one deduces that the boundary condition of $\psi_k \boldsymbol{v}$ shares the form as (5.8). This indicates the validity of $(5.7)_3$.

However, Lemma 2.8 is not legal for $\psi_k \boldsymbol{v}$ at the moment, because $\psi_k \boldsymbol{v}$ is not divergence-free, and it also does not lie on a smooth domain. Nevertheless, let $\tilde{\mathfrak{D}}_k$ be a bounded smooth domain, which contains $\mathfrak{D}_k$, with its boundary $\partial\tilde{\mathfrak{D}}_k \supset \partial\mathfrak{D}_k \cap \partial\mathcal{D}$. Guaranteed by the definition of $\mathfrak{D}_k$, one chooses each $\tilde{\mathfrak{D}}_k$ with $k > 0$ to be congruent with $\tilde{\mathfrak{D}}_1$, and every $\tilde{\mathfrak{D}}_k$ with $k < 0$ to be congruent with $\tilde{\mathfrak{D}}_{-1}$.

In order to eliminate the divergence part of $\psi_k \boldsymbol{v}$, we introduce auxiliary functions ξ_k, which satisfy

$$\begin{cases} \Delta \xi_k = v_3 \psi_k' & \text{in } \tilde{\mathfrak{D}}_k, \\ \dfrac{\partial \xi_k}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\tilde{\mathfrak{D}}_k, \\ \displaystyle\int_{\tilde{\mathfrak{D}}_k} \xi_k(x) dx = 0. \end{cases} \tag{5.9}$$

Below, we define

$$\boldsymbol{u}_k := \psi_k \boldsymbol{v} - \nabla \xi_k$$

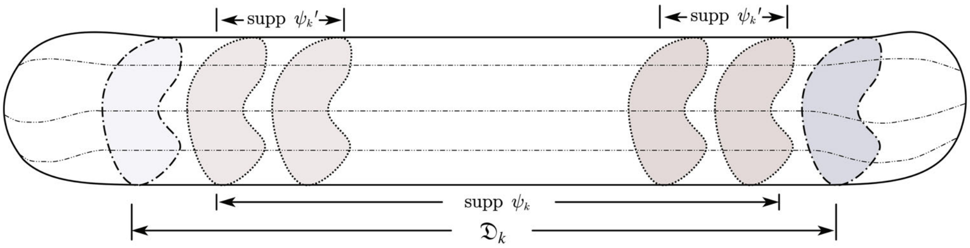

Figure 5 (Color online) A truncated smooth capsule $\tilde{\mathfrak{D}}_k$

for convenience. From (5.7) and (5.9), we know that $\mathbf{u}_k$ satisfies

$$\begin{cases} -\Delta \mathbf{u}_k + \nabla(\psi_k(\Pi - \overline{\Pi}_{\mathfrak{D}_k})) = \operatorname{div} \boldsymbol{F}_k + \boldsymbol{f}_k & \text{in } \tilde{\mathfrak{D}}_k, \\ \nabla \cdot \mathbf{u}_k = 0 & \text{in } \tilde{\mathfrak{D}}_k, \\ 2(\mathbb{S}\mathbf{u}_k \cdot \boldsymbol{n})_{\tan} + \alpha(\mathbf{u}_k)_{\tan} = \boldsymbol{h}_k, \quad \mathbf{u}_k \cdot \boldsymbol{n} = 0, & \text{on } \partial\tilde{\mathfrak{D}}_k. \end{cases} \tag{5.10}$$

Here,

$$\begin{aligned} \boldsymbol{F}_k &:= \psi_k \boldsymbol{F} + \nabla^2 \xi_k, \\ \boldsymbol{f}_k &:= \psi_k \boldsymbol{f} + (\Pi - \overline{\Pi}_{\mathfrak{D}_k})\psi_k' \boldsymbol{e_3} - 2\psi_k' \partial_{x_3} \boldsymbol{v} - \psi_k'' \boldsymbol{v} - \psi_k' \boldsymbol{F} \cdot \boldsymbol{e_3}, \\ \boldsymbol{h}_k &:= -2((\mathbb{S}\nabla\xi_k) \cdot \boldsymbol{n})_{\tan} - \alpha(\nabla\xi_k)_{\tan}. \end{aligned}$$

Now we are ready to show the regularity estimate of the quantities $\boldsymbol{F}_k$, $\boldsymbol{f}_k$ and $\boldsymbol{h}_k$ above.

Lemma 5.3. *The following estimate of $\boldsymbol{F}_k$, $\boldsymbol{f}_k$ and $\boldsymbol{h}_k$ holds*:

$$\|\boldsymbol{F}_k\|_{L^3(\tilde{\mathfrak{D}}_k)} + \|\boldsymbol{f}_k\|_{L^{\frac{3}{2}}(\tilde{\mathfrak{D}}_k)} + \|\boldsymbol{h}_k\|_{W^{-\frac{1}{3},3}(\partial\tilde{\mathfrak{D}}_k)} \leqslant C_{\mathcal{D}} \|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)} (1 + \|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}) + C\Phi\chi_{|k|\leqslant 1}. \tag{5.11}$$

Here, the constant $C_{\mathcal{D}}$ is uniform with k, and $\chi_{|k|\leqslant 1}$ is the characteristic function defined by

$$\chi_{|k|\leqslant 1} = \begin{cases} 1, & \text{if } k \in \{0, 1, -1\}, \\ 0, & \text{if } k \in \mathbb{Z} - \{0, 1, -1\}. \end{cases}$$

Proof. Noting that the support of ψ_k is uniformly bounded, we see that the estimates of $\boldsymbol{F}_k$ and $\boldsymbol{h}_k$ in (5.11) follow directly from the classical elliptic estimate of the system (5.9), which is

$$\|\xi_k\|_{W^{2,3}(\tilde{\mathfrak{D}}_k)} \leqslant C_{\mathcal{D}} \|\boldsymbol{v}\|_{L^3(\mathfrak{D}_k)} \leqslant C_{\mathcal{D}} \|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}. \tag{5.12}$$

The trace theorem of Sobolev functions indicates that

$$\|\boldsymbol{h}_k\|_{W^{-\frac{1}{3},3}(\partial\tilde{\mathfrak{D}}_k)} \leqslant C_{\alpha,\mathcal{D}} \|\xi_k\|_{W^{2,3}(\tilde{\mathfrak{D}}_k)}. \tag{5.13}$$

For the term $\boldsymbol{f}_k$, we only derive the estimate of the pressure term since the rest are transparent. Using Lemma 3.11 and Hölder's inequality, one deduces

$$\|(\Pi - \overline{\Pi}_{\mathfrak{D}_k})\psi_k'\|_{L^{\frac{3}{2}}(\tilde{\mathfrak{D}}_k)} \leqslant C\|\Pi - \overline{\Pi}_{\mathfrak{D}_k}\|_{L^2(\mathfrak{D}_k)} \leqslant C_1 \|\nabla\Pi\|_{H^{-1}(\mathfrak{D}_k)}. \tag{5.14}$$

Notice that by the definitions of $\mathfrak{D}_k$ and $\tilde{\mathfrak{D}}_k$, each $\mathfrak{D}_k$ ($k \in \mathbb{Z}$) is congruent with an element in $\{\mathfrak{D}_{-1}, \mathfrak{D}_0, \mathfrak{D}_1\}$, while every $\tilde{\mathfrak{D}}_k$ ($k \in \mathbb{Z}$) is congruent with an element in $\{\tilde{\mathfrak{D}}_{-1}, \tilde{\mathfrak{D}}_0, \tilde{\mathfrak{D}}_1\}$. Thus, the constants in the estimates (5.12)–(5.14) above could be chosen uniformly with respect to $k \in \mathbb{Z}$. Finally, by the equation

$$\nabla\Pi = \operatorname{div}(\nabla\boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{v} - \boldsymbol{a} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{a}) + \Delta\boldsymbol{a} + \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}}\right)\Phi\boldsymbol{e_3} - \boldsymbol{a} \cdot \nabla\boldsymbol{a}$$

with both

$$\Delta\boldsymbol{a} + \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}}\right)\Phi\boldsymbol{e_3}$$

and $\boldsymbol{a} \cdot \nabla\boldsymbol{a}$ vanishing in $\mathfrak{D}_k$ with $|k| \geqslant 2$, one concludes from (5.14) that

$$\begin{aligned} \|(\Pi - \overline{\Pi}_{\mathfrak{D}_k})\psi_k'\|_{L^{\frac{3}{2}}(\tilde{\mathfrak{D}}_k)} &\leqslant C(\|\nabla\boldsymbol{v}\|_{L^2(\mathfrak{D}_k)} + \|\boldsymbol{v}\|^2_{L^4(\mathfrak{D}_k)} + \Phi\|\boldsymbol{v}\|_{L^2(\mathfrak{D}_k)}) + C\Phi\chi_{|k|\leqslant 1} \\ &\leqslant C_{\alpha,\mathcal{D}} \|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)} (1 + \|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}) + C\Phi\chi_{|k|\leqslant 1}. \end{aligned} \tag{5.15}$$

Here, we have applied the Sobolev embedding theorem and interpolations of L^p spaces. This completes the proof of Lemma 5.3. □

Therefore, one concludes the following intermediate $W^{1,3}$-estimate of $\boldsymbol{v}$:

$$\|\boldsymbol{v}\|_{W^{1,3}(\mathcal{D})} \leqslant C_{\alpha,\mathcal{D}}\Phi \tag{5.16}$$

by combining (2.18) in Lemma 2.8 and (5.11) in Lemma 5.3, and then summing up with $k \in \mathbb{Z}$. The details are as follows:

$$\begin{aligned}\|\boldsymbol{v}\|^3_{W^{1,3}(\mathcal{D})} &\leqslant C\sum_{k\in\mathbb{Z}} \|\psi_k \boldsymbol{v}\|^3_{W^{1,3}(\tilde{\mathfrak{D}}_k)} \\ &\leqslant C\sum_{k\in\mathbb{Z}}(\|\mathbf{u}_k\|^3_{W^{1,3}(\tilde{\mathfrak{D}}_k)} + \|\xi_k\|^3_{W^{2,3}(\tilde{\mathfrak{D}}_k)}) \\ &\leqslant C_{\alpha,\mathcal{D}}\sum_{k\in\mathbb{Z}}(\|\boldsymbol{v}\|^3_{H^1(\mathfrak{D}_k)}(1+\|\boldsymbol{v}\|^3_{H^1(\mathfrak{D}_k)}) + \Phi^3\chi_{|k|\leqslant 1}).\end{aligned}$$

Here, we have applied the fact that any point in $\mathcal{D}$ is contained in at most two neighboring $\mathfrak{D}_k$. Noting that $\Phi \leqslant \Phi_0 = \Phi_0(\alpha,\mathcal{D})$, we see that

$$\|\boldsymbol{v}\|^3_{W^{1,3}(\mathcal{D})} \leqslant C_{\alpha,\mathcal{D}}\bigg((\Phi+\Phi^4)\sum_{k\in\mathbb{Z}}\|\boldsymbol{v}\|^2_{H^1(\mathfrak{D}_k)} + \Phi^3\bigg) \leqslant C_{\alpha,\mathcal{D}}\Phi^3.$$

Moreover, (5.16) and Hölder's inequality indicate that

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_{L^2(\mathcal{D})} \leqslant \|\boldsymbol{v}\|_{L^6(\mathcal{D})}\|\nabla\boldsymbol{v}\|_{L^3(\mathcal{D})} < \infty,$$

which further implies

$$\operatorname{div}\boldsymbol{F} + \boldsymbol{f} \in L^2(\mathcal{D}).$$

Similar to (5.11), now we can deduce

$$\|\operatorname{div}\boldsymbol{F}_k + \boldsymbol{f}_k\|_{L^2(\tilde{\mathfrak{D}}_k)} + \|\boldsymbol{h}_k\|_{H^{\frac{1}{2}}(\partial\tilde{\mathfrak{D}}_k)} \leqslant C_{\alpha,\mathcal{D}}\|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}(1+\|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}) + C\Phi\chi_{|k|\leqslant 1}. \tag{5.17}$$

From now on, Lemma 2.9 with $m=0$ is valid for the system (5.10). Combining (5.17) above and (2.19) in Lemma 2.9, one arrives at

$$\begin{aligned}\|\boldsymbol{v}\|_{H^2(\mathfrak{D}_k)} &\leqslant \|\mathbf{u}_k\|_{H^2(\tilde{\mathfrak{D}}_k)} + \|\xi_k\|_{H^3(\tilde{\mathfrak{D}}_k)} \\ &\leqslant C_{\alpha,\mathcal{D}}\|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}(1+\|\boldsymbol{v}\|_{H^1(\mathfrak{D}_k)}) + C\Phi\chi_{|k|\leqslant 1}.\end{aligned} \tag{5.18}$$

Now summing over $k \in \mathbb{Z}$, one proves (5.4) and (5.5) by an approach similar to the proof of (5.16). The estimate of $\nabla\Pi$ in (5.5) follows directly from the equations $(5.2)_1$ and (5.18) above. This completes the proof of Proposition 5.1. □

Remark 5.4. In the proof of Proposition 5.1, one notices that (5.14) and (5.15) can lead to the following uniform estimate of the pressure by summing over $k \in \mathbb{Z}$:

$$\sum_{k\in\mathbb{Z}}\|\Pi - \overline{\Pi}_{\mathfrak{D}_k}\|^2_{L^2(\mathfrak{D}_k)} \leqslant C_{\alpha,\mathcal{D}}\Phi^2 < \infty. \tag{5.19}$$

Moreover, since (5.19) is derived in the framework of the H^1-weak solution, it is valid for the case where $\partial\mathcal{D}$ is less regular.

5.2 Higher-order regularity and related estimates

Following the route of obtaining H^2 regularity of the solution, now we are ready to derive higher-order regularity of $(\boldsymbol{u}, p)$ via bootstrapping.

Proposition 5.5. *Let* $(\boldsymbol{u}, p)$ *be the weak solution to the problem* (1.4)–(1.6). *Then*

$$(\boldsymbol{u}, p) \in C^\infty(\overline{\mathcal{D}}).$$

Meanwhile,

$$\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{a}, \quad \Pi = p + \frac{\Phi \int_{-\infty}^{x_3} \eta(s)ds}{C_{P,R}} - \frac{\Phi \int_{-\infty}^{-x_3} \eta(s)ds}{C_{P,L}}$$

satisfies

$$\|\boldsymbol{v}\|_{H^{m+2}(\mathcal{D})} + \|\nabla \Pi\|_{H^m(\mathcal{D})} \leqslant C_{m,\alpha,\mathcal{D}} \Phi. \tag{5.20}$$

Proof. The proof follows from an induction argument. First, the case of $m = 0$ is already shown in Proposition 5.1. Once the regularity estimate (5.20) is achieved with the order $m \geqslant 0$, one deduces that

$$\|\nabla^{m+1}(\boldsymbol{v} \cdot \nabla \boldsymbol{v})\|_{L^2(\mathcal{D})} \leqslant C_{m,\mathcal{D}}(\|\boldsymbol{v}\|_{L^\infty(\mathcal{D})} \|\nabla^{m+2}\boldsymbol{v}\|_{L^2(\mathcal{D})} + \|\boldsymbol{v}\|^2_{W^{m+1,4}(\mathcal{D})}) \leqslant C_{m,\mathcal{D}} \|\boldsymbol{v}\|^2_{H^{m+2}(\mathcal{D})}. \tag{5.21}$$

Therefore, the Navier-Stokes system (1.1)–(1.3) is equivalent to

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla \Pi = \boldsymbol{g}, \quad \nabla \cdot \boldsymbol{v} = 0, & \text{in } \mathcal{D}, \\ 2(\mathbb{S}\boldsymbol{v} \cdot \boldsymbol{n})_{\tan} + \alpha \boldsymbol{v}_{\tan} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0, & \text{on } \partial\mathcal{D}, \end{cases} \tag{5.22}$$

where

$$\boldsymbol{g} = -(\boldsymbol{v} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \boldsymbol{v} + \boldsymbol{a} \cdot \nabla \boldsymbol{a}) + \Delta \boldsymbol{a} + \Phi \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}} \right) \boldsymbol{e_3}$$

enjoys

$$\|\boldsymbol{g}\|_{H^{m+1}(\mathcal{D})} \leqslant C_{m,\alpha,\mathcal{D}} \Phi < \infty$$

by direct calculations. Meanwhile, the problem (5.9)

$$\begin{cases} \Delta \xi_k = v_3 \psi_k' & \text{in } \tilde{\mathfrak{D}}_k, \\ \dfrac{\partial \xi_k}{\partial \boldsymbol{n}} = 0 & \text{on } \partial \tilde{\mathfrak{D}}_k, \\ \displaystyle\int_{\tilde{\mathfrak{D}}_k} \xi_k(x) dx = 0 \end{cases}$$

now admits a unique solution in $H^{m+4}(\tilde{\mathfrak{D}}_k)$ that satisfies

$$\|\xi_k\|_{H^{m+4}(\tilde{\mathfrak{D}}_k)} \leqslant C_{m,\mathcal{D}} \|\boldsymbol{v}\|_{H^{m+2}(\mathfrak{D}_k)}. \tag{5.23}$$

Here, the constant $C_{m,\mathcal{D}}$ is independent of k, because every $\tilde{\mathfrak{D}}_k$ is congruent with an element in $\{\tilde{\mathfrak{D}}_{-1}, \tilde{\mathfrak{D}}_0, \tilde{\mathfrak{D}}_1\}$. Recalling the construction of (5.10), we conclude that $\mathbf{u}_k := \psi_k \boldsymbol{v} - \nabla \xi_k$ satisfies

$$\begin{cases} -\Delta \mathbf{u}_k + \nabla(\psi_k(\Pi - \overline{\Pi}_{\mathfrak{D}_k})) = \boldsymbol{g}_k & \text{in } \tilde{\mathfrak{D}}_k, \\ \nabla \cdot \mathbf{u}_k = 0 & \text{in } \tilde{\mathfrak{D}}_k, \\ 2(\mathbb{S}\mathbf{u}_k \cdot \boldsymbol{n})_{\tan} + \alpha(\mathbf{u}_k)_{\tan} = \boldsymbol{h}_k, \quad \mathbf{u}_k \cdot \boldsymbol{n} = 0, & \text{on } \partial \tilde{\mathfrak{D}}_k \end{cases}$$

with

$$\begin{aligned} \boldsymbol{g}_k &:= \operatorname{div}(\psi_k \boldsymbol{F} + \nabla^2 \xi_k) + \psi_k \boldsymbol{f} + (\Pi - \overline{\Pi}_{\mathfrak{D}_k}) \psi_k' \boldsymbol{e_3} - 2\psi_k' \partial_{x_3} \boldsymbol{v} - \psi_k'' v_3 \boldsymbol{e_3} - \psi_k' \boldsymbol{F} \cdot \boldsymbol{e_3}, \\ \boldsymbol{h}_k &:= -2((\mathbb{S}\nabla \xi_k) \cdot \boldsymbol{n})_{\tan} - \alpha(\nabla \xi_k)_{\tan}. \end{aligned}$$

By induction, together with (5.21) and (5.23), one deduces that

$$\|\boldsymbol{g}_k\|_{H^{m+1}(\tilde{\mathfrak{D}}_k)} + \|\boldsymbol{h}_k\|_{H^{m+\frac{3}{2}}(\partial \tilde{\mathfrak{D}}_k)} \leqslant C_{m,\alpha,\mathcal{D}} \|\boldsymbol{v}\|_{H^{m+2}(\mathfrak{D}_k)} (1 + \|\boldsymbol{v}\|_{H^{m+2}(\mathfrak{D}_k)}) + C_m \Phi \chi_{|k| \leqslant 1} \tag{5.24}$$

by the approach in the proof of Lemma 5.3. Using the higher-order regularity for linear Stokes equations in Lemma 2.9, together with the estimates (5.23) and (5.24), one proves the following H^{m+3}-estimate in $\tilde{\mathfrak{D}}_k$:

$$\begin{aligned}\|\boldsymbol{v}\|_{H^{m+3}(\mathfrak{D}_k)} &\leqslant \|\mathbf{u}_k\|_{H^{m+3}(\tilde{\mathfrak{D}}_k)} + \|\xi_k\|_{H^{m+4}(\tilde{\mathfrak{D}}_k)}\\ &\leqslant C_{m,\alpha,\mathcal{D}}\|\boldsymbol{v}\|_{H^{m+2}(\mathfrak{D}_k)}(1+\|\boldsymbol{v}\|_{H^{m+2}(\mathfrak{D}_k)}) + C_m\Phi\chi_{|k|\leqslant 1}.\end{aligned}$$

Summing over $k\in\mathbb{Z}$, one concludes

$$\|\boldsymbol{v}\|_{H^{m+3}(\mathcal{D})} \leqslant C_{\alpha,m,\mathcal{D}}\Phi.$$

Finally, the estimate of $\nabla\Pi$ in (5.20) follows directly from the equation $(5.22)_1$ and the estimate above. This completes the proof of Proposition 5.5, which indicates the validity of (1.10) in Theorem 1.6. □

5.3 Exponential decay of the weak solution

In this subsection, we show the H^1-norm exponential decay property of the solution. Our proof is carried out under the framework of the H^1-weak solution, which means that we only assume the solution satisfies the estimate in Theorem 3.8. However, with the help of the higher-order uniform estimates of the solution in Proposition 5.5, the proof of the exponential decay property would be much simpler. Nevertheless, our proof in this subsection is also valid for the stationary Navier-Stokes problem on domains which are less regular, i.e., an infinite pipe only with a $C^{1,1}$ boundary.

Proposition 5.6. *Let the conditions of Theorem* 1.4 *be satisfied and* $(\boldsymbol{v},\Pi)$ *is given in* (5.1)*. Then there exist positive constants* C *and* σ *depending only on* α *and* $\mathcal{D}$ *such that*

$$\|\boldsymbol{u}-\boldsymbol{g}_\Phi^L\|_{H^1(\Sigma_L\times(-\infty,-\zeta))} + \|\boldsymbol{u}-\boldsymbol{g}_\Phi^R\|_{H^1(\Sigma_R\times(\zeta,\infty))} \leqslant C\|\boldsymbol{v}\|_{H^1(\mathcal{D})}\exp(-\sigma\zeta) \tag{5.25}$$

for any $\zeta > Z+1$.

Proof. We only prove the estimate of the term $\|\boldsymbol{u}-\boldsymbol{g}_\Phi^R\|_{H^1(\Sigma_R\times(\zeta,\infty))}$ since the remaining term is essentially identical. In $\Sigma_R\times(Z,\infty)$, the equation of $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{a}$ is

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \boldsymbol{a}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \nabla\Pi - \Delta\boldsymbol{v} = 0. \tag{5.26}$$

This is because

$$\Delta\boldsymbol{a} + \left(\frac{\eta(x_3)}{C_{P,R}} + \frac{\eta(-x_3)}{C_{P,L}}\right)\Phi\boldsymbol{e_3} - \boldsymbol{a}\cdot\nabla\boldsymbol{a} = \left(\Delta g_\Phi^R + \frac{\Phi}{C_{P,R}}\right)\boldsymbol{e_3} = 0 \quad \text{in } \Sigma_R\times(Z,\infty).$$

In the following proof, we drop (upper or lower) indexes "R" for convenience. For any $Z<\zeta\leqslant\zeta'<\zeta_1$, taking the inner product with $\boldsymbol{v}$ on both sides of (5.26) and integrating on $\Sigma\times(\zeta',\zeta_1)$, one has

$$\underbrace{\int_{\Sigma\times(\zeta',\zeta_1)}\boldsymbol{v}\cdot\Delta\boldsymbol{v}dx}_{\text{left-hand side (LHS)}} = \underbrace{\int_{\Sigma\times(\zeta',\zeta_1)}(\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \boldsymbol{a}\cdot\nabla\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{a} + \nabla\Pi)\cdot\boldsymbol{v}dx.}_{\text{right-hand side (RHS)}} \tag{5.27}$$

To handle the left-hand side of (5.27), one first recalls the derivation of (4.6), which indicates that

$$\begin{aligned}&\int_{\Sigma\times(\zeta',\zeta_1)}\boldsymbol{v}\cdot\Delta\boldsymbol{v}dx\\ &\quad= -2\int_{\Sigma\times(\zeta',\zeta_1)}|\mathbb{S}\boldsymbol{v}|^2dx - \alpha\int_{\partial\Sigma\times(\zeta',\zeta_1)}|\boldsymbol{v}_{\tan}|^2dS\\ &\qquad - \sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta'\}}v_i(\partial_{x_3}v_i+\partial_{x_i}v_3)dx_h + \sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta_1\}}v_i(\partial_{x_3}v_i+\partial_{x_i}v_3)dx_h.\end{aligned} \tag{5.28}$$

On the other hand, one can derive the following inequality similar to (4.11):

$$\int_{\Sigma\times(\zeta',\zeta_1)} \boldsymbol{v}\cdot\Delta\boldsymbol{v}dx \leqslant -\int_{\Sigma\times(\zeta',\zeta_1)} |\nabla\boldsymbol{v}|^2dx + C_{\alpha,\mathcal{D}}\int_{\partial\Sigma\times(\zeta',\zeta_1)} |\boldsymbol{v}_{\tan}|^2dS$$
$$-\sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta'\}} v_i\partial_{x_3}v_idx_h + \sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta_1\}} v_i\partial_{x_3}v_idx_h. \tag{5.29}$$

Therefore, by calculating

$$(5.28)\times C_{\alpha,\mathcal{D}} + (5.29)\times\alpha,$$

one deduces that the left-hand side of (5.27) satisfies

$$\mathrm{LHS} \leqslant -\alpha\int_{\Sigma\times(\zeta',\zeta_1)} |\nabla\boldsymbol{v}|^2dx + C_{\alpha,\mathcal{D}}\bigg(\sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta_1\}} v_i\partial_{x_i}v_3dx_h - \sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta'\}} v_i\partial_{x_i}v_3dx_h\bigg)$$
$$+(\alpha+C_{\alpha,\mathcal{D}})\bigg(\sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta_1\}} v_i\partial_{x_3}v_idx_h - \sum_{i=1}^3\int_{\Sigma\times\{x_3=\zeta'\}} v_i\partial_{x_3}v_idx_h\bigg). \tag{5.30}$$

Using integration by parts on the right-hand side of (5.27), one arrives at

$$\mathrm{RHS} = \int_{\Sigma\times\{x_3=\zeta_1\}} \bigg(\frac{1}{2}(v_3+g_\Phi)|\boldsymbol{v}|^2 + v_3\Pi + g_\Phi(v_3)^2\bigg)dx_h$$
$$-\int_{\Sigma\times\{x_3=\zeta'\}} \bigg(\frac{1}{2}(v_3+g_\Phi)|\boldsymbol{v}|^2 + v_3\Pi + g_\Phi(v_3)^2\bigg)dx_h$$
$$-\int_{\Sigma\times(\zeta',\zeta_1)} \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{a}dx. \tag{5.31}$$

Now we are ready to perform $\zeta_1\to\infty$. To do this, one must be careful with the integrations on $\Sigma\times\{x_3=\zeta_1\}$ in both (5.30) and (5.31). Recalling the estimates of $(\boldsymbol{v},\Pi)$ in Theorem 3.8 and Remark 5.4, one derives

$$\|\boldsymbol{v}\|^2_{H^1(\mathcal{D})} + \|\boldsymbol{v}\|^4_{L^4(\mathcal{D})} + \sum_{k\in\mathbb{Z}}\|\Pi-\overline{\Pi}_{\mathfrak{D}_k}\|^2_{L^2(\mathfrak{D}_k)} \leqslant C_{\alpha,\mathcal{D}}\Phi^2 < \infty. \tag{5.32}$$

Choosing

$$M := \frac{C_{\alpha,\mathcal{D}}\Phi^2}{Z},$$

one concludes that for any $k>1$, there exists a slice $\Sigma\times\{x_3=\zeta_{1,k}\}$ which satisfies

$$\Sigma\times\{x_3=\zeta_{1,k}\}\subset\mathcal{D}\cap\bigg\{x\in\mathbb{R}^3:\bigg(\frac{3k}{2}-\frac{1}{2}\bigg)Z\leqslant x_3\leqslant\bigg(\frac{3k}{2}+\frac{1}{2}\bigg)Z\bigg\}\subset\mathfrak{D}_k,$$

and it holds that

$$\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} (|\nabla\boldsymbol{v}|^2+|\boldsymbol{v}|^4+|\Pi-\overline{\Pi}_{\mathfrak{D}_k}|^2)dx_h \leqslant M.$$

Otherwise, one has

$$\|\boldsymbol{v}\|^2_{H^1(\mathfrak{D}_k)} + \|\boldsymbol{v}\|^4_{L^4(\mathfrak{D}_k)} + \|\Pi-\overline{\Pi}_{\mathfrak{D}_k}\|^2_{L^2(\mathfrak{D}_k)} > ZM = C_{\alpha,\mathcal{D}}\Phi^2,$$

which creates a paradox to (5.32). Choosing $k_0>0$ being sufficiently large such that the sequence satisfies $\{\zeta_{1,k}\}_{k=k_0}^\infty\subset[\zeta',\infty)$, clearly one has $\zeta_{1,k}\nearrow\infty$ as $k\to\infty$. Moreover, using the trace theorem of functions in the Sobolev space H^1, one has

$$\int_\Sigma |\boldsymbol{v}(x_h,x_3)|^2dx_h \leqslant C\int_{z>x_3}\int_\Sigma (|\boldsymbol{v}|^2+|\nabla\boldsymbol{v}|^2)(x_h,z)dx_hdz \to 0 \quad \text{as } x_3\to\infty.$$

Noting that $\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} v_3 dx_h = 0$ for $k \geqslant k_0$, we deduce the following by the Poincaré inequality:

$$\begin{aligned}
&\bigg|\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} v_3 \Pi dx_h\bigg| \\
&\quad = \bigg|\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} v_3(\Pi - \overline{\Pi}_{\mathfrak{D}_k}) dx_h\bigg| \\
&\quad \leqslant \bigg(\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |\boldsymbol{v}|^2 dx_h\bigg)^{1/2} \bigg(\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |\Pi - \overline{\Pi}_{\mathfrak{D}_k}|^2 dx_h\bigg)^{1/2} \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Meanwhile, one finds

$$\begin{aligned}
&\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |\boldsymbol{v}|(|\nabla\boldsymbol{v}| + |\boldsymbol{v}|^2) dx_h \\
&\quad \leqslant \bigg(\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} (|\nabla\boldsymbol{v}|^2 + |\boldsymbol{v}|^4) dx_h\bigg)^{1/2} \bigg(\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |\boldsymbol{v}|^2 dx_h\bigg)^{1/2} \to 0 \quad \text{as } k\to\infty
\end{aligned}$$

and

$$\int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |g_\Phi||\boldsymbol{v}|^2 dx_h \leqslant \|g_\Phi\|_{L^\infty(\mathcal{D}_R)} \int_{\Sigma\times\{x_3=\zeta_{1,k}\}} |\boldsymbol{v}|^2 dx_h \to 0 \quad \text{as } k\to\infty.$$

Choosing $\zeta_1 = \zeta_{1,k}$ ($k \geqslant k_0$) in (5.30) and (5.31), respectively, and performing $k\to\infty$, one can deduce that

$$\begin{aligned}
&\alpha \int_{\Sigma\times(\zeta',\infty)} |\nabla\boldsymbol{v}|^2 dx \\
&\quad \leqslant \underbrace{\int_{\Sigma\times(\zeta',\infty)} \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{a} dx}_{R_1} + C_{\alpha,\mathcal{D}} \int_{\Sigma\times\{x_3=\zeta'\}} (|\boldsymbol{v}|(|\boldsymbol{v}|^2 + |g_\Phi||\boldsymbol{v}| + |\nabla\boldsymbol{v}|) + v_3\Pi) dx_h.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, the Poincaré inequality in Lemma 2.5 and the construction of the profile vector $\boldsymbol{a}$, one derives

$$R_1 \leqslant \|\boldsymbol{a}\|_{L^\infty(\mathcal{D})} \bigg(\int_{\Sigma\times(\zeta',\infty)} |\nabla\boldsymbol{v}|^2 dx\bigg)^{1/2} \bigg(\int_{\Sigma\times(\zeta',\infty)} |\boldsymbol{v}|^2 dx\bigg)^{1/2} \leqslant C_0\Phi \int_{\Sigma\times(\zeta',\infty)} |\nabla\boldsymbol{v}|^2 dx,$$

which indicates the following estimate provided Φ is small enough such that $C_0\Phi < \alpha$:

$$\int_{\Sigma\times(\zeta',\infty)} |\nabla\boldsymbol{v}|^2 dx \leqslant C_{\alpha,\mathcal{D}} \int_{\Sigma\times\{x_3=\zeta'\}} (|\boldsymbol{v}|(|\boldsymbol{v}|^2 + |g_\Phi||\boldsymbol{v}| + |\nabla\boldsymbol{v}|) + v_3\Pi) dx_h. \tag{5.33}$$

Denoting

$$\mathcal{G}(\zeta') := \int_{\Sigma\times(\zeta',\infty)} |\nabla\boldsymbol{v}|^2 dx, \tag{5.34}$$

and integrating (5.33) with ζ' on (ζ,∞), one arrives at

$$\int_\zeta^\infty \mathcal{G}(\zeta') d\zeta' \leqslant C_{\alpha,\mathcal{D}} \bigg(\int_{\Sigma\times(\zeta,\infty)} (|\boldsymbol{v}|(|\boldsymbol{v}|^2 + |g_\Phi||\boldsymbol{v}| + |\nabla\boldsymbol{v}|)) dx + \bigg|\int_{\Sigma\times(\zeta,\infty)} v_3\Pi dx\bigg|\bigg). \tag{5.35}$$

Applying the Poincaré inequality in Lemma 2.5, one deduces

$$\int_{\Sigma\times(\zeta,\infty)} |\boldsymbol{v}|(|\boldsymbol{v}|^2 + |g_\Phi||\boldsymbol{v}| + |\nabla\boldsymbol{v}|) dx \leqslant C_{\alpha,\mathcal{D}} \int_{\Sigma\times(\zeta,\infty)} |\nabla\boldsymbol{v}|^2 dx. \tag{5.36}$$

Moreover, using a similar approach as in the proof of Proposition 4.2, one notices that

$$\bigg|\int_{\Sigma\times(\zeta,\infty)} v_3\Pi dx\bigg| \leqslant \sum_{m=1}^\infty \bigg|\int_{\Omega^+_{\zeta+m}} v_3\Pi dx\bigg|$$

$$
\begin{aligned}
&\leqslant C \sum_{m=1}^{\infty} (\|g_\Phi\|_{L^\infty(\Omega^+_{\zeta+m})} \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega^+_{\zeta+m})} + \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega^+_{\zeta+m})} + \|\nabla \boldsymbol{v}\|^3_{L^2(\Omega^+_{\zeta+m})}) \\
&\leqslant C \int_{\Sigma\times(\zeta,\infty)} |\nabla \boldsymbol{v}|^2 dx. \qquad (5.37)
\end{aligned}
$$

Substituting (5.36) and (5.37) into (5.35), one arrives at

$$\int_\zeta^\infty \mathcal{G}(\zeta')d\zeta' \leqslant C_{\alpha,\mathcal{D}}\mathcal{G}(\zeta) \quad \text{for any } \zeta > Z.$$

This implies that

$$\mathcal{N}(\zeta) := \int_\zeta^\infty \mathcal{G}(\zeta')d\zeta'$$

is well-defined for all $\zeta > Z$, and

$$\mathcal{N}(\zeta) \leqslant -C_{\alpha,\mathcal{D}}\mathcal{N}'(\zeta) \quad \text{for any } \zeta > Z. \qquad (5.38)$$

Multiplying the factor $\mathrm{e}^{C^{-1}_{\alpha,\mathcal{D}}\zeta}$ on both sides of (5.38) and integrating on $[Z, \zeta]$, one deduces

$$\mathcal{N}(\zeta) \leqslant C_{\alpha,\mathcal{D}} \exp(-C^{-1}_{\alpha,\mathcal{D}}\zeta) \quad \text{for any } \zeta > Z.$$

According to the definition (5.34), one knows that $\mathcal{G}$ is both non-negative and non-increasing. Thus,

$$\mathcal{G}(\zeta) \leqslant \int_{\zeta-1}^{\zeta} \mathcal{G}(\zeta')d\zeta' \leqslant \mathcal{N}(\zeta-1) \leqslant C\exp(-C^{-1}_{\alpha,\mathcal{D}}\zeta) \quad \text{for any } \zeta > Z+1.$$

We complete the proof of (5.25) by choosing $\sigma = C^{-1}_{\alpha,\mathcal{D}}$. □

5.4 On the exponential decay for higher-order derivatives

In this subsection, we focus on the higher-order asymptotic behavior of the aforementioned unique smooth solution to the problem. One sees that the solution $\boldsymbol{u}$ converges to the Poiseuille flow at the far field with an exponential speed. Based on the H^1 decay property in Proposition 5.6, we finish the proof of the estimate (1.11) in Theorem 1.6.

Using Sobolev embedding, we first need to show the following decay of the solution in the H^m-norms with $m \geqslant 2$:

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{H^m(\Sigma_L\times(-\infty,-\zeta))} + \|\boldsymbol{v}\|_{H^m(\Sigma_R\times(\zeta,\infty))} \\
&\quad \leqslant C_{m,\alpha,\mathcal{D}}(\|\boldsymbol{v}\|_{H^1(\Sigma_L\times(-\infty,-\zeta+2Z))} + \|\boldsymbol{v}\|_{H^1(\Sigma_R\times(\zeta-2Z,\infty))}) \quad \text{for all } \zeta > 3Z.
\end{aligned}
$$

This is derived by using the method in the proof of Proposition 5.5, but summing over $k \in \mathbb{Z}$ such that

$$\operatorname{supp}\psi_k \cap ((-\infty,-\zeta)\cup(\zeta,\infty)) \neq \emptyset.$$

Then the proof is completed by the H^1 decay estimate (5.25).

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