Journal of Functional Analysis 286 (2024) 110393

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Finite speed axially symmetric Navier-Stokes flows passing a cone

Zijin Li [a], Xinghong Pan [b], Xin Yang [c,d], Chulan Zeng [c], Qi S. Zhang [c,*], Na Zhao [e]

[a] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*
[b] *School of Mathematics and Key Laboratory of MIIT, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China*
[c] *Department of mathematics, University of California, Riverside, CA 92521, USA*
[d] *School of Mathematics, Southeast University, Nanjing, 211189, China* [1]
[e] *School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China*

ARTICLE INFO

Article history:
Received 13 February 2023
Accepted 17 February 2024
Available online 6 March 2024
Communicated by Michele Coti Zelati

MSC:
35Q30
76D03
76D05

Keywords:
Axially symmetric Navier-Stokes equations
Global strong solutions
Exterior conic regions
Partial smallness

ABSTRACT

Let D be the exterior of a cone inside a ball, with its altitude angle at most $\pi/6$ in $\mathbb{R}^3$, which touches the x_3 axis at the origin. For any initial value $v_0 = v_{0,r}e_r + v_{0,\theta}e_\theta + v_{0,3}e_3$ in a $C^2(\overline{D})$ class, which has the usual even-odd-odd symmetry in the x_3 variable and has the partial smallness only in the swirl direction: $|rv_{0,\theta}| \le \frac{1}{100}$, the axially symmetric Navier-Stokes equations (ASNS) with Navier-Hodge-Lions slip boundary condition have a finite-energy solution that stays bounded for all time. In particular, no finite-time blowup of the fluid velocity occurs. Compared with standard smallness assumptions on the initial velocity, no size restriction is made on the components $v_{0,r}$ and $v_{0,3}$. In a broad sense, this result appears to solve 2/3 of the regularity problem of ASNS in such domains in the class of solutions with the above symmetry. Equivalently, this result is connected to the general open question which asks that if an absolute smallness

* Corresponding author.
E-mail addresses: zijinli@nuist.edu.cn (Z. Li), xinghong_87@nuaa.edu.cn (X. Pan), xiny@ucr.edu, xinyang@seu.edu.cn (X. Yang), czeng011@ucr.edu (C. Zeng), qizhang@math.ucr.edu (Q.S. Zhang), zhaona@shufe.edu.cn (N. Zhao).
[1] Current address.

https://doi.org/10.1016/j.jfa.2024.110393

of one component of the initial velocity implies the global smoothness, see e.g. page 873 in Chemin et al. (2017) [6]. Our result seems to give a positive answer in a special setting.

As a byproduct, we also construct an unbounded solution of the forced Navier Stokes equation in a special cusp domain that has finite energy. The forcing term, with the scaling factor of -1, is in the standard regularity class, and it can be generated by an electric current in a long and straight wire (i.e. Ampère force). This result confirms the intuition that if the channel of a fluid is very thin, arbitrarily high speed in the classical sense can be attained under a mildly singular, physically reasonable force.

Contents

1. Introduction

The goal of the paper is to construct a class of global bounded solutions to the axially symmetric Navier-Stokes equations, abbreviated as ASNS henceforth.

$$\begin{cases} (\Delta - \frac{1}{r^2})v_r - (v_r\partial_r + v_3\partial_{x_3})v_r + \frac{(v_\theta)^2}{r} - \partial_r P - \partial_t v_r = 0, \\ (\Delta - \frac{1}{r^2})v_\theta - (v_r\partial_r + v_3\partial_{x_3})v_\theta - \frac{v_\theta v_r}{r} - \partial_t v_\theta = 0, \\ \Delta v_3 - (v_r\partial_r + v_3\partial_{x_3})v_3 - \partial_{x_3} P - \partial_t v_3 = 0, \\ \frac{1}{r}\partial_r(rv_r) + \partial_{x_3} v_3 = 0. \end{cases} \tag{1.1}$$

Here, $v = v_r e_r + v_\theta e_\theta + v_3 e_3$ is the velocity in the cylindrical system with the standard basis $\{e_r, e_\theta, e_3\}$, where for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$ and

$$e_r = (x_1/r, x_2/r, 0), \quad e_\theta = (-x_2/r, x_1/r, 0), \quad e_3 = (0, 0, 1). \tag{1.2}$$

The components v_r, v_θ and v_3 are independent of the azimuthal angle θ. Although ASNS is a special case of the full 3D Navier-Stokes equations,

$$\Delta v - (v \cdot \nabla)v - \nabla P - \partial_t v = 0, \quad \operatorname{div} v = 0, \tag{1.3}$$

the regularity problem of the former is still open in general. In the last several decades, there has been an outburst of research on ASNS, see e.g. [18,39,7,8,17,14,10,19,40,42] and the references therein. Especially after it was realized in [19] that ASNS is essentially a critical system, there is some expectation that the regularity problem is becoming accessible one way or the other.

A little of the expectation is achieved in [42] where the regularity problem is solved for a cusp domain under the Navier-slip boundary condition. This is the first time that the regularity problem of ASNS is settled when the essential difficulty is beyond that in 2D. Actually, the regularity problem of the 3D Navier-Stokes equations is also solved in [23] under the helical symmetry assumption of the solution. It is such an assumption that makes the classical 2D Ladyzhenskaya's inequality available in 3D. With that being said, the fundamental obstacle of the 3D regularity problem is absent in this situation.

One may feel that the cusp domain in [42] is somewhat special. In the current paper, we consider the ASNS in some wider domains, those outside a cone (see Fig. 1), which seems to be the next most feasible case. The problem we are studying can be used to model water flows around a cone shaped reef. Although we are not able to fully solve the regularity problem in our main result, Theorem 1.5, since there is a size assumption on the initial velocity, this assumption is only applied in the swirl direction and no size assumption is made on the other components of the initial velocity.

Since there are many well-established results of global smoothness for the Navier Stokes equations involving size assumptions for the initial value, we hereby explain the main new feature of this paper. The standard global smoothness result for ASNS in the literature can be summarized as follows. There exists a function $\lambda = \lambda(s)$, whose value goes to $+\infty$ as $s \to +\infty$, such that for any small $\epsilon > 0$, the solution to the ASNS is globally smooth if the initial condition v_0 satisfies

$$\|v_{0,\theta}\|_X < \epsilon, \quad \|v_{0,r}\|_Y + \|v_{0,3}\|_Y < \lambda(\epsilon^{-1}).$$

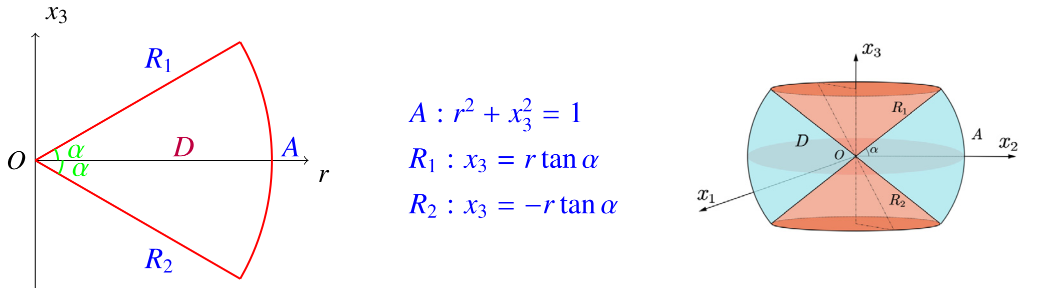

Fig. 1. Domain D in cylindrical coordinates.

Here X is a scaling-invariant suitable space of various choices, and $\|\cdot\|_Y$ is a quantity which may involve both velocity and vorticity. Notice that the non-swirl components $v_{0,r}$ and $v_{0,3}$ of the initial velocity are also restricted in size, unless the swirl component $v_{0,\theta} = 0$. In contrast, these restrictions are removed in our Theorem 1.5 below. This result is also connected to the general open question, which asks that if an absolute smallness of one component of the initial velocity implies the global smoothness, see e.g. page 873 in [6] in which the space $X = \dot{H}^{\frac{1}{2}}$. Our result seems to give a positive answer in the special setting stated in Theorem 1.5.

Now we make more precise description of the domains in this paper which are the exterior of certain cones inside a ball that touches the x_3 axis at the origin. We remark that similar regions were also introduced before to study other fluid problems, such as the singular formation for Euler flows [13], but these regions are bounded away from the x_3-axis.

Definition 1.1. Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ be any fixed angle. The domain D with boundary surfaces R_1, R_2 and A is defined in the cylindrical coordinates as follows (also see Fig. 1):

$$D = \left\{(r, \theta, x_3) : 0 < r^2 + x_3^2 < 1,\ -r\tan\alpha < x_3 < r\tan\alpha,\ \theta \in [0, 2\pi)\right\}. \tag{1.4}$$

Moreover, for convenience of notation, we denote

$$\partial^R D = R_1 \cup R_2, \quad \partial^A D = A,$$

where the superscripts R and A stand for the radial boundary and the annular boundary respectively.

The associated boundary condition is

$$v \cdot n = 0, \quad \omega \times n = 0, \quad \text{on} \quad \partial D, \tag{1.5}$$

where n is the unit outward normal on the smooth part of ∂D and ω is the vorticity defined as

$$\omega = \text{curl } v = \nabla \times v. \tag{1.6}$$

Condition (1.5) is a special case in a family of boundary conditions proposed by Navier [26]. This condition has been studied extensively in the literature and was attributed to different authors. For example, it was studied in [38]. Later, it was called the Navier-Hodge boundary condition in [25], and the Navier-Lions boundary condition in [15]. For this reason, we will name it as Navier-Hodge-Lions boundary condition in this paper, which is abbreviated as the NHL boundary condition thereafter. For more details on the history of this boundary condition and other types of Navier boundary conditions, see also ([15,41,9,24,30,31,3]).

Due to Leray [21], if $D = \mathbb{R}^3$, $v_0 \in L^2(\mathbb{R}^3)$, the Cauchy problem (1.3) has a solution in the energy space (cf. (1.7) below). By finite energy, we mean the solutions are in the energy space $\mathbf{E} = L^2_t H^1_x \cap L^\infty_t L^2_x$. Here and throughout, the norm in $\mathbf{E}$ for a function v on $D \times [0, T]$ is taken as

$$\|v\|^2_{\mathbf{E}} = \int_0^T \int_D |\nabla v|^2 dx dt + \sup_{t \in [0,T]} \int_D |v(x,t)|^2 dx. \tag{1.7}$$

Here, $T > 0$ and the function v can be vector-valued or scalar-valued, depending on the context. The solutions with finite energy include the so-called Leray-Hopf solutions which need to satisfy the strong energy inequality. In general, it is not known if Leray-Hopf solutions stay bounded or regular for all $t > 0$. Recently, by allowing a super-critical forcing term in (1.3), it was shown in [2] that even with zero initial value and identical forcing term, Leray-Hopf solutions may not be unique.

In this paper, we will focus on a special case of (1.3), namely when v and P are independent of the azimuthal angle θ in the cylindrical coordinate system (r, θ, x_3). Although ASNS seems more complicated than the full 3-dimensional equation, a simplification happens in the 2nd equation where the pressure term disappears. For a succinct derivation of the ASNS (1.1) using the tensor notations, we refer the readers to [42]. If the swirl $v_\theta = 0$, then it is well-known that finite energy solutions to the Cauchy problem of (1.1) in $\mathbb{R}^3$ are smooth for all time $t > 0$, see e.g. [18,39,20]. In the presence of swirl, it is still not known in general if finite energy solutions blow up in finite time.

By the partial regularity result in [5], possible singularity for suitable weak solutions of ASNS can only appear at the x_3 axis. See also [22] for a simplified proof and [4] for the same statement but without the "suitable" requirement. Moreover, in [7,8,17,35], it was shown that if

$$|v(x,t)| \le \frac{C}{r}, \tag{1.8}$$

then finite energy solutions to the Cauchy problem of ASNS are smooth for all time. Here, C is any positive constant. Later, there are some logarithmic improvements on the order of the criterion (1.8), see e.g. [28,34,33,11]. Also see [37] for a similar improvement in full

3D Navier-Stokes equations. In contrast, the energy bound scales as $-1/2$. So even with axial symmetry, there is a finite scaling gap which makes the ASNS supercritical, just like the full equations. Promisingly in [19], the authors revealed the following property.

The vortex stretching term of the ASNS is critical after a suitable change of dependent variables.

Thus, the aforementioned scaling gap is zero, which makes the regularity problem of ASNS appears less formidable. Nevertheless, all major open problems are still open.

The main result in [19] includes the following statement. Let $\delta_0 \in (0, \frac{1}{2})$ and $C_* > 1$. If

$$\sup_{0\le t<T} |r v_\theta(r, x_3, t)| \le C_* |\ln r|^{-2}, \quad r \le \delta_0, \tag{1.9}$$

then the above v is regular globally in time. Note that a priori we have $|r v_\theta(r, x_3, t)| \le C$ by the maximal principle applied on equation (1.10) of Γ:

$$\Delta\Gamma - b \cdot \nabla\Gamma - \frac{2}{r}\partial_r \Gamma - \partial_t \Gamma = 0, \tag{1.10}$$

where $\Gamma = r v_\theta$ and $b = v_r e_r + v_3 e_3$. So there is still a gap of logarithmic nature from regularity. Later, the power index -2 in (1.9) was improved to $-\frac{3}{2}$ in [40].

Now we specify the meaning of solutions to ASNS (1.1) associated with the NHL boundary condition (1.5). In the rest of this paper, functions and vector fields are always assumed to be axially symmetric with respect to the x_3-axis unless stated otherwise. Fix any $T > 0$ and any $v_0 \in H^2(D)$ that is divergence free in D and satisfies the NHL boundary condition (1.5). Consider

$$\begin{cases} \Delta v - (v\cdot\nabla)v - \nabla P - \partial_t v = 0 & \text{in} \quad D \times (0, T], \\ \nabla \cdot v = 0 & \text{in} \quad D \times (0, T], \\ v \cdot n = 0, \quad \omega \times n = 0 & \text{on} \quad \partial D \times (0, T], \\ v(\cdot, 0) = v_0(\cdot) & \text{in} \quad D, \end{cases} \tag{1.11}$$

where $\omega = \nabla \times v$. Define the space of testing vector fields to be

$$\begin{aligned} \mathscr{S}_T := \Big\{ & f \in H^1_t L^2_x \cap L^2_t H^2_x\big(D \times [0, T]\big) : \nabla \cdot f = 0 \text{ in } \; D \times [0, T], \\ & f \cdot n = 0 \text{ on } \; \partial D \times [0, T] \Big\}. \end{aligned} \tag{1.12}$$

If there exist $v \in \mathscr{S}_T$ and $P \in L^2_t H^1_x(D \times [0, T])$ such that (v, P) satisfies (1.11), then we test (1.11) by any function $f \in \mathscr{S}_T$ to obtain (see Section A.4 for detailed computations)

$$\begin{aligned}&\int_D v(x,T)\cdot f(x,T)\,dx+\int_0^T\int_D(\nabla\times v)\cdot(\nabla\times f)\,dx\,dt\\&=\int_D v_0(x)\cdot f(x,0)\,dx-\int_0^T\int_D[(v\cdot\nabla)v]\cdot f\,dx\,dt+\int_0^T\int_D v\cdot(\partial_t f)\,dx\,dt.\end{aligned}\tag{1.13}$$

If we replace f by v, then (1.13) yields the following identity:

$$\int_D|v(x,T)|^2\,dx+2\int_0^T\int_D|\nabla\times v(x,t)|^2\,dx\,dt=\int_D|v_0(x)|^2\,dx.\tag{1.14}$$

We point out that the left-hand side of (1.14) is not the energy norm (1.7) of v. Actually, without further assumptions, it is not clear if (1.14) implies the uniform (in time) finite-energy of the solution since the L^2 norm of ∇v may not be controlled by the L^2 norm of $\nabla\times v$, see the discussion in Section 4.1.

In this paper, we are looking for strong solutions of (1.11) which are defined as below.

Definition 1.2. If there exist $v\in L_t^2H_x^2\cap H_t^1L_x^2\big(D\times[0,T]\big)$ and $P\in L_t^2H_x^1\big(D\times[0,T]\big)$ such that (v,P) satisfies (1.11) in L_{tx}^2 sense, then v or (v,P) is called a strong solution of (1.11) on $D\times[0,T]$.

Note that if (v,P) is a strong solution, then (v,P) satisfies (1.11) almost everywhere. In addition, both the integration identities (1.13) and (1.14) are valid for v. For the bounded domain D in (1.4) with $\alpha\in\big(0,\frac{\pi}{6}\big]$ and under the NHL boundary condition (1.5), we manage to obtain a strong solution to ASNS (1.1) under the assumptions (i) and (ii) in the main result of this paper, Theorem 1.5, which removes the logarithmic term in (1.9). We emphasize that the assumption (1.17) is only made on the initial swirl $v_{0,\theta}$ and no smallness restriction is imposed on the other components $v_{0,r}$ and $v_{0,3}$. Assumption (i) is a symmetry condition which we describe now.

Definition 1.3. Let $v=v_re_r+v_\theta e_\theta+v_3e_3$ be a vector field in $\mathbb{R}^3$. We say v has the even-odd-odd symmetry if v_r is even, and v_θ and v_3 are odd symmetric in x_3.

This symmetry condition will be not only used to find a strong solution, but also utilized to establish the uniform (in time) energy inequality (1.19) in Theorem 1.5. Next, we introduce the admissible class $\mathscr{A}$ of the initial vector fields that we consider in this paper. Since the original domain D touches the x_3 axis with an angle, the singularity of the velocity might have more chance to occur. Moreover, the solution may not be expected to have higher regularity than $L_t^2H_x^1$. In order to acquire more regularity and to prove the boundedness of the velocity, we first cut the corner of D and then study the problem in approximating domains D_m ($m\geq 2$), which are defined as

$$D_m = \left\{ (r, \theta, x_3) : \frac{1}{m^2} < r^2 + x_3^2 < 1,\ -r \tan\alpha < x_3 < r \tan\alpha,\ \theta \in [0, 2\pi) \right\}. \tag{1.15}$$

The NHL boundary condition (1.5) associated with D_m is:

$$v \cdot n = 0, \quad \omega \times n = 0, \quad \text{on} \quad \partial D_m. \tag{1.16}$$

Due to the above strategy, it is natural to choose the elements in $\mathscr{A}$ to be the limits of vector fields on D_m.

Definition 1.4 *(Admissible classes $\mathscr{A}_m$ and $\mathscr{A}$).* Fix any angle $\alpha \in \left(0, \frac{\pi}{2}\right)$.

(1) For any integer $m \geq 2$, we define the admissible class $\mathscr{A}_m$ on D_m to be the space of vector fields $v_0^{(m)}$ in $C^2(\overline{D_m})$ that are divergence free in D_m and satisfy the NHL condition (1.16) on ∂D_m.
(2) For the domain D, we define the admissible class $\mathscr{A}$ on it to be the space of vector fields v_0 in $C^2(\overline{D})$ such that there exist vector fields $\left\{v_0^{(m)}\right\}_{m \geq 2}$ such that $v_0^{(m)} \in \mathscr{A}_m$ and

$$\lim_{m \to \infty} \left\| v_0 - v_0^{(m)} \right\|_{C^2(\overline{D_m})} = 0.$$

Now we are ready to state the main result of this paper.

Theorem 1.5. *Let the domain D be as defined in (1.4) with the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Suppose the initial velocity v_0 lies in the admissible class $\mathscr{A}$ with the following two properties:*

(i) *v_0 has the even-odd-odd symmetry as in Definition 1.3;*
(ii) *the swirl component of the initial velocity satisfies*

$$\sup_D r|v_{0,\theta}| \leq \frac{1}{100}. \tag{1.17}$$

Then for any $T > 0$, equation (1.1) with the initial data v_0 and the NHL boundary condition (1.5) has a strong solution (v, P) on $D \times [0, T]$ such that v is bounded uniformly in time and possesses the even-odd-odd symmetry. More precisely,

$$\|v\|_{L^\infty_{tx}(D\times[0,T])} + \|v\|_{H^1_t L^2_x(D\times[0,T])} + \|v\|_{L^2_t H^2_x(D\times[0,T])} + \|P\|_{L^2_t H^1_x(D\times[0,T])} \leq C, \tag{1.18}$$

where C is a constant that only depends on α and $\|v_0\|_{C^2(\overline{D})}$. In addition, the following energy inequality holds:

$$\int_D |v(x,T)|^2\,dx + \frac{2}{3}\int_0^T \int_D |\nabla v(x,t)|^2\,dx\,dt \leq \int_D |v_0(x)|^2\,dx. \tag{1.19}$$

On the other hand, if $(\tilde{v}, \tilde{P})$ *is another strong solution on* $D \times [0, T]$ *with the even-odd-odd symmetry, then* $\tilde{v}$ *coincides with the above strong solution* v.

Remark 1.6. We remark that without the symmetry assumption (i) in the above theorem, there is indication that the energy inequality (1.19) may fail. One example is the stationary solution

$$v = \frac{1}{r} e_\theta,$$

which satisfies equation (1.1) with $P = -\frac{1}{2r^2}$, and the NHL boundary condition (1.5), in the domain D_m for any $m \geq 2$. See also Section 7.

We also want to mention that there do exist vector fields $v_0 = v_{0,\rho} e_\rho + v_{0,\phi} e_\phi + v_{0,\theta} e_\theta$ in $\mathscr{A}$ which satisfy the assumptions (i) and (ii) in Theorem 1.5, and for which the size of $v_{0,\rho}$ and $v_{0,\phi}$ can be chosen arbitrarily large. We will provide such an initial vector field v_0 in Example 2.2.

Remark 1.7. Similar to the weak-strong uniqueness property (see e.g. Theorem 6.10 in Section 6.3 in [32]) of the classical Navier-Stokes equations (1.3), one can apply that idea to establish such a property for ASNS (1.1) under the NHL boundary condition (1.5), with a small modification as follows. Since a strong solution only satisfies the integral identity (1.14), a weak solution is required to satisfy the corresponding inequality:

$$\int_D |v(x,T)|^2 \, dx + 2 \int_0^T \int_D |\nabla \times v(x,t)|^2 \, dx \, dt \leq \int_D |v_0(x)|^2 \, dx.$$

We would like to mention that the above inequality is different from the classical one since its left hand side only involves the vorticity $\nabla \times v$ rather than the gradient ∇v.

Let us describe the organization of the paper. After some preparations in Section 2, we will prove, in Section 3, the existence and uniqueness of strong solutions in approximating domains D_m. The core of the paper is contained in Section 4 and Section 5 where we will prove the required uniform a priori bounds on the solutions found in Section 3. After these two sections, the proof of the main result, Theorem 1.5, will be completed in Section 6. Finally, as a byproduct of studying the NHL boundary condition, we will construct a special class of blowup solutions of (1.1) on some cusp domains in Section 7. In particular, for a thin cusp channel, we show in Proposition 7.2 that the vector field $\frac{\eta(t)}{r} e_\theta$ solves the forced ASNS (7.5) with forces $-\frac{\eta'(t)}{r} e_\theta$, where $\eta(t)$ is a smooth cut-off function in time. As a magnetohydrodynamic intuition, one can interpret the situation as the Ampère magnetic forces $-\frac{\eta'(t)}{r} e_\theta$, generated by an electric current in a long wire, can produce infinite speed of the conductive fluid in the classical sense. Note that these forces are subcritical under the standard scaling.

Here are some key ideas in the proofs. The first step is to rewrite the ASNS and the vorticity equations in the spherical coordinate system. It is well known for a long time, see e.g. paragraph 4 in the contemporary exposition [36], that the Navier-Stokes equations are supercritical. In particular, the vorticity equations contain the supercritical vortex stretching terms which block the path to the standard energy estimates, without size restrictions on all components. Our new input is the discovery of two new quantities K and F (see (2.14)):

$$K = \frac{\omega_\rho}{\rho}, \quad F = \frac{\omega_\phi}{\rho},$$

involving the vorticity for which the vortex stretching terms become critical. In addition, the boundary behaviors of these quantities are manageable so that an energy estimate can be achieved under only the partial smallness condition (ii) in Theorem 1.5. One may wonder, if the well known quantities $\Omega = \omega_\theta / r$ (see [39]) and $J = \omega_r / r$ (see [10]) in the cylindrical system are still useful in our situation. It turns out that Ω is still necessary but we are not able to control the boundary terms coming out from the equation of J. The next step is to derive an energy estimate for the system of equations for K, F and Ω (see (2.15)). Since there are a large number of terms in the system, which need to be handled separately, and which may satisfy various boundary conditions, the calculation will be relatively long. Although the modified vortex equations for F, K, Ω are essentially critical, some of the bad terms still appear bigger in size than the good viscosity term. For example, the term $6K/\rho^2$ in $(2.15)_1$ can not be controlled by ΔK using the standard Hardy's inequality in 3D domains. This is also why we need the extra restrictions on the angle of the domain and the even-odd-odd symmetry of the data. With the energy estimate in hand, we can prove the boundedness of the velocity v by using a modified version of the Biot-Savart law and the Moser's iteration. Here are some crucial steps to obtain the uniform estimate on $\|v\|_{L^\infty_{tx}(D_m \times [0,T])}$.

- Step 1: We will derive an energy inequality about v in Section 4.1. This energy inequality provides a uniform bound on $\|v\|_{E_{m,T}}$.
- Step 2: In Sections 4.2–4.4, we will take advantage of the Biot-Savart law and the condition $\alpha \in \left(0, \frac{\pi}{6}\right]$ to control the $L^2(D_m)$ norms of $\nabla(v_\rho/\rho)(\cdot,t)$ and $\nabla(v_\phi/\rho)(\cdot,t)$ by $\|\Omega(\cdot,t)\|_{L^2(D_m)}$, and control the $L^2(D_m)$ norms of $\frac{1}{\rho}\nabla(v_\rho/\rho)(\cdot,t)$ and $\frac{1}{\rho}\nabla(v_\phi/\rho)(\cdot,t)$ by $\|\nabla\Omega(\cdot,t)\|_{L^2(D_m)}$.
- Step 3: Thanks to the smallness condition $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)} \le \frac{1}{95}$, the estimates in Step 1 will be used in Section 4.5 to obtain an upper bound, which is uniform in m and T, on $\|(K,F,\Omega)\|_{L^\infty_t L^2_x(D_m \times [0,T])}$.
- Step 4: According to the uniform bound on $\|(K,F,\Omega)\|_{L^\infty_t L^2_x(D_m \times [0,T])}$, we will derive in Section 4.6 a uniform bound on $\|v/\rho\|_{L^\infty_t L^6_x(D_m \times [0,T])}$.
- Step 5: Finally in Section 4.7, we will bound $\|v\|_{L^\infty_{tx}(D_m \times [0,T])}$ in terms of $\|v_0\|_{C^2(\overline{D_m})}$, $\|v\|_{E_{m,T}}$, $\|(K,F,\Omega)\|_{L^\infty_t L^2_x(D_m \times [0,T])}$ and $\|v/\rho\|_{L^\infty_t L^6_x(D_m \times [0,T])}$. Due to the uniform

estimates in Steps 1, 3 and 4, the bound on $\|v\|_{L^\infty_{tx}(D_m\times[0,T])}$ will also be uniform in m and T.

We finish the introduction with a list of some notations and conventions to be used throughout this paper.

- Functions or vector fields in this paper are always assumed to be axially symmetric unless stated otherwise.
- The velocity field is usually called v and the vorticity $\nabla\times v$ is denoted as ω. We use subscripts to denote their components in either the cylindrical or spherical coordinate systems (see Section 2). For instance, $v_\rho = v\cdot e_\rho$, $\omega_\theta = \omega\cdot e_\theta$, $\omega_\phi = \omega\cdot e_\phi$. Here, θ refers to the azimuthal (longitude) angle and ϕ is the angle between the radius vector and the positive x_3-axis. In addition, we write $b = v_r e_r + v_3 e_3 = v_\rho e_\rho + v_\phi e_\phi$.
- $L^p(\Omega)$, $p\geq 1$, denotes the usual Lebesgue space on a domain Ω which may be a spatial, temporal or space-time domain. Let X be a Banach space defined for functions on $\Omega\subset\mathbb{R}^3$. $L^p(0,T;X)$ represents the Bochner-Banach space of functions f on the space time domain $D\times[0,T]$ with the norm $\left(\int_0^T \|f(\cdot,t)\|_X^p dt\right)^{1/p}$. We also use $L^p_x L^q_t$ or $L^q_t L^p_x$ to denote the mixed p,q norm in space time.
- Let $\Omega\subset\mathbb{R}^3$ be an open domain, then $H^1(\Omega) = W^{1,2}(\Omega) = \{f : f, \nabla f\in L^2(\Omega)\}$ and $H^2(\Omega) = W^{2,2}(\Omega) = \{f : f, \nabla f, \nabla^2 f\in L^2(\Omega)\}$, denote the standard Sobolev spaces on Ω. Meanwhile, for any time interval $I\subset\mathbb{R}$, the notation $H^1(I)$ means the Sobolev space $W^{1,2}(I)$.
- Interchangeable notations $\operatorname{div} v = \nabla\cdot v$, $\operatorname{curl} v = \nabla\times v$ will be used.
- $B(x,r)$ denotes the ball of radius r centered at x in a Euclidean space; and $B_X(f,r)$ denotes the open ball in a normed space X, centered at $f\in X$ with radius r.
- We use C or C_i $(i\geq 1)$ with or without index to denote generic constants which may change from line to line. Sometimes, we will make the dependence of constants on parameters explicitly. For example, the notation $C = C(a,b\dots)$ or $C = C_{a,b,\dots}$ means that the constant C only depends on $a,b\dots$.

2. Preliminaries

Although the Navier-Stokes equations under the spherical coordinates are well-known, various notations exist in literatures. In this section, we will first fix the notations and derive the basic equations for the key quantities K, F and Ω in Section 2.1. We point out that the equation (2.7) for the velocity v and the equation (2.13) for the vorticity ω may look slightly differently from other literatures since we have rewritten some terms based on the divergence free condition. Then we will introduce some inequalities of Poincaré's or Hardy's type which will be used in latter sections. Furthermore, we will establish the a priori L^∞ bound for another crucial quantity Γ.

2.1. Reformulation of equations in spherical system with unknowns $K = \omega_\rho/\rho$, $F = \omega_\phi/\rho$ and $\Omega = \omega_\theta/(\rho \sin\phi)$

Due to the geometry of the domain D and the boundary condition (1.5), it may be more beneficial to adopt the spherical coordinates (ρ, ϕ, θ), where ρ is the radial distance and ϕ is the angle between the radius vector and the positive x_3 axis. The relation between the cylindrical coordinates and the spherical coordinates is

$$\begin{pmatrix} r \\ \theta \\ x_3 \end{pmatrix} = \begin{pmatrix} \rho \sin\phi \\ \theta \\ \rho\cos\phi \end{pmatrix}. \tag{2.1}$$

For any axially symmetric vector field v, we denote

$$v = v_\rho(\rho, \phi, t)e_\rho + v_\phi(\rho, \phi, t)e_\phi + v_\theta(\rho, \phi, t)e_\theta,$$

where

$$e_\rho = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}, \quad e_\phi = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \tag{2.2}$$

Then

$$\begin{cases} e_\rho = \sin\phi\, e_r + \cos\phi\, e_3, \\ e_\phi = \cos\phi\, e_r - \sin\phi\, e_3, \end{cases} \qquad \begin{cases} v_\rho = \sin\phi\, v_r + \cos\phi\, v_3, \\ v_\phi = \cos\phi\, v_r - \sin\phi\, v_3. \end{cases} \tag{2.3}$$

Under the spherical coordinates, the domain D in (1.4) is equivalent to the following (also see Fig. 2)

$$D = \big\{(\rho, \phi, \theta) : 0 < \rho < 1,\ \frac{\pi}{2} - \alpha < \phi < \frac{\pi}{2} + \alpha,\ \theta \in [0, 2\pi)\big\}. \tag{2.4}$$

The boundary condition (1.5) becomes

$$\begin{cases} v_\phi = \omega_\rho = \omega_\theta = 0, \quad \text{on} \quad \partial^R D; \\ v_\rho = \omega_\phi = \omega_\theta = 0, \quad \text{on} \quad \partial^A D, \end{cases} \tag{2.5}$$

and the initial vector field v_0 can be written as

$$v_0 = v_{0,\rho}(\rho, \phi)e_\rho + v_{0,\phi}(\rho, \phi)e_\phi + v_{0,\theta}(\rho, \phi)e_\theta. \tag{2.6}$$

We can convert (1.1) from the cylindrical coordinates to the spherical coordinates. For simplicity in notation, we denote $b = v_r e_r + v_3 e_3$, or equivalently, in the spherical coordinates,

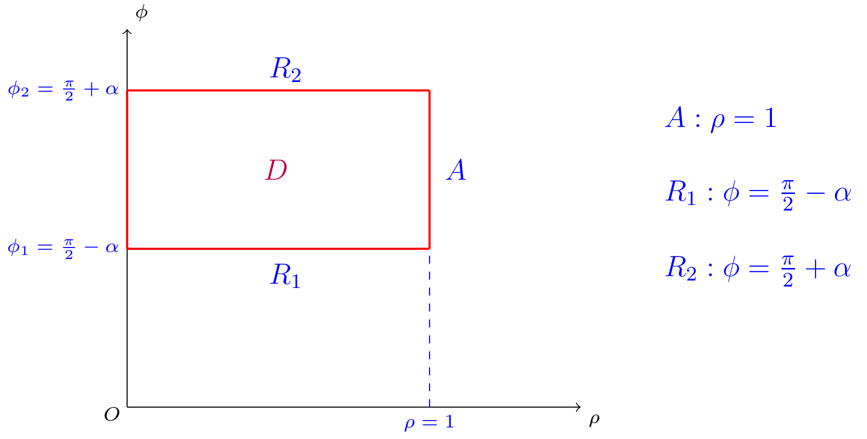

Fig. 2. Domain D in spherical coordinates.

$$b = v_\rho e_\rho + v_\phi e_\phi.$$

Then (1.1) can be rewritten as the following well-known system for which we give a short derivation in Appendix A.1.

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)v_\rho - b\cdot\nabla v_\rho + \frac{1}{\rho}(v_\phi^2 + v_\theta^2) - \partial_\rho P - \partial_t v_\rho = 0, \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)v_\phi - b\cdot\nabla v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho - \frac{1}{\rho}v_\rho v_\phi + \frac{\cot\phi}{\rho}v_\theta^2 - \frac{1}{\rho}\partial_\phi P - \partial_t v_\phi = 0, \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)v_\theta - b\cdot\nabla v_\theta - \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)v_\theta - \partial_t v_\theta = 0, \\ \frac{1}{\rho^2}\partial_\rho(\rho^2 v_\rho) + \frac{1}{\rho\sin\phi}\partial_\phi(\sin\phi\, v_\phi) = 0. \end{cases} \tag{2.7}$$

We remark that under the spherical coordinates, the assumption (i) in Theorem 1.5 means that $v_{0,\rho}$ is even, and $v_{0,\phi}$ and $v_{0,\theta}$ are odd symmetric with respect to the plane $\{\phi = \frac{\pi}{2}\}$, respectively. In other words,

$$v_{0,\rho}(\rho,\phi) = v_{0,\rho}(\rho,\pi-\phi), \quad v_{0,\phi}(\rho,\phi) = -v_{0,\phi}(\rho,\pi-\phi), \quad v_{0,\theta}(\rho,\phi) = -v_{0,\theta}(\rho,\pi-\phi). \tag{2.8}$$

The quantity $\Gamma := r v_\theta$, in the cylindrical coordinate case, can now be expressed in the spherical coordinates as

$$\Gamma = \rho\sin\phi\, v_\theta. \tag{2.9}$$

It then follows from (1.10) that Γ satisfies the equation below.

$$\Delta\Gamma - b\cdot\nabla\Gamma - \frac{2}{\rho}\,\partial_\rho\Gamma - \frac{2\cot\phi}{\rho^2}\,\partial_\phi\Gamma - \partial_t\Gamma = 0. \tag{2.10}$$

Moreover, the restriction (1.17) is converted to be

$$\sup_D \rho \sin\phi\, |v_{0,\theta}| \le \frac{1}{100}. \tag{2.11}$$

The vorticity $\omega = \nabla \times v$ can be written as $\omega = \omega_\rho e_\rho + \omega_\phi e_\phi + \omega_\theta e_\theta$, where

$$\begin{cases} \omega_\rho = \frac{1}{\rho}(\partial_\phi + \cot\phi) v_\theta = \frac{1}{\rho \sin\phi}\, \partial_\phi(\sin\phi\, v_\theta), \\ \omega_\phi = -(\partial_\rho + \frac{1}{\rho}) v_\theta = -\frac{1}{\rho}\, \partial_\rho(\rho v_\theta), \\ \omega_\theta = (\partial_\rho + \frac{1}{\rho}) v_\phi - \frac{1}{\rho}\partial_\phi v_\rho = \frac{1}{\rho}\, \partial_\rho(\rho v_\phi) - \frac{1}{\rho}\, \partial_\phi v_\rho. \end{cases} \tag{2.12}$$

Meanwhile, ω satisfies the following well-known system for which we also give a short derivation in appendix A.2.

$$\begin{cases} \big(\Delta + \frac{2}{\rho}\, \partial_\rho + \frac{2}{\rho^2}\big)\omega_\rho - b \cdot \nabla \omega_\rho + \omega \cdot \nabla v_\rho - \partial_t \omega_\rho = 0, \\ \big(\Delta - \frac{1}{\rho^2 \sin^2\phi}\big)\omega_\phi - b \cdot \nabla \omega_\phi + \frac{2}{\rho^2}\partial_\phi \omega_\rho + \omega \cdot \nabla v_\phi + \frac{1}{\rho}(v_\rho \omega_\phi - \omega_\rho v_\phi) - \partial_t \omega_\phi = 0, \\ \big(\Delta - \frac{1}{\rho^2 \sin^2\phi}\big)\omega_\theta - b \cdot \nabla \omega_\theta + \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta - \frac{1}{\rho^2}\partial_\phi(v_\theta^2) + \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2) - \partial_t \omega_\theta = 0, \\ \frac{1}{\rho^2}\partial_\rho(\rho^2 \omega_\rho) + \frac{1}{\rho \sin\phi}\partial_\phi(\sin\phi\, \omega_\phi) = 0. \end{cases} \tag{2.13}$$

Due to the presence of some super-critical terms in the above vorticity equation (2.13), it is actually more effective to consider modified quantities K, F and Ω which are defined by

$$K = \frac{\omega_\rho}{\rho}, \quad F = \frac{\omega_\phi}{\rho}, \quad \Omega = \frac{\omega_\theta}{\rho \sin\phi}. \tag{2.14}$$

It follows from (2.13) that K, F and Ω satisfy the system below:

$$\begin{cases} \big(\Delta + \frac{4}{\rho}\partial_\rho + \frac{6}{\rho^2}\big)K - b \cdot \nabla K + \omega \cdot \nabla\big(\frac{v_\rho}{\rho}\big) - \partial_t K = 0, \\ \big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1 - \cot^2\phi}{\rho^2}\big)F - b \cdot \nabla F + \frac{2}{\rho^2}\partial_\phi K + \omega \cdot \nabla\big(\frac{v_\phi}{\rho}\big) - \partial_t F = 0, \\ \big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2\cot\phi}{\rho^2}\partial_\phi\big)\Omega - b \cdot \nabla \Omega - \frac{2 v_\theta}{\rho \sin\phi}\, (K + \cot\phi\, F) - \partial_t \Omega = 0, \\ \frac{1}{\rho^2}\, \partial_\rho(\rho^3 K) + \frac{1}{\sin\phi}\, \partial_\phi(\sin\phi\, F) = 0. \end{cases} \tag{2.15}$$

The derivations of (2.12), (2.13) and (2.15) can be found in Appendix A.3. Meanwhile, since

$$K + \cot\phi\, F = \frac{\omega_\rho}{\rho} + \cot\phi\, \frac{\omega_\phi}{\rho} = \frac{1}{\rho^2}\, \partial_\phi v_\theta - \frac{\cot\phi}{\rho}\, \partial_\rho v_\theta,$$

the third equation for Ω in (2.15) is equivalent to

$$\left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2\cot\phi}{\rho^2}\partial_\phi\right)\Omega - b \cdot \nabla \Omega - \partial_t \Omega = \frac{1}{\rho^3 \sin\phi}\, \partial_\phi(v_\theta^2) - \frac{\cot\phi}{\rho^2 \sin\phi}\, \partial_\rho(v_\theta^2). \tag{2.16}$$

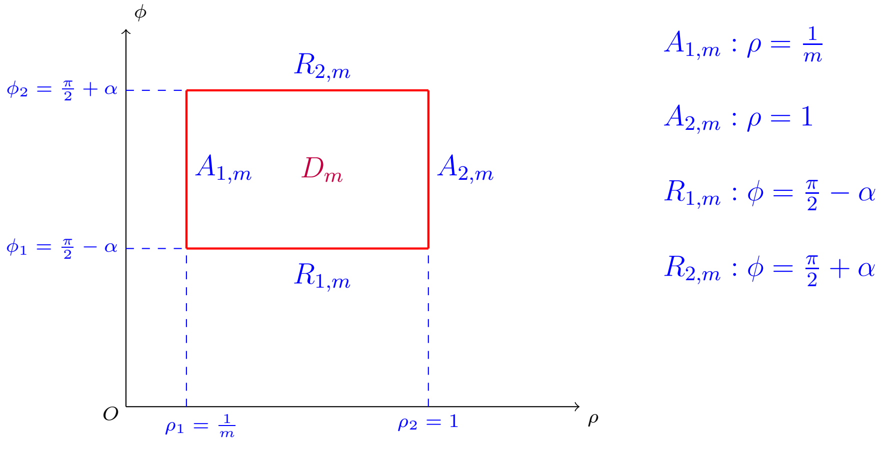

Fig. 3. Domain D_m in spherical coordinates.

Noticing that the system (2.15) contains two vortex stretching terms $\omega \cdot \nabla\left(\frac{v_\rho}{\rho}\right)$ and $\omega \cdot \nabla\left(\frac{v_\phi}{\rho}\right)$, we hope to find relations between $\frac{v_\rho}{\rho}$, $\frac{v_\phi}{\rho}$ and K, F, Ω so that we can close the energy estimate. Similar to the cylindrical case, one is able to establish equations between $\frac{v_\rho}{\rho}$, $\frac{v_\phi}{\rho}$ and Ω, see Section 4.2. In this manner, the vortex stretching terms become critical, which allows us to prove the main result.

2.2. Boundary conditions in approximating domains D_m in spherical coordinates

Under the spherical coordinates, the domain D_m in (1.15) is equivalent to the following (also see Fig. 3):

$$D_m = \left\{(\rho, \phi, \theta) : \frac{1}{m} < \rho < 1,\ \frac{\pi}{2} - \alpha < \phi < \frac{\pi}{2} + \alpha,\ \theta \in [0, 2\pi)\right\}. \tag{2.17}$$

In addition, for convenience of notation, we denote the four pieces of the boundary ∂D_m to be $R_{1,m}$, $R_{2,m}$, $A_{1,m}$ and $A_{2,m}$, and write $\partial^R D_m = R_{1,m} \cup R_{2,m}$, $\partial^A D_m = A_{1,m} \cup A_{2,m}$.

Then the NHL boundary condition (1.16) associated with D_m becomes:

$$\begin{cases} v_\phi = \omega_\rho = \omega_\theta = 0, & \text{on} \quad \partial^R D_m; \\ v_\rho = \omega_\phi = \omega_\theta = 0, & \text{on} \quad \partial^A D_m. \end{cases} \tag{2.18}$$

Making use of the vorticity formula (2.12), we see (2.18) is equivalent to

$$\begin{cases} v_\phi = \partial_\phi v_\rho = \partial_\phi(\sin\phi\, v_\theta) = 0, & \text{on} \quad \partial^R D_m; \\ v_\rho = \partial_\rho(\rho v_\phi) = \partial_\rho(\rho v_\theta) = 0, & \text{on} \quad \partial^A D_m. \end{cases} \tag{2.19}$$

Based on (2.18) and (2.19), we can also obtain the boundary conditions for Γ, K, F and Ω by direct computation. We collect all these results in the lemma below.

Lemma 2.1. *Let D_m be the domain as in (2.17). Then the following boundary conditions hold.*

$$\begin{cases} \partial_\phi v_\rho = 0, \quad v_\phi = 0, \quad \partial_\phi v_\theta = -\cot\phi\, v_\theta, \quad \partial_\phi \Gamma = 0, \\ \omega_\rho = \omega_\theta = K = \Omega = 0, \quad \partial_\phi \omega_\phi = -\cot\phi\, \omega_\phi, \quad \partial_\phi F = -\cot\phi\, F, \end{cases} \quad \text{on } \partial^R D_m, \tag{2.20}$$

and

$$\begin{cases} v_\rho = 0, \quad \partial_\rho v_\phi = -v_\phi/\rho, \quad \partial_\rho v_\theta = -v_\theta/\rho, \quad \partial_\rho \Gamma = 0, \\ \omega_\phi = \omega_\theta = F = \Omega = 0, \quad \partial_\rho \omega_\rho = -2\omega_\rho/\rho \quad \partial_\rho K = -3K/\rho, \end{cases} \quad \text{on } \partial^A D_m. \tag{2.21}$$

Before ending this subsection, we construct an element $v_0 = v_{0,\rho} e_\rho + v_{0,\phi} e_\phi + v_{0,\theta} e_\theta$ in the admissible set $\mathscr{A}$ (see Definition 1.4) such that $v_{0,\rho}$ and $v_{0,\phi}$ can be chosen arbitrarily large while $v_{0,\theta}$ can be chosen arbitrarily small. In addition, v_0 enjoys the even-odd-odd symmetry as in (2.8).

Example 2.2. Let $\alpha \in \left(0, \frac{\pi}{2}\right)$. We first choose

$$f(\rho) = \rho^7(\rho-1)^3, \quad g(\phi) = \sin^3\left(\frac{\pi}{\alpha}\left(\phi - \frac{\pi}{2}\right)\right), \quad h(s) = s^3(s-1).$$

Then for any real numbers λ_1 and λ_2, we define $v_0 = v_{0,\rho} e_\rho + v_{0,\phi} e_\phi + v_{0,\theta} e_\theta$, where

$$v_{0,\rho} = \frac{\lambda_1}{\rho^2 \sin\phi} f(\rho) g'(\phi), \quad v_{0,\phi} = -\frac{\lambda_1}{\rho \sin\phi} f'(\rho) g(\phi),$$
$$v_{0,\theta} = \frac{\lambda_2}{\rho \sin\phi}\left(\int_0^\rho h(s)\, ds\right) \sin\left(\frac{\pi}{2\alpha}\left(\phi - \frac{\pi}{2}\right)\right).$$

We claim that v belongs to $\mathscr{A}$ and has the even-odd-odd symmetry as in (2.8). Moreover, by taking λ_1 sufficiently large and λ_2 sufficiently small, $v_{0,\rho}$ and $v_{0,\phi}$ can be chosen arbitrarily large while $v_{0,\theta}$ can be chosen arbitrarily small.

In order to show $v \in \mathscr{A}$, for any $m \geq 2$, we first choose $g(\phi)$ to be the same function as the above example, and choose

$$f_m(\rho) = \rho^4\left(\rho - \frac{1}{m}\right)^3 (\rho-1)^3, \quad h_m(s) = s^2\left(s - \frac{1}{m}\right)(s-1).$$

Then we define $v_0^{(m)} = v_{0,\rho}^{(m)} e_\rho + v_{0,\phi}^{(m)} e_\phi + v_{0,\theta}^{(m)} e_\theta$, where

$$v_{0,\rho}^{(m)} = \frac{\lambda_1}{\rho^2 \sin\phi} f_m(\rho) g'(\phi), \quad v_{0,\phi}^{(m)} = -\frac{\lambda_1}{\rho \sin\phi} f_m'(\rho) g(\phi),$$

$$v_{0,\theta}^{(m)} = \frac{\lambda_2}{\rho \sin\phi} \left(\int_0^\rho h_m(s)\, ds \right) \sin\left(\frac{\pi}{2\alpha}\left(\phi - \frac{\pi}{2}\right)\right).$$

Then for each $m \geq 2$, one can directly check that $v_0^{(m)}$ satisfies the NHL boundary condition (2.19). In addition,

$$\begin{aligned} \operatorname{div} v_0^{(m)} &= \frac{1}{\rho^2}\partial_\rho\big(\rho^2 v_{0,\rho}^{(m)}\big) + \frac{1}{\rho \sin\phi}\partial_\phi\big(\sin\phi\, v_{0,\phi}^{(m)}\big) \\ &= \frac{1}{\rho^2 \sin\phi}\Big[\partial_\rho\Big(\rho^2 \sin\phi\, v_{0,\rho}^{(m)}\Big) + \partial_\phi\Big(\rho \sin\phi\, v_{0,\phi}^{(m)}\Big)\Big] = 0. \end{aligned}$$

Thus, $v_0^{(m)} \in \mathscr{A}_m$. Meanwhile, it is obvious that

$$\lim_{m\to\infty} \big\| v_0 - v_0^{(m)} \big\|_{C^2(\overline{D_m})} = 0.$$

Therefore, $v \in \mathscr{A}$.

2.3. Two weighted Poincaré inequalities on $\mathbb{R}$

In this subsection, we will introduce some weighted Poincaré inequalities, in the spirit of [29], which are needed in the sequel. Given $a, b \in \mathbb{R}$ with $a < b$, let $p \in C^\infty\big([a,b]\big)$ and assume

$$\min_{y\in[a,b]} p(y) > 0.$$

Denote the numbers p_A and p_B by

$$\begin{cases} p_A = \max\limits_{[a,b]} \left(\frac{1}{2}\frac{p''}{p} - \frac{3}{4}\frac{(p')^2}{p^2} \right), \\ p_B = \max\limits_{[a,b]} \frac{1}{2}\left(\frac{(p')^2}{p^2} - \frac{p''}{p} \right). \end{cases} \tag{2.22}$$

Lemma 2.3. *Define a functional Φ as*

$$\Phi(u) = \frac{\int_a^b p(y)\big(u'(y)\big)^2\, dy}{\int_a^b p(y) u^2(y)\, dy}, \quad \forall u \in \mathcal{A},$$

where $\mathcal{A} = \big\{u \in H^1(a,b) \setminus \{0\} : \int_a^b p(y)u(y)\, dy = 0\big\}$. Then

$$\inf_{u\in\mathcal{A}} \Phi(u) \geq \frac{\pi^2}{(b-a)^2} - p_A,$$

where p_A is defined in (2.22).

The proof of this lemma follows directly from the proof of the lemma on page 3 in Section 2 in [29]. By choosing $p(y) = \sin y$ on an interval $\left[\frac{\pi}{2} - \alpha,\, \frac{\pi}{2} + \alpha\right]$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$, we immediately obtain the following corollary.

Corollary 2.4. *Let $0 < \alpha < \frac{\pi}{2}$, $a = \frac{\pi}{2} - \alpha$, $b = \frac{\pi}{2} + \alpha$. Then for any $u \in H^1(a,b) \setminus \{0\}$ with $\int_a^b \sin y\, u(y)\, dy = 0$, we have*

$$\int_a^b \sin y\, u^2(y)\, dy \le C_{\alpha,A} \int_a^b \sin y \left(u'(y)\right)^2 dy,$$

where

$$C_{\alpha,A} = \frac{(b-a)^2}{\pi^2 + 2\alpha^2} = \frac{4\alpha^2}{\pi^2 + 2\alpha^2}. \tag{2.23}$$

So far, the Poincaré inequalities cover functions whose weighted integral on $[a, b]$ is equal to 0. In the next two results, we will consider the situation when the functions are equal to 0 on the boundary of the interval.

Lemma 2.5. *Define a functional Ψ as*

$$\Psi(u) = \frac{\int_a^b p(y)\left(u'(y)\right)^2 dy}{\int_a^b p(y) u^2(y)\, dy}, \quad \forall u \in \mathcal{B},$$

where $\mathcal{B} = H_0^1(a,b) \setminus \{0\}$. Then

$$\inf_{u\in\mathcal{B}} \Psi(u) \ge \frac{\pi^2}{(b-a)^2} - p_B,$$

where p_B is defined in (2.22).

In Lemma 2.5, by choosing $p(y) = \sin y$ on an interval $\left[\frac{\pi}{2} - \alpha,\, \frac{\pi}{2} + \alpha\right]$ for $\alpha \in \left(0, \frac{\pi}{4}\right]$, we conclude the following result right away.

Corollary 2.6. *Let $0 < \alpha \le \frac{\pi}{4}$, $a = \frac{\pi}{2} - \alpha$, $b = \frac{\pi}{2} + \alpha$. Then for any $u \in H_0^1(a,b) \setminus \{0\}$, we have*

$$\int_a^b \sin y\, u^2(y)\, dy \le C_{\alpha,B} \int_a^b \sin y \left(u'(y)\right)^2 dy,$$

where

$$C_{\alpha,B} = \frac{(b-a)^2}{\pi^2 - \frac{2\alpha^2}{\cos^2\alpha}} = \frac{4\alpha^2}{\pi^2 - \frac{2\alpha^2}{\cos^2\alpha}}. \tag{2.24}$$

Note that when $\alpha \in \left(0, \frac{\pi}{4}\right]$, both $C_{\alpha,A}$ and $C_{\alpha,B}$ are increasing functions in α. In particular,

$$\begin{cases} C_{\pi/6,A} = \frac{2}{19}, \quad C_{\pi/4,A} = \frac{2}{9}, \\ C_{\pi/6,B} = \frac{3}{25}, \quad C_{\pi/4,B} = \frac{1}{3}. \end{cases} \tag{2.25}$$

Proof of Lemma 2.5. The idea of this proof is similar to that of the lemma on page 3 in Section 2 in [29]. Define

$$\mathcal{B}_1 = \left\{ u \in H_0^1(a,b) : \int_a^b p(y)u^2(y)\,dy = 1 \right\}.$$

Then $\mathcal{B}_1 \subset \mathcal{B}$. By standard argument, there exists some $u_* \in \mathcal{B}_1$ such that the operator Ψ attains its infimum over $\mathcal{B}$ at u_*. Denote $\lambda = \Psi(u_*)$. Then

$$\inf_{u\in\mathcal{B}} \Psi(u) = \Psi(u_*) = \lambda > 0.$$

Now for any $h \in H_0^1(a,b)$, $u_* + th$ is still in $\mathcal{B}$ for any sufficiently small t. Define

$$g(t) = \Psi(u_* + th).$$

Then $g'(0) = 0$. This implies that

$$\int_a^b p u_*' h'\,dy - \lambda \int_a^b p u_* h\,dy = 0, \quad \forall\, h \in H_0^1(a,b). \tag{2.26}$$

So u_* is a weak solution of

$$\left(p u_*'\right)' + \lambda p u_* = 0, \quad \text{in} \quad (a,b). \tag{2.27}$$

Since p is smooth and bounded from below by a positive constant, it follows from classical regularity theory that $u_* \in C^\infty\left([a,b]\right)$. So u_* is a classical solution to the following equation with Dirichlet boundary condition.

$$\begin{cases} u_*'' + \frac{p'}{p} u_*' + \lambda u_* = 0, \quad \text{in} \quad (a,b), \\ u_*(a) = u_*(b) = 0. \end{cases} \tag{2.28}$$

Testing (2.28) by u_* and using integration by parts,

$$\int_a^b \left(u_*'\right)^2 dy = \frac{1}{2}\int_a^b \frac{p'}{p}\left(u_*^2\right)' dy + \lambda \int_a^b u_*^2\, dy$$

$$= \int_a^b \left[-\frac{1}{2}\left(\frac{p'}{p}\right)' + \lambda\right] u_*^2\, dy$$

$$\leq (\lambda + p_B) \int_a^b u_*^2\, dy,$$

where p_B is as defined in (2.22). Hence,

$$\lambda + p_B \geq \frac{\int_a^b \left(u_*'\right)^2 dy}{\int_a^b u_*^2\, dy}.$$

Since $u_*(a) = u_*(b) = 0$, it is well-known that the quotient on the right-hand side of the above inequality is bounded from below by $\pi^2/(b-a)^2$. Thus,

$$\lambda \geq \frac{\pi^2}{(b-a)^2} - p_B. \quad \square$$

2.4. A Hardy's type inequality in D_m

Let the region D_m be as defined in (2.4) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{2}\right)$. If a scalar-valued function $f \in H^1(D_m)$ with 0 boundary value, that is $f \in H_0^1(D_m)$, then it follows from the classical Hardy's inequality that $\left\|\frac{f}{\rho}\right\|_{L^2(D_m)} = \left\|\frac{f}{|x|}\right\|_{L^2(D_m)} \leq 2\|\nabla f\|_{L^2(D_m)}$. But if a function does not vanish on the boundary, then the norm of the gradient ∇f alone does not suffice to control the norm of f/ρ. The next result says that in the special domains D_m, after adding the norm of a lower-order term, only the norm of partial gradient, $\partial_\rho f$, is needed to control the norm of f/ρ with constants independent of m. Such an estimate may be known, but we could not find the specific form in the literature when the domain is a finite cone.

Lemma 2.7. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then for any scalar-valued function $f \in H^1(D_m)$ and for any $\epsilon > 0$,*

$$\int_{D_m} \frac{f^2}{\rho^2}\, dx \leq (4+\epsilon) \int_{D_m} |\partial_\rho f|^2\, dx + \left(40 + \frac{16}{\epsilon}\right) \int_{D_m} f^2\, dx. \tag{2.29}$$

Proof. By converting the integral into spherical coordinates, we have

$$\int_{D_m} \frac{f^2}{\rho^2}\, dx = 2\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi \left(\int_{\frac{1}{m}}^{1} f^2(\rho,\phi)\, d\rho\right) d\phi. \tag{2.30}$$

Using integration by parts,

$$\int_{\frac{1}{m}}^{1} f^2(\rho,\phi)\,d\rho \le f^2(1,\phi) - 2\int_{\frac{1}{m}}^{1} \rho f \partial_\rho f\,d\rho$$

Plugging this estimate into (2.30) yields

$$\int_{D_m} \frac{f^2}{\rho^2}\,dx \le I_1 - I_2, \tag{2.31}$$

where

$$I_1 = 2\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi\, f^2(1,\phi)\,d\phi, \quad I_2 = 4\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{1}{m}}^{1} \rho \sin\phi\, f \partial_\rho f\,d\rho\,d\phi.$$

For I_2, by changing back to the Euclidean coordinates and using Cauchy-Schwarz inequality, we find

$$|I_2| \le \frac{1}{2}\int_{D_m} \frac{f^2}{\rho^2}\,dx + 2\int_{D_m} |\partial_\rho f|^2\,dx. \tag{2.32}$$

In order to estimate I_1, we fix a cutoff function $\eta \in C^\infty(\mathbb{R})$ such that $0 \le \eta \le 1$,

$$\eta(t) = \begin{cases} 0 & \text{if} \quad t \le \frac{3}{4}, \\ 1 & \text{if} \quad t \ge 1, \end{cases}$$

and $\sup\limits_{t\in\mathbb{R}} |\eta'(t)| \le 10$. Then

$$\begin{aligned} I_1 &= 2\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi \left[f^2(1,\phi)\eta(1) - f^2(3/4,\phi)\eta(3/4)\right] d\phi \\ &= 2\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi \int_{\frac{3}{4}}^{1} \partial_\rho \Big[f^2(\rho,\phi)\eta(\rho)\Big]\,d\rho\,d\phi. \end{aligned}$$

Since $|\eta'| \le 10$, it follows from the above expression that

$$I_1 \le 20\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{3}{4}}^{1} \sin\phi\, f^2(\rho,\phi)\,d\rho\,d\phi + 4\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{3}{4}}^{1} \sin\phi\, |f \partial_\rho f|\,d\rho\,d\phi.$$

Since ρ has the lower bound $\frac{3}{4}$ in the above integral, we further deduce that

$$I_1 \leq 40\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{3}{4}}^{1} \rho^2 \sin\phi\, f^2(\rho,\phi)\, d\rho\, d\phi + 8\pi \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{3}{4}}^{1} \rho^2 \sin\phi\, |f \partial_\rho f|\, d\rho\, d\phi.$$

Changing back to the Euclidean coordinates and applying Cauchy-Schwarz inequality, we find

$$I_1 \leq 20 \int_{D_m} f^2\, dx + 4 \int_{D_m} |f \partial_\rho f|\, dx \leq \left(20 + \frac{8}{\epsilon}\right) \int_{D_m} f^2\, dx + \frac{\epsilon}{2} \int_{D_m} |\partial_\rho f|^2\, dx. \tag{2.33}$$

Putting (2.33) and (2.32) into (2.31) leads to (2.29). $\square$

Let v be a vector field on D_m. It has two decompositions under the Euclidean coordinates and the spherical coordinates respectively:

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_\rho e_\rho + v_\phi e_\phi + v_\theta e_\theta.$$

Then it is well-known that $|\nabla v|^2 = \sum_{i=1}^{3} |\nabla v_i|^2$. But according to formula (A.8), the relation $|\nabla v|^2 = |\nabla v_\rho|^2 + |\nabla v_\phi|^2 + |\nabla v_\theta|^2$ may not hold. Nonetheless, we can take advantage of Lemma 2.7 to show the equivalence between the $H^1(D_m)$ norm of v and the sum of $H^1(D_m)$ norms of its components v_ρ, v_ϕ and v_θ.

Corollary 2.8. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{2}\right)$. Let $v = v_\rho e_\rho + v_\phi e_\phi + v_\theta e_\theta$ be a vector field on D_m. Then v belongs to $H^1(D_m)$ if and only if all its components v_ρ, v_ϕ and v_θ belong to $H^1(D_m)$. In addition, there exists some constant $C > 1$, which only depends on α, such that*

$$\frac{1}{C}\|v\|_{H^1(D_m)} \leq \|v_\rho\|_{H^1(D_m)} + \|v_\phi\|_{H^1(D_m)} + \|v_\theta\|_{H^1(D_m)} \leq C\|v\|_{H^1(D_m)}. \tag{2.34}$$

Proof. Firstly, since $\{e_\rho, e_\phi, e_\theta\}$ forms an orthogonal basis in $\mathbb{R}^3$, $|v|^2 = |v_\rho|^2 + |v_\phi|^2 + |v_\theta|^2$, so

$$\|v\|^2_{L^2(D_m)} = \|v_\rho\|^2_{L^2(D_m)} + \|v_\phi\|^2_{L^2(D_m)} + \|v_\theta\|^2_{L^2(D_m)}.$$

On the other hand, according to formula (A.8), under the basis (A.7), the gradient ∇v can be represented as

$$\nabla v = \begin{pmatrix} \partial_\rho v_\rho & \frac{1}{\rho}(\partial_\phi v_\rho - v_\phi) & -\frac{1}{\rho}\, v_\theta \\ \partial_\rho v_\phi & \frac{1}{\rho}(\partial_\phi v_\phi + v_\rho) & -\frac{\cot\phi}{\rho}\, v_\theta \\ \partial_\rho v_\theta & \frac{1}{\rho}\partial_\phi v_\theta & \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi) \end{pmatrix}. \tag{2.35}$$

Meanwhile,

$$|\nabla v_\rho|^2 = |\partial_\rho v_\rho|^2 + \left|\frac{1}{\rho}\partial_\phi v_\rho\right|^2, \quad |\nabla v_\phi|^2 = |\partial_\rho v_\phi|^2 + \left|\frac{1}{\rho}\partial_\phi v_\phi\right|^2, \quad |\nabla v_\theta|^2 = |\partial_\rho v_\theta|^2 + \left|\frac{1}{\rho}\partial_\phi v_\theta\right|^2.$$

Noticing that in the domain D_m, ρ and $\cot\phi$ are bounded:

$$\frac{1}{m} < \rho < 1, \quad 0 \le \cot\phi < \cot\alpha,$$

so it is straightforward to check that v belongs to $H^1(D_m)$ if and only if all its components v_ρ, v_ϕ and v_θ belong to $H^1(D_m)$. Moreover, one can apply Lemma 2.7 and Cauchy-Schwarz inequality to (2.35) to establish (2.34). $\square$

2.5. A priori L^∞ bound for $\Gamma = \rho\sin\phi\, v_\theta$ in D_m

In this section, we study the quantity Γ, defined as in (2.9), in the approximating space-time domain $D_m \times [0, T]$, where $0 < T < \infty$. Define the energy space $E_{m,T}$ as

$$E_{m,T} = L_t^\infty L_x^2 \cap L_t^2 H_x^1\big(D_m \times [0, T]\big) \tag{2.36}$$

which is equipped with the following norm:

$$\|v\|_{E_{m,T}}^2 = \int_0^T \int_{D_m} |\nabla v(x,t)|^2\, dx\, dt + \sup_{t\in[0,T]} \int_{D_m} |v(x,t)|^2\, dx. \tag{2.37}$$

The function v can be either vector-valued or scalar-valued, depending on the context. We denote by $E_{m,T}^\sigma$ the subspace of $E_{m,T}$ which consists of vectors which are divergence free and whose normal component vanishes on the boundary of D_m.

$$E_{m,T}^\sigma = \big\{v \in E_{m,T} :\ \nabla\cdot v = 0 \text{ in } D_m \text{ and } v\cdot n = 0 \text{ on } \partial D_m \text{ for a.e. } t \in [0, T]\big\}. \tag{2.38}$$

If a function v is independent of time, we may also say it belongs to $E_{m,T}$ or $E_{m,T}^\sigma$ by regarding it as a stationary function.

Based on the equation (2.10) and the boundary conditions in Lemma 2.1, Γ is determined by the following problem:

$$\begin{cases} \Delta\Gamma - b\cdot\nabla\Gamma - \frac{2}{\rho}\partial_\rho\Gamma - \frac{2\cot\phi}{\rho^2}\partial_\phi\Gamma - \partial_t\Gamma = 0, & \text{in} \quad D_m \times (0, T]; \\ \partial_n\Gamma = 0, \quad \text{on} \quad \partial D_m \times (0, T]; \\ \Gamma(x, 0) = \Gamma_0(x), \quad x \in D_m, \end{cases} \tag{2.39}$$

where $b = v_\rho e_\rho + v_\phi e_\phi$, Γ_0 is the initial value defined as $\Gamma_0 = \rho\sin\phi\, v_{0,\theta}(x)$, and $\partial_n\Gamma$ means the directional derivative of Γ along the exterior normal direction of ∂D_m, except at the corners. In this section, we will study the solvability of (2.39) and the regularity of its solution. As a preparation, we first introduce an embedding result.

In general, for any 3D domain Ω and for any function v that lies in the energy space $L_t^\infty L_x^2 \cap L_t^2 H_x^1\big(\Omega \times [0,T]\big)$ automatically belongs to $L_{tx}^{10/3}\big(\Omega \times [0,T]\big)$ by standard interpolation. But if the function v is axially symmetric and the domain Ω, say $\Omega = D_m$, is bounded and has a positive distance to the x_3 axis, then we can regard v as a function on a 2D domain Ω' in the ρ-ϕ space. Thus, the 2D Ladyzhenskaya's inequality (or more precisely, the Gagliardo-Nirenberg inequality) is applicable and we are able to improve the regularity of v from $L_{tx}^{10/3}$ to L_{tx}^4. We point out that the range of α in the following Lemma 2.9 and 2.10 is larger than the one in the main theorem.

Lemma 2.9. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \big(0, \frac{\pi}{2}\big)$. Then for any $T > 0$, the energy space $E_{m,T}$ is embedded in $L_{tx}^4\big(D_m \times [0,T]\big)$. In addition, there exists a constant $C = C(\alpha, m)$ such that*

$$\|f\|_{L_{tx}^4(D_m \times [0,T])} \leq C\big(T^{1/4} + 1\big)\|f\|_{E_{m,T}}, \quad \forall f \in E_{m,T}. \tag{2.40}$$

Proof. Since the volume element $\rho^2 \sin\phi \, d\rho d\phi d\theta$ on D_m is equivalent to the two-dimensional volume element $d\rho d\phi$ on the ρ-ϕ plane, we can apply the 2D Gagliardo-Nirenberg inequality to f in D_m to conclude that

$$\begin{aligned}&\|f(\cdot,t)\|_{L^4(D_m)} \leq C\Big(\|f(\cdot,t)\|_{L^2(D_m)}^{\frac{1}{2}}\|\nabla f(\cdot,t)\|_{L_2(D_m)}^{\frac{1}{2}} + \|f(\cdot,t)\|_{L^2(D_m)}\Big),\\ &\quad \text{for a.e. } t \in [0,T],\end{aligned}$$

where C is some constant that only depends on α and m. As a result, we deduce that

$$\begin{aligned}&\|f(\cdot,t)\|_{L^4(D_m)}^4 \leq C\Big(\|f(\cdot,t)\|_{L^2(D_m)}^2\|\nabla f(\cdot,t)\|_{L_2(D_m)}^2 + \|f(\cdot,t)\|_{L^2(D_m)}^4\Big),\\ &\quad \text{for a.e. } t \in [0,T].\end{aligned}$$

Then (2.40) follows from integrating the above estimate in t on $[0,T]$. □

Now we are ready to present the main result of this subsection.

Lemma 2.10. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \big(0, \frac{\pi}{2}\big)$. Let $T > 0$ and $b = v_\rho e_\rho + v_\phi e_\phi \in E_{m,T}^\sigma$. Assume the initial velocity $v_0 \in H^2(D_m)$ is divergence free and satisfies the NHL boundary condition (2.18). Then the problem (2.39) possesses a unique bounded weak solution Γ in the energy space $E_{m,T}$ which satisfies*

$$\|\Gamma\|_{L^\infty(D_m \times [0,T])} \leq \|\Gamma_0\|_{L^\infty(D_m)}, \tag{2.41}$$

and

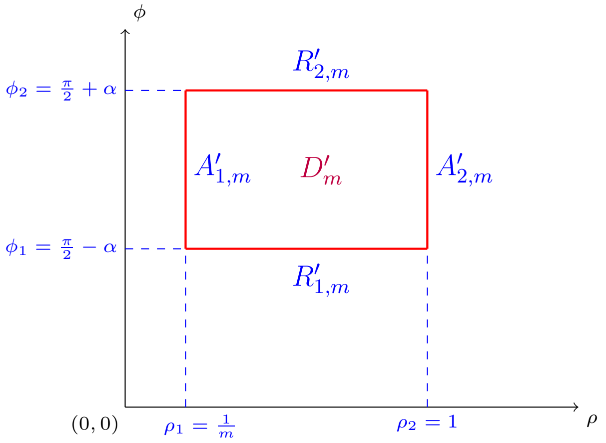

Fig. 4. Domain D'_m in ρ-ϕ coordinates.

$$\|\Gamma\|_{E_{m,T}} \le Ce^{CT}\|\Gamma_0\|_{L^2(D_m)}, \tag{2.42}$$

where $\Gamma_0 = \rho \sin\phi\, v_{0,\theta}$ *and* C *is a positive constant which only depends on* α *and* m.

Proof. We use the dimension reduction method and Lemma 2.9 to justify the conclusions. Firstly, we view Γ as a function of variables ρ, ϕ and t, and regard D_m as a 2D domain D'_m on the ρ-ϕ plane, which is defined as below (see Fig. 4).

$$D'_m = \Big\{(\rho,\phi) : \frac{1}{m} < \rho < 1,\ \frac{\pi}{2} - \alpha < \phi < \frac{\pi}{2} + \alpha\Big\}. \tag{2.43}$$

Then equation (2.39) can be rewritten as below:

$$\begin{cases} \partial_\rho^2\Gamma + \frac{1}{\rho^2}\partial_\phi^2\Gamma - v_\rho\partial_\rho\Gamma - \left(\frac{1}{\rho}v_\phi + \frac{\cot\phi}{\rho^2}\right)\partial_\phi\Gamma - \partial_t\Gamma = 0, \quad \text{in} \quad D'_m \times (0,T]; \\ \partial_n\Gamma = 0, \quad \text{on} \quad \partial D'_m \times (0,T]; \\ \Gamma(x,0) = \Gamma_0(x), \quad x \in D'_m. \end{cases} \tag{2.44}$$

Thanks to Lemma 2.9, both v_ρ and v_ϕ belong to $L^4_{tx}(D'_m \times (0,T])$, and any function Γ in the energy space $E_{m,T}$ also belongs to $L^4_{tx}(D'_m \times (0,T])$ which is the critical space for (2.44) in 2D space. Since the distance of D'_m to the x_3 axis is at least $\frac{1}{m}$, the existence of a weak solution Γ of (2.39) in $E_{m,T}$ follows from the classical theory. Meanwhile, since D'_m is a 2D domain and both v_ρ and v_ϕ belong to L^4_{tx}, the weak maximum principle is applicable for (2.39), see e.g. Theorem 2.1 in [16]. As a result, the uniqueness of the solution and the estimate (2.41) are justified.

Since the above solution Γ lies in $L^\infty \cap E_{m,T}$, it can be served as a test function to (2.39). Meanwhile, since $b \in E^\sigma_{m,T}$, which implies $\nabla \cdot b = 0$ in $D_m \times (0,T]$ and $b \cdot n = 0$ on $\partial D_m \times (0,T]$, then it holds that $\int_{D_m} (b \cdot \nabla\Gamma)\Gamma\, dx = 0$. As a result, (2.42) follows from the standard energy estimate. $\square$

3. Existence of strong solutions in D_m

In this section, we study the existence of solutions in the approximating space-time domains $D_m \times [0, T]$, where $m \geq 2$ and $0 < T < \infty$. We point out that the local existence of the solution in the energy space $E_{m,T}$ has already been proven in literature, see e.g. [25] which even covers more general Lipschitz domains. In the current situation, the local existence can be extended to the global one since D_m is away from x_3 axis. Our main goal here is to prove the existence of the solution with higher regularity. Actually, we will establish the existence of the bounded strong solution v on $D_m \times [0, T]$ for any $T > 0$. The proof of higher regularity of solutions, although somewhat unsurprising, requires some detailed analysis because the domains are not smooth. Those, who would like to have a quick view of the key idea in the proof for the main Theorem 1.5, can skip this section for now and jump to Section 4. We also remark that if the NHL boundary condition is replaced by the Dirichlet boundary condition, then the local existence of strong solutions on general bounded Lipschitz domains has been established in [12] using the semi-group theory. But it may take much effort to adapt that method to treat the more complicated NHL boundary condition.

Besides the existence of the strong solution, we will also show that if the initial data enjoys the even-odd-odd symmetry, defined as in Definition (1.3), then this symmetry will be preserved in time for the strong solution. For convenience of notations, we define

$$E^{\sigma,s}_{m,T} = \big\{ v \in E^{\sigma}_{m,T} : \ v \text{ has the even-odd-odd symmetry} \big\}, \tag{3.1}$$

where s stands for symmetry. In the following, we will first construct a local solution in Proposition 3.1 and then extend it to be a global one in Corollary 3.3.

Proposition 3.1. *Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $m \geq 2$. Assume the initial velocity $v_0 \in H^2(D_m)$ is divergence free in D_m and satisfies the NHL condition (2.18) on ∂D_m. Then there exists some time $T > 0$ and a strong solution (v, P) of (2.7) on $D_m \times [0, T]$ with the initial data v_0 and the NHL condition (2.18) such that*

$$v \in E^{\sigma}_{m,T} \cap H^1_t L^2_x \cap L^2_t H^2_x \cap L^\infty_{tx}\big(D_m \times [0, T]\big), \quad P \in L^2_t H^1_x(D_m \times [0, T]). \tag{3.2}$$

Moreover, if $(\hat{v}, \hat{P})$ is another strong solution, then $\hat{v}$ coincides with v on $D_m \times [0, T]$. As a result, if v_0 possesses the even-odd-odd symmetry, i.e. $v_0 \in E^{\sigma,s}_{m,T}$, then so does v.

Remark 3.2. In Euclidean coordinates, the strong solution v of (2.7) in Proposition 3.1 is understood in the same sense as that in Definition 1.2 with D being replaced by D_m.

Proof of Proposition 3.1. Firstly, we decompose the given initial data v_0 and the initial vorticity $\omega_0 := \operatorname{curl} v_0$ as

$$v_0 = v_{0,\rho} e_\rho + v_{0,\phi} e_\phi + v_{0,\theta} e_\theta, \quad \omega_0 = \omega_{0,\rho} e_\rho + \omega_{0,\phi} e_\phi + \omega_{0,\theta} e_\theta.$$

Meanwhile, we denote

$$A_0 := 1 + \|v_{0,\theta}\|_{L^\infty(D_m)} + \|\omega_{0,\theta}\|_{L^6(D_m)}. \tag{3.3}$$

In the following proof, C denotes a generic constant which may depend on α and m. The values of C may be different from line to line. If a constant C also depends on other quantities, we will state it explicitly. Now we give an outline of the proof:

(i) For any $T > 0$ and for any scalar functions v_ρ and v_ϕ such that the vector field $b := v_\rho e_\rho + v_\phi e_\phi$ belongs to $E^\sigma_{m,T}$, we use b as a given data in the equation for v_θ (see (2.7)). This linearized equation, with suitable boundary condition and $v_{0,\theta}$ as the initial value, determines a vector field $v_\theta e_\theta \in E^\sigma_{m,T} \cap L^\infty_{tx}(D_m \times (0,T])$.

(ii) Use the above b and v_θ as given data in the equation for $\widetilde{\Omega}$ (see (2.16)) with 0 boundary value and $\omega_{0,\theta}/(\rho\sin\phi)$ as initial value, one finds $\widetilde{\Omega}$ in $E_{m,T} \cap L^\infty_t L^q_x\big(D_m \times [0,T]\big)$ for any $q \geq 1$. Then we define $\widetilde{\omega}_\theta = \rho \sin\phi\, \widetilde{\Omega}$ and treat it as the angular vorticity.

(iii) Based on the $\widetilde{\omega}_\theta$ constructed above and the Biot-Savart law with a suitable boundary condition, we determine a vector

$$\tilde{b} = \tilde{v}_\rho e_\rho + \tilde{v}_\phi e_\phi \in E^\sigma_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T]).$$

Thus, the correspondence between b and $\tilde{b}$ determines a map $\mathbb{L}$:

$$\mathbb{L} b = \tilde{b}, \tag{3.4}$$

from the space $E^\sigma_{m,T} \cap \text{span}\{e_\rho, e_\phi\}$ to itself. As a summary of steps so far, a diagram of the process is given below:

$$\text{Diagram:} \quad b \Rightarrow v_\theta \Rightarrow \widetilde{\Omega} \Rightarrow \widetilde{\omega}_\theta \equiv \rho \sin\phi\, \widetilde{\Omega} \Rightarrow \tilde{b}.$$

(iv) Next, we will find a suitably large number M such that $\mathbb{L}$ is a contraction mapping on the space $\overline{B_{E^\sigma_{m,T}}(0,M)} \bigcap \text{span}\{e_\rho, e_\phi\}$ as long as T is sufficiently small. Thus, we obtain a fixed point b of $\mathbb{L}$ thanks to the contraction mapping theorem.

(v) Based on the fixed point b of $\mathbb{L}$ in the above step, we define $v \equiv b + v_\theta e_\theta$ and $\omega = \nabla \times v$, where v_θ is the function constructed in step (i). Then we show that ω_θ coincides with the previously constructed $\widetilde{\omega}_\theta$. Based on this, we manage to prove $v \in E^\sigma_{m,T} \cap H^1_t L^2_x \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T])$ and find a pressure term P in $L^2_t H^1_x(D_m \times [0,T])$ such that (v,P) is a strong solution of (2.7) on $D_m \times [0,T]$ subject to the initial data v_0 and the NHL boundary condition (2.18).

(vi) Finally, the uniqueness of the strong solution v will be addressed. As a byproduct, we will justify the preservation of the even-odd-odd symmetry of the initial data.

In the following argument, details of the above steps will be carried out.

Step 1. Construction of v_θ.

Fix any $b := v_\rho e_\rho + v_\phi e_\phi \in E^\sigma_{m,T}$. Based on the v_θ equation in (2.7), we determine v_θ by the following initial boundary value problem:

$$\begin{cases} \left(\Delta - \frac{1}{\rho^2 \sin^2\phi}\right) v_\theta - b\cdot\nabla v_\theta - \frac{1}{\rho}\big(v_\rho + \cot\phi\, v_\phi\big) v_\theta - \partial_t v_\theta = 0, \quad \text{in} \quad D_m \times (0,T]; \\ \partial_\phi v_\theta = -\cot\phi\, v_\theta, \quad \text{on} \quad \partial^R D_m \times (0,T], \quad \partial_\rho v_\theta = -\frac{1}{\rho} v_\theta, \quad \text{on} \quad \partial^A D_m \times (0,T]; \\ v_\theta(x,0) = v_{0,\theta}(x), \quad x \in D_m, \end{cases} \tag{3.5}$$

where $v_{0,\theta}$ is the θ-component of the given initial data v_0. Since the boundary condition for v_θ is of Robin type which is more complicated than the Neumann condition, we instead consider the equation for Γ, defined as

$$\Gamma = \rho \sin\phi\, v_\theta,$$

which satisfies the homogeneous Neumann boundary condition. More precisely, Γ is determined by the following problem based on (3.5).

$$\begin{cases} \Delta\Gamma - b\cdot\nabla\Gamma - \frac{2}{\rho}\partial_\rho\Gamma - \frac{2\cot\phi}{\rho^2}\partial_\phi\Gamma - \partial_t\Gamma = 0, \quad \text{in} \quad D_m \times (0,T]; \\ \partial_n\Gamma = 0, \quad \text{on} \quad \partial D_m \times (0,T]; \\ \Gamma(x,0) = \Gamma_0(x), \quad x \in D_m, \end{cases} \tag{3.6}$$

where $\Gamma_0 := \rho\sin\phi\, v_{0,\theta}$. According to Lemma 2.10, (3.6) possesses a unique bounded weak solution Γ in $E_{m,T}$ which satisfies the estimates (2.41) and (2.42). Since ρ is bounded from above and below, then (2.41) and (2.42) imply that

$$\begin{aligned} &\|v_\theta\|_{L^\infty(D_m\times[0,T])} \le C\|v_{0,\theta}\|_{L^\infty(D_m)}, \\ &\|v_\theta\|_{E_{m,T}} \le Ce^{CT}\|v_{0,\theta}\|_{L^2(D_m)}. \end{aligned} \tag{3.7}$$

Step 2. Constructing an intermediate angular vorticity $\widetilde{\omega}_\theta$.

With the vector field b and the corresponding v_θ from Step 1, we will introduce a function

$$\widetilde{\omega}_\theta := \rho\sin\phi\,\widetilde{\Omega}, \tag{3.8}$$

where $\widetilde{\Omega}$ is determined by the following problem (also see (2.16)):

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2\cot\phi}{\rho^2}\partial_\phi\right)\widetilde{\Omega} - b\cdot\nabla\widetilde{\Omega} - \partial_t\widetilde{\Omega} \\ \quad = \frac{1}{\rho^2\sin\phi}\left(\frac{1}{\rho}\partial_\phi(v_\theta^2) - \cot\phi\,\partial_\rho(v_\theta^2)\right), \ \text{in} \ D_m\times(0,T]; \\ \widetilde{\Omega} = 0, \ \text{on} \ \partial D_m \times (0,T]; \\ \widetilde{\Omega}(x,0) = \omega_{0,\theta}(x)/(\rho\sin\phi), \ x\in D_m. \end{cases} \tag{3.9}$$

Here, $\omega_{0,\theta}$ is the θ-component of $\omega_0 := \operatorname{curl} v_0$. The reason that we study the equation (3.9) of $\widetilde{\Omega}$ instead of the equation of $\widetilde{\omega}_\theta$ (see (3.14)) is to avoid the term $\frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\widetilde{\omega}_\theta$ in (3.14).

Claim A. *The problem (3.9) has a unique weak solution $\widetilde{\Omega}$ in the energy space $E_{m,T}$. In addition, the energy of $\widetilde{\Omega}$ has the following upper bound:*

$$\|\widetilde{\Omega}\|_{E_{m,T}} \le C A_0^2 e^{CT}, \tag{3.10}$$

where $C = C(\alpha, m)$ and A_0 is as defined in (3.3).

Proof of Claim A. Firstly, we denote the function on the right-hand side of (3.9) to be R_1, that is

$$R_1 := \frac{1}{\rho^2 \sin\phi}\Big(\frac{1}{\rho}\partial_\phi(v_\theta^2) - \cot\phi\, \partial_\rho(v_\theta^2)\Big).$$

Thanks to the estimates (3.7), we know $v_\theta \in E_{m,T} \cap L^\infty_{tx}(D_m \times [0,T])$, which implies $R_1 \in L^2_{tx}(D_m \times [0,T])$ and

$$\|R_1\|_{L^2(D_m\times[0,T])} \le Ce^{CT}\|v_{0,\theta}\|_{L^\infty(D_m)}\|v_{0,\theta}\|_{L^2(D_m)} \le Ce^{CT}A_0^2. \tag{3.11}$$

Next, similar to the proof of Lemma 2.10, we regard the problem (3.9) as a 2D problem on the domain D'_m which is defined as in (2.43). Then the energy space $E_{m,T}$ is embedded into $L^4_{tx}(D_m \times [0,T])$ due to Lemma 2.9. So the vector field b in the drift term $b\cdot\nabla\widetilde{\Omega}$ is in the critical class. As a result, the existence part in Claim A follows from standard parabolic theory. To address the uniqueness part, we assume there are two weak solutions $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ in the energy space $E_{m,T}$ and then consider the equation for their difference $\widetilde{\Omega}_1 - \widetilde{\Omega}_2$. Then it follows from the standard energy estimate that $\widetilde{\Omega}_1 - \widetilde{\Omega}_2 \equiv 0$ on $D_m \times [0,T]$. Finally, the estimate (3.10) can be established by testing (3.9) with $\widetilde{\Omega}$ and taking advantage of the estimate (3.11). Hence, Claim A is verified.

For the solution $\widetilde{\Omega}$ in the above claim, we can actually obtain higher integrability of $\widetilde{\Omega}$ which will be used later. Since $v_0 \in H^2(D_m)$ and satisfies the NHL boundary condition (2.18), then $\widetilde{\Omega}(\cdot,0) = \omega_{0,\theta}(x)/(\rho\sin\phi) \in H^1(D_m)$ with 0 boundary value. So by regarding it as a function on the 2D domain D'_m, we find $\widetilde{\Omega}(\cdot,0) \in L^q(D_m)$ for any $q \ge 1$ due to the 2D Sobolev inequality. Then using the standard energy estimate for $\widetilde{\Omega}^q$ and the fact that the drift terms are integrated out, we have

$$\|\widetilde{\Omega}\|_{L^\infty_t L^q_x(D_m\times[0,T])} \le \exp\Big(Cq\|v_\theta\|^4_{L^\infty(D_m\times[0,T])}T\Big)\Big(\|\widetilde{\Omega}(\cdot,0)\|_{L^q(D_m)} + 1\Big).$$

Since $\|v_\theta\|_{L^\infty(D_m\times[0,T])} \le C\|v_{0,\theta}\|_{L^\infty(D_m)} \le CA_0$, we deduce

$$\|\widetilde{\Omega}\|_{L^\infty_t L^q_x(D_m\times[0,T])} \le e^{CqA_0^4T}\Big(\|\widetilde{\Omega}(\cdot,0)\|_{L^q(D_m)} + 1\Big). \tag{3.12}$$

In particular, if q is restricted in the interval $[1,6]$, then

$$\|\widetilde{\Omega}\|_{L_t^\infty L_x^q(D_m\times[0,T])} \le Ce^{CA_0^4T}A_0, \quad \forall 1\le q\le 6. \tag{3.13}$$

After the construction of $\widetilde{\Omega}$, we define

$$\widetilde{\omega}_\theta = \rho\sin\phi\,\widetilde{\Omega}.$$

Then it is the unique weak solution of the following problem (3.14) in the energy space $E_{m,T}$.

$$\begin{cases} (\Delta - \frac{1}{\rho^2\sin^2\phi})\widetilde{\omega}_\theta - b\cdot\nabla\widetilde{\omega}_\theta + \frac{1}{\rho}(v_\rho+\cot\phi\, v_\phi)\widetilde{\omega}_\theta - \partial_t\widetilde{\omega}_\theta \\ \qquad\qquad\qquad = \frac{1}{\rho^2}\partial_\phi(v_\theta^2) - \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2), \text{ in } D_m\times(0,T]; \\ \widetilde{\omega}_\theta = 0, \text{ on } \partial D_m\times(0,T]; \\ \widetilde{\omega}_\theta(x,0) = \omega_{0,\theta}(x), \ x\in D_m. \end{cases} \tag{3.14}$$

Note that $\widetilde{\omega}_\theta$ may not be equal to curl b yet. Next, we will use $\widetilde{\Omega}$ to construct a vector field $\tilde{b}$ according to the Biot-Savart law $\Delta\tilde{b} = -\nabla\times\widetilde{\omega}_\theta$. Eventually, the map that assigns b to $\tilde{b}$ will be shown to have a fixed point. For such a fixed point b, we will prove in Step 5 that $\operatorname{curl} b = \widetilde{\omega}_\theta$.

Step 3. Introducing a map $\mathbb{L}$ from $E_{m,T}^\sigma \bigcap \operatorname{span}\{e_\rho, e_\phi\}$ into itself.

Using the function $\widetilde{\Omega}$ in Step 2 and the Biot-Savart law in the spherical system (see (4.12) and (4.13) in Section 4.2), we construct two functions $\tilde{v}_\rho, \tilde{v}_\phi \in E_{m,T}$ by solving the elliptic problems (3.15) and (3.16) respectively in $H^1(D_m)$ for a.e. $t\in[0,T]$.

$$\begin{cases} \Big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\Big)\tilde{v}_\rho = -\frac{1}{\sin\phi}\,\partial_\phi(\sin^2\phi\,\widetilde{\Omega}), \text{ in } D_m; \\ \partial_\phi\tilde{v}_\rho = 0 \text{ on } \partial^R D_m, \quad \tilde{v}_\rho = 0 \text{ on } \partial^A D_m. \end{cases} \tag{3.15}$$

$$\begin{cases} \Big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\Big)\tilde{v}_\phi = \frac{1}{\rho^3}\,\partial_\rho(\rho^4\sin\phi\,\widetilde{\Omega}), \text{ in } D_m; \\ \tilde{v}_\phi = 0 \text{ on } \partial^R D_m, \quad \partial_\rho\tilde{v}_\phi = -\frac{1}{\rho}\tilde{v}_\phi \text{ on } \partial^A D_m. \end{cases} \tag{3.16}$$

In particular, when $t=0$, recalling that $\widetilde{\Omega}(x,0) = \omega_{0,\theta}(x)/(\rho\sin\phi)$ in (3.9), then by defining

$$\tilde{v}_\rho(x,0) = v_{0,\rho}(x) \quad \text{and} \quad \tilde{v}_\phi(x,0) = v_{0,\phi}(x) \quad \text{on} \quad \overline{D_m}, \tag{3.17}$$

one can verify that $\tilde{v}_\rho(x,0)$ and $\tilde{v}_\phi(x,0)$ satisfy (3.15) and (3.16) respectively when $t=0$.

Claim B. *For a.e. $t\in[0,T]$, (3.15) (resp. (3.16)) has a unique solution $\tilde{v}_\rho(\cdot,t)$ (resp. $\tilde{v}_\phi(\cdot,t)$) in the space $H^1(D_m)$. Moreover, both $\tilde{v}_\rho(\cdot,t)$ and $\tilde{v}_\phi(\cdot,t)$ belong to $H^2(D_m)$ and satisfy the following estimates:*

$$\|\tilde{v}_\rho(\cdot,t)\|_{H^1(D_m)} + \|\tilde{v}_\phi(\cdot,t)\|_{H^1(D_m)} \le C\|\widetilde{\Omega}(\cdot,t)\|_{L^2(D_m)}, \tag{3.18}$$

$$\|\tilde{v}_\rho(\cdot,t)\|_{H^2(D_m)} + \|\tilde{v}_\phi(\cdot,t)\|_{H^2(D_m)} \le C\|\widetilde{\Omega}(\cdot,t)\|_{H^1(D_m)}, \tag{3.19}$$

where $C = C(\alpha, m)$.

Proof of Claim B. Firstly, since $\widetilde{\Omega} \in E_{m,T}$, we can find a set $S_T \subseteq [0,T]$ such that $[0,T] \setminus S_T$ has measure 0 and for any $t \in S_T$, $\widetilde{\Omega}(\cdot,t) \in H^1(D_m)$. Fix any $t \in S_T$, the functions on the right-hand side of (3.20) and (3.21) are in $L^2(D_m)$. Noting the signs of the potential terms in (3.15) and (3.16) are not helpful when proving the existence and uniqueness of the solutions, so we introduce

$$\tilde{f}(\cdot) := \rho\tilde{v}_\rho(\cdot,t) \quad \text{and} \quad \tilde{g}(\cdot) := \rho\tilde{v}_\phi(\cdot,t)$$

which are determined by the following problems:

$$\begin{cases} \Delta\tilde{f} = -\frac{\rho}{\sin\phi}\,\partial_\phi(\sin^2\phi\,\widetilde{\Omega}), & \text{in } D_m; \\ \partial_\phi\tilde{f} = 0 \ \text{ on } \partial^R D_m, & \tilde{f} = 0 \ \text{ on } \partial^A D_m. \end{cases} \tag{3.20}$$

$$\begin{cases} \left(\Delta - \frac{1+\cot^2\phi}{\rho^2}\right)\tilde{g} = \frac{1}{\rho^2}\,\partial_\rho(\rho^4\sin\phi\,\widetilde{\Omega}), & \text{in } D_m; \\ \tilde{g} = 0 \ \text{ on } \partial^R D_m, & \partial_\rho\tilde{g} = 0 \ \text{ on } \partial^A D_m. \end{cases} \tag{3.21}$$

Now the potential term in (3.15) disappears and the potential term in (3.21) has the good sign, so the existence and uniqueness of the solutions of (3.20) and (3.21) in the space $H^1(D_m)$ can be established using classical methods, e.g. the Lax-Milgram theory. Next, we will show both $\tilde{f}$ and $\tilde{g}$ belong to the stronger space $H^2(D_m)$. Analogous to the proof of Lemma 2.10, we view (3.20) and (3.21) as 2D elliptic problems on the rectangular domain D'_m in the ρ-ϕ plane, see Fig. 4. Then the problems become

$$\begin{cases} \left(\partial_\rho^2 + \frac{1}{\rho^2}\partial_\phi^2 + \frac{2}{\rho}\partial_\rho + \frac{\cot\phi}{\rho^2}\partial_\phi\right)\tilde{f} = -\frac{\rho}{\sin\phi}\,\partial_\phi(\sin^2\phi\,\widetilde{\Omega}), & \text{in } D'_m; \\ \partial_\phi\tilde{f} = 0 \ \text{ on } \partial^R D'_m, & \tilde{f} = 0 \ \text{ on } \partial^A D'_m. \end{cases} \tag{3.22}$$

$$\begin{cases} \left(\partial_\rho^2 + \frac{1}{\rho^2}\partial_\phi^2 + \frac{2}{\rho}\partial_\rho + \frac{\cot\phi}{\rho^2}\partial_\phi - \frac{1+\cot^2\phi}{\rho^2}\right)\tilde{g} = \frac{1}{\rho^2}\,\partial_\rho(\rho^4\sin\phi\,\widetilde{\Omega}), & \text{in } D'_m; \\ \tilde{g} = 0 \ \text{ on } \partial^R D'_m, & \partial_\rho\tilde{g} = 0 \ \text{ on } \partial^A D'_m. \end{cases} \tag{3.23}$$

Since $\widetilde{\Omega}(\cdot,t) \in L^2(D'_m)$ and $\frac{1}{m} \le \rho \le 1$, we can use the standard interior regularity theory to estimate the $H^1(D'_m)$ (resp. $H^2(D'_m)$) norms of $\tilde{f}$ and $\tilde{g}$ in terms of the $L^2(D'_m)$ (resp. $H^1(D'_m)$) norms of $\widetilde{\Omega}(\cdot,t)$. In addition, since D'_m is a rectangle in ρ-ϕ plane and the boundary conditions of $\tilde{f}$ and $\tilde{g}$ are of mixed Dirichlet-Neumann type, we can apply appropriate reflection near the boundary of D'_m (two reflections are needed near any corner) to reduce the boundary regularity estimates into interior regularity estimates. Thus, we know both $\tilde{f}$ and $\tilde{g}$ belong to $H^2(D_m)$ and

$$\|\tilde{f}\|_{H^1(D_m)} + \|\tilde{g}\|_{H^1(D_m)} \le C\|\widetilde{\Omega}(\cdot,t)\|_{L^2(D_m)}, \tag{3.24}$$

$$\|\tilde{f}\|_{H^2(D_m)} + \|\tilde{g}\|_{H^2(D_m)} \le C\|\widetilde{\Omega}(\cdot,t)\|_{H^1(D_m)}. \tag{3.25}$$

Now changing back to $\tilde{v}_\rho(\cdot,t)$ and $\tilde{v}_\phi(\cdot,t)$ from $\tilde{f}$ and $\tilde{g}$, we conclude that both $\tilde{v}_\rho(\cdot,t)$ and $\tilde{v}_\phi(\cdot,t)$ belong to $H^2(D_m)$ and they satisfy the estimates (3.18) and (3.19). Hence, Claim B is justified.

For any $t \in S_T$ and for the function $\tilde{f}$ defined in the above proof, if we apply the Moser iteration on (3.20), then one can find

$$\|\tilde{f}\|_{L^\infty(D_m)} \le C\big(1 + \|\widetilde{\Omega}(\cdot,t)\|_{L^6(D_m)}\big)\big(1 + \|\tilde{f}\|_{L^6(D_m)}\big). \tag{3.26}$$

By Sobolev inequality, $\|\tilde{f}\|_{L^6(D_m)} \le C\|\tilde{f}\|_{H^1(D_m)}$. Then we combine the estimates (3.26) with (3.24) to obtain

$$\begin{aligned}\|\tilde{f}\|_{L^\infty(D_m)} &\le C\big(1 + \|\widetilde{\Omega}(\cdot,t)\|_{L^6(D_m)}\big)\big(1 + \|\widetilde{\Omega}(\cdot,t)\|_{L^2(D_m)}\big)\\ &\le C\big(1 + \|\widetilde{\Omega}(\cdot,t)\|_{L^6(D_m)}\big)^2.\end{aligned}$$

By similar argument, the above inequality also holds if the function $\tilde{f}$ is replaced by $\tilde{g}$. Therefore,

$$\|\tilde{v}_\rho(\cdot,t)\|_{L^\infty(D_m)} + \|\tilde{v}_\phi(\cdot,t)\|_{L^\infty(D_m)} \le C\big(1 + \|\widetilde{\Omega}(\cdot,t)\|_{L^6(D_m)}\big)^2. \tag{3.27}$$

Recalling the estimates (3.10) and (3.13) for $\widetilde{\Omega}$, we know $\widetilde{\Omega} \in E_{m,T} \cap L_t^\infty L_x^6(D_m \times [0,T])$ and

$$\|\widetilde{\Omega}\|_{E_{m,T}} + \|\widetilde{\Omega}\|_{L_t^\infty L_x^6(D_m\times[0,T])} \le CA_0^2 e^{CA_0^4 T}.$$

Consequently, we deduce from (3.18), (3.19) and (3.27) that

$$\tilde{v}_\rho, \tilde{v}_\phi \in L_t^\infty H_x^1 \cap L_t^2 H_x^2 \cap L_{tx}^\infty\big(D_m \times [0,T]\big) \tag{3.28}$$

and

$$\|\tilde{v}_\rho\|_{L_t^\infty H_x^1(D_m\times[0,T])} + \|\tilde{v}_\rho\|_{L_t^2 H_x^2(D_m\times[0,T])} + \|\tilde{v}_\rho\|_{L_{tx}^\infty(D_m\times[0,T])} \le CA_0^4 e^{CA_0^4 T}, \tag{3.29}$$

$$\|\tilde{v}_\phi\|_{L_t^\infty H_x^1(D_m\times[0,T])} + \|\tilde{v}_\phi\|_{L_t^2 H_x^2(D_m\times[0,T])} + \|\tilde{v}_\phi\|_{L_{tx}^\infty(D_m\times[0,T])} \le CA_0^4 e^{CA_0^4 T}. \tag{3.30}$$

Define

$$\tilde{b} = \tilde{v}_\rho e_\rho + \tilde{v}_\phi e_\phi. \tag{3.31}$$

Then the above steps determine the map $\mathbb{L}$ (3.4) from b to $\tilde{b}$. Next, we will prove $\tilde{b} \in E^{\sigma}_{m,T}$. Due to the regularity property (3.28) and the boundary conditions in (3.15) and (3.16) for $\tilde{v}_\rho$ and $\tilde{v}_\phi$, it remains to show $\operatorname{div}\tilde{b} = 0$ for a.e. $t \in [0,T]$. Instead of showing $\operatorname{div}\tilde{b} = 0$ directly, we will take advantage of the fact that $\rho^2\operatorname{div}\tilde{b}$ satisfies a simple equation (3.32) with a good boundary condition, which allows us to conclude $\rho^2\operatorname{div}\tilde{b} = 0$ for a.e. $t \in [0,T]$.

In fact, we fix any $t \in S_T$, where S_T is the set defined in the proof of Claim B, and then define

$$h(\cdot) = \rho^2\operatorname{div}\tilde{b}(\cdot,t), \quad \text{on} \quad D_m.$$

By direct calculation, it follows from the equations (3.15) and (3.16) for $\tilde{v}_\rho$ and $\tilde{v}_\phi$ that

$$\begin{cases} \Delta h = 0, & \text{in} \quad D_m, \\ \partial_n h = 0, & \text{on} \quad \partial D_m. \end{cases} \tag{3.32}$$

Testing (3.32) with h, we have

$$\|\nabla h\|_{L^2(D_m)} = 0,$$

which implies $h \equiv C$ is a constant on D_m. Next, we will prove this constant C must be 0. Based on the divergence formula (A.3) in spherical coordinates,

$$h(\cdot) = \rho^2\operatorname{div}\tilde{b}(\cdot,t) = \partial_\rho\big(\rho^2\tilde{v}_\rho(\cdot,t)\big) + \frac{1}{\sin\phi}\,\partial_\phi\big(\rho\sin\phi\,\tilde{v}_\phi(\cdot,t)\big). \tag{3.33}$$

Denote $\phi_1 = \frac{\pi}{2} - \alpha$ and $\phi_2 = \frac{\pi}{2} + \alpha$. For any $\rho \in \left[\frac{1}{m}, 1\right]$, we multiply (3.33) by $\sin\phi$ and then integrate both sides with respect to ϕ from ϕ_1 to ϕ_2. Then due to the fact that $\tilde{v}_\phi = 0$ on $\partial^R D_m$, we know the second term on the right-hand side disappears. Thus, we obtain

$$\int_{\phi_1}^{\phi_2} h(\rho,\phi)\sin\phi\,d\phi = \partial_\rho\Bigg(\rho\int_{\phi_1}^{\phi_2} \rho\tilde{v}_\rho(\rho,\phi,t)\sin\phi\,d\phi\Bigg).$$

Define

$$H(\rho) = \rho\int_{\phi_1}^{\phi_2} \tilde{v}_\rho(\rho,\phi,t)\sin\phi\,d\phi, \quad \forall\,\rho \in [1/m, 1].$$

In order to show h is identically 0, it suffices to prove

$$H(\rho) = 0, \quad \forall\,\rho \in [1/m, 1]. \tag{3.34}$$

Denote $\tilde{f}(\cdot) = \rho\tilde{v}_\rho(\cdot,t)$ on D_m as we did in the proof of Claim B, then $H(\rho) = \int_{\phi_1}^{\phi_2} \tilde{f}(\rho,\phi)\sin\phi\, d\phi$. Meanwhile, it follows from (3.22) that $\tilde{f}$ can be regarded as a solution of the following equation on the 2D domain D'_m.

$$\begin{cases} \left(\partial_\rho^2 + \frac{1}{\rho^2}\partial_\phi^2 + \frac{2}{\rho}\partial_\rho + \frac{\cot\phi}{\rho^2}\partial_\phi\right)\tilde{f} = -\frac{\rho}{\sin\phi}\,\partial_\phi(\sin^2\phi\,\widetilde{\Omega}), \quad \text{in} \quad D'_m; \\ \partial_\phi \tilde{f} = 0 \quad \text{on} \quad \partial^R D'_m, \qquad \tilde{f} = 0 \quad \text{on} \quad \partial^A D'_m. \end{cases} \tag{3.35}$$

For any $\rho \in \left(\frac{1}{m}, 1\right)$, by taking advantage of the boundary conditions $\partial_\phi \tilde{f} = \widetilde{\Omega} = 0$ on $\partial^R D'_m$ and the relation

$$(\partial_\phi^2 \tilde{f})\sin\phi + \cos\phi\,\partial_\phi\tilde{f} = \partial_\phi\big(\sin\phi\,\partial_\phi\tilde{f}\big),$$

we can multiply (3.35) by $\sin\phi$ and then integrate both sides with respect to ϕ from ϕ_1 to ϕ_2 to obtain

$$H''(\rho) + \frac{2}{\rho}H'(\rho) = 0, \quad \forall\,\rho \in (1/m, 1). \tag{3.36}$$

In addition, we have $H(\rho_1) = H(\rho_2) = 0$ since $\tilde{v}_\rho = 0$ on $\partial^A D'_m$. By solving (3.36) with the Dirichlet boundary condition, we conclude $H \equiv 0$ on $[\rho_1, \rho_2]$. As a result, $\nabla\cdot\tilde{b} = h/\rho^2 = 0$, completing this step. Meanwhile, thanks to (3.34), we also obtain the following byproduct:

$$\int_{\phi_1}^{\phi_2} \tilde{v}_\rho(\rho,\phi,t)\sin\phi\, d\phi = 0, \quad \forall\,\rho \in [1/m, 1], \quad t > 0. \tag{3.37}$$

Step 4. We prove $\mathbb{L}$ is a contraction map from $\overline{B_{E^\sigma_{m,T}}(0,M)} \bigcap \operatorname{span}\{e_\rho, e_\phi\}$ into itself for some large M and small T.

For any $b \in E^\sigma_{m,T} \cap \operatorname{span}\{e_\rho, e_\phi\}$, denote $\tilde{b} = \mathbb{L}b$. We point out that although the initial value of b is not required to be $v_{0,\rho}e_\rho + v_{0,\phi}e_\phi$, where v_0 is the given initial velocity in Proposition 3.1, the initial value of $\tilde{b}$ is guaranteed to be $v_{0,\rho}e_\rho + v_{0,\phi}e_\phi$ according to the construction of $\mathbb{L}$ (see (3.17)). In addition, based on (3.5), the constructed v_θ is also ensured to have the initial value $v_{0,\theta}$, where v_0 is again the given initial velocity. As a result, when $T \le 1$, it follows from the estimates (3.29) and (3.30) that

$$\|\tilde{b}\|_{E^\sigma_{m,T}} \le CA_0^4 e^{CA_0^4},$$

where A_0 is as defined in (3.3). Now we denote M to be the above upper bound:

$$M := CA_0^4 e^{CA_0^4}. \tag{3.38}$$

Then $\mathbb{L}$ maps $\overline{B_{E^\sigma_{m,T}}(0,M)} \bigcap \operatorname{span}\{e_\rho, e_\phi\}$ into itself. We fix such an M and then we will prove $\mathbb{L}$ is a contraction map if T is sufficiently small.

For $i = 1, 2$, let $b^{(i)} = v_\rho^{(i)} e_\rho + v_\phi^{(i)} e_\phi \in E^\sigma_{m,T}$, and denote $\Gamma^{(i)}$, $v_\theta^{(i)}$, $\widetilde{\Omega}^{(i)}$, $\tilde{v}_\rho^{(i)}$, $\tilde{v}_\phi^{(i)}$ and $\tilde{b}^{(i)}$ to be the functions constructed as in the previous steps 1-3. According to the equations (3.15) and (3.16) with $\tilde{v}_\rho$, $\tilde{v}_\phi$ and $\widetilde{\Omega}$ being replaced by $\tilde{v}_\rho^{(i)}$, $\tilde{v}_\phi^{(i)}$ and $\widetilde{\Omega}^{(i)}$ for $i = 1, 2$ respectively, we have

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)\left(\tilde{v}_\rho^{(2)} - \tilde{v}_\rho^{(1)}\right) = -\frac{1}{\sin\phi}\,\partial_\phi\left[\sin^2\phi\left(\widetilde{\Omega}^{(2)} - \widetilde{\Omega}^{(1)}\right)\right], \quad \text{in} \quad D_m; \\ \partial_\phi\left(\tilde{v}_\rho^{(2)} - \tilde{v}_\rho^{(1)}\right) = 0 \quad \text{on} \quad \partial^R D_m, \quad \left(\tilde{v}_\rho^{(2)} - \tilde{v}_\rho^{(1)}\right) = 0 \quad \text{on} \quad \partial^A D_m. \end{cases}$$

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\right)\left(\tilde{v}_\phi^{(2)} - \tilde{v}_\phi^{(1)}\right) = \frac{1}{\rho^3}\,\partial_\rho\left[\rho^4\sin\phi\left(\widetilde{\Omega}^{(2)} - \widetilde{\Omega}^{(1)}\right)\right], \quad \text{in} \quad D_m; \\ \left(\tilde{v}_\phi^{(2)} - \tilde{v}_\phi^{(1)}\right) = 0 \quad \text{on} \quad \partial^R D_m, \quad \partial_\rho\left(\tilde{v}_\phi^{(2)} - \tilde{v}_\phi^{(1)}\right) = -\frac{1}{\rho}\left(\tilde{v}_\phi^{(2)} - \tilde{v}_\phi^{(1)}\right) \quad \text{on} \quad \partial^A D_m. \end{cases}$$

Then similar to the derivation of (3.18), we find

$$\|\tilde{b}^{(2)} - \tilde{b}^{(1)}\|_{E_{m,T}} \leq C\|\widetilde{\Omega}^{(2)} - \widetilde{\Omega}^{(1)}\|_{L_t^\infty L_x^2(D_m\times[0,T])}. \tag{3.39}$$

Denote $f = \widetilde{\Omega}^{(2)} - \widetilde{\Omega}^{(1)}$. Then based on the equations in (3.9) with $\widetilde{\Omega}$ and b being replaced by $\widetilde{\Omega}^{(i)}$ and $b^{(i)}$ for $i = 1, 2$, we know f is a weak solution to the following problem:

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2\cot\phi}{\rho^2}\partial_\phi\right) f - b^{(2)}\cdot\nabla f - (b^{(2)} - b^{(1)})\cdot\nabla\widetilde{\Omega}^{(1)} - \partial_t f \\ \quad = \frac{1}{\rho^2\sin\phi}\left(\frac{1}{\rho}\partial_\phi\left[\left(v_\theta^{(2)}\right)^2 - \left(v_\theta^{(1)}\right)^2\right] - \cot\phi\,\partial_\rho\left[\left(v_\theta^{(2)}\right)^2 - \left(v_\theta^{(1)}\right)^2\right]\right), \quad \text{in} \quad D_m\times(0,T]; \\ f = 0, \quad \text{on} \quad \partial D_m\times(0,T]; \\ f(x,0) = 0, \quad x\in D_m. \end{cases} \tag{3.40}$$

Testing (3.40) with f and using integration by parts, we have

$$\int_0^T\int_{D_m} |\nabla f|^2\,dx\,dt + \frac{1}{2}\int_{D_m} |f(x,T)|^2\,dx = I_1 + I_2, \tag{3.41}$$

where

$$I_1 = -\int_0^T\int_{D_m} \left[(b^{(2)} - b^{(1)})\cdot\nabla\widetilde{\Omega}^{(1)}\right] f\,dx\,dt,$$

$$I_2 = \int_0^T\int_{D_m} \frac{1}{\rho^2\sin\phi}\left[\left(v_\theta^{(2)}\right)^2 - \left(v_\theta^{(1)}\right)^2\right]\left(\frac{1}{\rho}\,\partial_\phi f - \cot\phi\,\partial_\rho f\right) dx\,dt.$$

Applying the integration by parts and the Hölder's inequality, we know

$$
\begin{aligned}
I_1 &\leq \|\widetilde{\Omega}^{(1)}\|_{L^5_{tx}(D_m\times[0,T])}\|b^{(2)}-b^{(1)}\|_{L^{10/3}_{tx}(D_m\times[0,T])}\|\nabla f\|_{L^2_{tx}(D_m\times[0,T])} \\
&\leq T^{1/5}\|\widetilde{\Omega}^{(1)}\|_{L^\infty_t L^5_x(D_m\times[0,T])}\|b^{(2)}-b^{(1)}\|_{E_{m,T}}\|\nabla f\|_{L^2_{tx}(D_m\times[0,T])}.
\end{aligned}
$$

Then it follows from the Cauchy-Schwarz inequality and the estimate (3.13) with $q=5$ that

$$
I_1 \leq \frac{1}{4}\|\nabla f\|^2_{L^2_{tx}(D_m\times[0,T])} + CA_0^2 e^{CA_0^4 T}T^{2/5}\|b^{(2)}-b^{(1)}\|^2_{E_{m,T}}. \tag{3.42}
$$

Next, we estimate I_2. By Hölder's inequality, we find

$$
I_2 \leq C\|\nabla f\|_{L^2_{tx}(D_m\times[0,T])}\big\|v_\theta^{(2)}+v_\theta^{(1)}\big\|_{L^\infty_{tx}(D_m\times[0,T])}\big\|v_\theta^{(2)}-v_\theta^{(1)}\big\|_{L^2_{tx}(D_m\times[0,T])}.
$$

By Cauchy-Schwarz inequality and the bound (3.7), we have

$$
I_2 \leq \frac{1}{4}\|\nabla f\|^2_{L^2_{tx}(D_m\times[0,T])} + CA_0^2\big\|v_\theta^{(2)}-v_\theta^{(1)}\big\|^2_{L^2_{tx}(D_m\times[0,T])}. \tag{3.43}
$$

Plugging (3.42) and (3.43) into (3.41) leads to

$$
\|f\|^2_{E_{m,T}} \leq CA_0^2 e^{CA_0^4 T}T^{2/5}\|b^{(2)}-b^{(1)}\|^2_{E_{m,T}} + CA_0^2\big\|v_\theta^{(2)}-v_\theta^{(1)}\big\|^2_{L^2_{tx}(D_m\times[0,T])}. \tag{3.44}
$$

Combining (3.39) with (3.44) yields

$$
\|\tilde{b}^{(2)}-\tilde{b}^{(1)}\|_{E_{m,T}} \leq CA_0^2 e^{CA_0^4 T}T^{2/5}\|b^{(2)}-b^{(1)}\|^2_{E_{m,T}} + CA_0^2\big\|v_\theta^{(2)}-v_\theta^{(1)}\big\|^2_{L^2_{tx}(D_m\times[0,T])}. \tag{3.45}
$$

So it remains to estimate $\|v_\theta^{(2)}-v_\theta^{(1)}\|_{L^2_{tx}(D_m\times[0,T])}$ or equivalently $\|\Gamma^{(2)}-\Gamma^{(1)}\|_{L^2_{tx}(D_m\times[0,T])}$. Denote $g=\Gamma^{(2)}-\Gamma^{(1)}$. Then according to (3.6), it holds that

$$
\begin{cases}
\Delta g - b^{(2)}\cdot\nabla g - \big(b^{(2)}-b^{(1)}\big)\cdot\nabla\Gamma^{(1)} - \frac{2}{\rho}\partial_\rho g - \frac{2\cot\phi}{\rho^2}\partial_\phi g - \partial_t g = 0, \quad \text{in} \quad D_m\times(0,T]; \\
\partial_n g = 0, \quad \text{on} \quad \partial D_m\times(0,T]; \\
g(x,0)=0, \quad x\in D_m.
\end{cases} \tag{3.46}
$$

Testing (3.46) by g, then we have

$$
\int_0^T\int_{D_m}|\nabla g|^2\,dx\,dt + \frac{1}{2}\int_{D_m}|g(x,T)|^2\,dx = J_1 + J_2, \tag{3.47}
$$

where

$$J_1 = -\int_0^T \int_{D_m} \left[(b^{(2)} - b^{(1)}) \cdot \nabla \Gamma^{(1)} \right] g \, dx \, dt,$$

$$J_2 = -\int_0^T \int_{D_m} \left(\frac{2}{\rho} \partial_\rho g + \frac{2 \cot \phi}{\rho^2} \partial_\phi g \right) g \, dx \, dt.$$

We first estimate J_1. Applying the integration by parts and the Hölder's inequality yields

$$\begin{aligned} J_1 &\leq \|\Gamma^{(1)}\|_{L^\infty_{tx}(D_m \times [0,T])} \|b^{(2)} - b^{(1)}\|_{L^2_{tx}(D_m \times [0,T])} \|\nabla g\|_{L^2_{tx}(D_m \times [0,T])} \\ &\leq T^{\frac{1}{2}} \|b^{(2)} - b^{(1)}\|_{E_{m,T}} \|\Gamma^{(1)}\|_{L^\infty_{tx}(D_m \times [0,T])} \|\nabla g\|_{L^2_{tx}(D_m \times [0,T])}. \end{aligned}$$

It then follows from the estimate (3.7) and the Cauchy-Schwarz inequality that

$$J_1 \leq \frac{1}{4} \|\nabla g\|^2_{L^2_{tx}(D_m \times [0,T])} + C A_0^2 T \|b^{(2)} - b^{(1)}\|^2_{E_{m,T}}. \tag{3.48}$$

Next, we estimate J_2 by Cauchy-Schwarz inequality to get

$$J_2 \leq \frac{1}{4} \|\nabla g\|^2_{L^2_{tx}(D_m \times [0,T])} + C \|g\|^2_{L^2_{tx}(D_m \times [0,T])}. \tag{3.49}$$

Plugging (3.48) and (3.49) into (3.47) leads to

$$\frac{1}{2} \int_0^T \int_{D_m} |\nabla g|^2 \, dx \, dt + \frac{1}{2} \int_{D_m} |g(x,T)|^2 \, dx \leq C A_0^2 T \|b^{(2)} - b^{(1)}\|^2_{E_{m,T}} + C \|g\|^2_{L^2_{tx}(D_m \times [0,T])}.$$

Now by Gronwall's inequality, we obtain

$$\|g\|^2_{E_{m,T}} \leq C A_0^2 T e^{CT} \|b^{(2)} - b^{(1)}\|^2_{E_{m,T}},$$

which implies

$$\|v_\theta^{(2)} - v_\theta^{(1)}\|^2_{L^2_{tx}(D_m \times [0,T])} \leq T \|v_\theta^{(2)} - v_\theta^{(1)}\|^2_{E_{m,T}} \leq C A_0^2 T^2 e^{CT} \|b^{(2)} - b^{(1)}\|^2_{E_{m,T}}. \tag{3.50}$$

Finally, substituting (3.50) into (3.45) leads to

$$\|\tilde{b}^{(2)} - \tilde{b}^{(1)}\|_{E_{m,T}} \leq C \big(A_0^2 e^{C A_0^4 T} T^{2/5} + A_0^4 e^{CT} T^2 \big) \|b^{(2)} - b^{(1)}\|^2_{E_{m,T}}.$$

Now by choosing

$$T \leq e^{-C A_0^4}, \tag{3.51}$$

where C is some large constant that only depends on α and m, we obtain

$$\|\tilde{b}^{(2)} - \tilde{b}^{(1)}\|_{E_{m,T}} \leq \frac{1}{2}\|b^{(2)} - b^{(1)}\|^2_{E_{m,T}}.$$

Hence, for any M and T that satisfies (3.38) and (3.51), $\mathbb{L}$ is a contraction map. Thanks to the contraction mapping theorem, $\mathbb{L}$ has a fixed point b that lies in $\overline{B_{E^\sigma_{m,T}}(0,M)} \bigcap \mathrm{span}\{e_\rho, e_\phi\}$. In addition, by taking advantage of the fact that $b = \tilde{b}$ and (3.28),

$$b \in L^\infty_t H^1_x \cap L^2_t H^2_x \cap L^\infty_{tx}\big(D_m \times [0,T]\big).$$

Step 5. Existence of a strong solution v such that

$$v \in E^\sigma_{m,T} \cap H^1_t L^2_x \cap L^2_t H^2_x \cap L^\infty_{tx}\big(D_m \times [0,T]\big), \quad (\nabla \times v)_\theta \in L^\infty_t L^6_x\big(D_m \times [0,T]\big).$$

Based on the fixed point b defined in the previous step, we define $v = b + v_\theta e_\theta$, where v_θ is the function constructed in Step 1 based on b. We will first show $v_\theta \in L^2_t H^2_x\big(D_m \times [0,T]\big)$. Recall the equation for Γ (3.6):

$$\begin{cases} \Delta\Gamma - b\cdot\nabla\Gamma - \frac{2}{\rho}\partial_\rho\Gamma - \frac{2\cot\phi}{\rho^2}\partial_\phi\Gamma - \partial_t\Gamma = 0, \quad \text{in} \quad D_m \times (0,T]; \\ \partial_n\Gamma = 0, \quad \text{on} \quad \partial D_m \times (0,T]; \\ \Gamma(x,0) = \Gamma_0(x), \quad x \in D_m. \end{cases}$$

Now the function b is in $L^\infty(D_m \times [0,T])$, so it follows from the standard theory that Γ is $L^2_t H^2_x$ in $D_{\mathrm{int}} \times [0,T]$, where D_{int} is any interior domain of D_m, i.e. $\overline{D_{\mathrm{int}}} \subset D_m$. Moreover, by the reflection argument as that in Step 4.1, we can show Γ is $L^2_t H^2_x$ on the whole region $D_m \times [0,T]$. As a result, $v \in E^\sigma_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}\big(D_m \times [0,T]\big)$.

Define $\omega = \nabla \times v$ and write $\omega = \omega_\rho e_\rho + \omega_\phi e_\phi + \omega_\theta e_\theta$. Then $\omega \in L^2_t H^1_x(D_m \times [0,T])$. Let $\widetilde{\omega}_\theta$ be given by (3.14). We remark that although $\tilde{b}$ is constructed from $\widetilde{\omega}_\theta$ according to the Biot-Savart law $\Delta\tilde{b} = -\nabla \times \widetilde{\omega}_\theta$ (also see (3.15) and (3.16)), it is not obvious that $\nabla \times \tilde{b} = \widetilde{\omega}_\theta$. As a result, although $b = \tilde{b}$, it is not readily seen that $\nabla \times b = \widetilde{\omega}_\theta$. Next, we will carry out a detailed argument to show that ω_θ indeed coincides with $\widetilde{\omega}_\theta$ so that ω_θ also satisfies (3.14). Firstly, since $\omega = \nabla \times v$, then it follows from (2.12) that $\omega_\theta e_\theta = \nabla \times b$, where $b = v_\rho e_\rho + v_\phi e_\phi$. Thus,

$$\Delta b = -\nabla \times (\omega_\theta e_\theta). \tag{3.52}$$

On the other hand, since b is divergence free, we can use formula (A.12) to find

$$\Delta b = \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\right)v_\rho e_\rho + \left[\left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho\right]e_\phi. \tag{3.53}$$

Recall that b is the fixed point of the mapping $\mathbb{L}$, v_ρ and v_ϕ are given by (3.15) and (3.16) respectively. As a result,

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)v_\rho = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\widetilde{\omega}_\theta), \\ \Delta v_\phi = -\frac{2}{\rho}\partial_\rho v_\phi - \frac{1-\cot^2\phi}{\rho^2}v_\phi + \frac{1}{\rho^3}\,\partial_\rho(\rho^3\widetilde{\omega}_\theta). \end{cases} \tag{3.54}$$

Meanwhile, it follows from (2.12) that $\omega_\theta = \frac{1}{\rho}\,\partial_\rho(\rho v_\phi) - \frac{1}{\rho}\,\partial_\phi v_\rho$, which implies

$$\partial_\phi v_\rho = \partial_\rho(\rho v_\phi) - \rho\omega_\theta. \tag{3.55}$$

Putting (3.54) and (3.55) into (3.53) yields

$$\Delta b = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\widetilde{\omega}_\theta)e_\rho + \left(\frac{1}{\rho^3}\,\partial_\rho(\rho^3\widetilde{\omega}_\theta) - \frac{2}{\rho}\omega_\theta\right)e_\phi.$$

Applying formula (2.12) again (replacing v by $\widetilde{\omega}_\theta e_\theta$), we find

$$\nabla\times(\widetilde{\omega}_\theta e_\theta) = \frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\widetilde{\omega}_\theta)e_\rho - \frac{1}{\rho}\partial_\rho(\rho\widetilde{\omega}_\theta)e_\phi.$$

Combining the above two relations, we know

$$\Delta b = -\nabla\times(\widetilde{\omega}_\theta e_\theta) + \frac{2}{\rho}(\widetilde{\omega}_\theta - \omega_\theta)e_\phi.$$

Since we have already derived in (3.52) that $\Delta b = -\nabla\times(\omega_\theta e_\theta)$, the above equation implies

$$\nabla\times(ue_\theta) - \frac{2}{\rho}ue_\phi = 0, \tag{3.56}$$

where $u := \widetilde{\omega}_\theta - \omega_\theta$. By computing $\nabla\times(ue_\theta)$ based on formula (2.12) (replacing v by ue_θ), it follows from (3.56) that

$$\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,u)e_\rho - \frac{1}{\rho^3}\partial_\rho(\rho^3u)e_\phi = 0.$$

So $\partial_\phi(\sin\phi\,u) = \partial_\rho(\rho^3u) = 0$. Define $\tilde{u} = \rho^3\sin\phi\,u$. Then

$$\partial_\phi\tilde{u} = \partial_\rho\tilde{u} = 0 \quad \text{in} \quad D_m\times(0,T]. \tag{3.57}$$

On the boundary ∂D_m, $\widetilde{\omega}_\theta = 0$ by the construction (3.14). Meanwhile, since $\omega_\theta = \frac{1}{\rho}\,\partial_\rho(\rho v_\phi) - \frac{1}{\rho}\,\partial_\phi v_\rho$, it follows from the constructions of v_ρ and v_ϕ in (3.15) and (3.16) that $\omega_\theta = 0$ on ∂D_m. Hence,

$$\tilde{u} = 0 \quad \text{on} \quad \partial D_m\times(0,T]. \tag{3.58}$$

Since $\tilde{u} \in L^2_t H^1_x(D_m \times [0,T])$, we deduce from (3.57) and (3.58) that $\tilde{u} = 0$ in D_m for a.e. $t \in (0,T]$. This implies that $\widetilde{\omega}_\theta = \omega_\theta$ in D_m for a.e. $t \in (0,T]$. Now the interior regularity of $\widetilde{\omega}_\theta$ and ω_θ indicates that $\widetilde{\omega}_\theta = \omega_\theta$ in $D_m \times (0,T]$. For the initial data, it again follows from the constructions of $\widetilde{\omega}_\theta$, v_ρ and v_ϕ that $\widetilde{\omega}_\theta(\cdot,0) = \omega_{0,\theta}(\cdot) = \omega_\theta(\cdot,0)$. Thus,

$$\widetilde{\omega}_\theta = \omega_\theta \quad \text{in} \quad D_m \times [0,T].$$

In particular, ω_θ also satisfies (3.14):

$$\begin{cases} (\Delta - \frac{1}{\rho^2 \sin^2\phi})\omega_\theta - b\cdot\nabla\omega_\theta + \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta - \partial_t\omega_\theta \\ \qquad\qquad\qquad\qquad = \frac{1}{\rho^2}\partial_\phi(v_\theta^2) - \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2), \quad \text{in} \quad D_m \times (0,T]; \\ \omega_\theta = 0, \quad \text{on} \quad \partial D_m \times (0,T]; \\ \omega_\theta(x,0) = \omega_{0,\theta}(x), \quad \text{in} \quad D_m. \end{cases} \tag{3.59}$$

Meanwhile, it follows from (3.13) that $\omega_\theta \in L^\infty_t L^6_x(D_m \times [0,T])$ and

$$\|\omega_\theta\|_{L^\infty_t L^6_x(D_m\times[0,T])} \le Ce^{CA_0^4 T} A_0. \tag{3.60}$$

Finally, we will take advantage of (3.59) to find a pressure term P such that (v,P) satisfies (2.7) and the NHL boundary condition (2.18) pointwisely so that (v,P) is a strong solution. First, we recall a vector calculus identity (see equation (2.45) on page 429 in [42]) in the cylindrical coordinates:

$$\begin{aligned} &\nabla \times \left(\Delta b - (b\cdot\nabla)b + \frac{v_\theta^2}{r}e_r - \partial_t b\right) \\ &= \left[\left(\Delta - \frac{1}{r^2}\right)\omega_\theta - b\cdot\nabla\omega_\theta + 2\frac{v_\theta}{r}\partial_{x_3}v_\theta + \frac{v_r}{r}\omega_\theta - \partial_t\omega_\theta\right]e_\theta. \end{aligned} \tag{3.61}$$

Next, we will convert this identity in the form of spherical coordinates. Noticing

$$r = \rho\sin\phi, \quad e_r = \sin\phi\, e_\rho + \cos\phi\, e_\phi, \quad v_r = \sin\phi\, v_\rho + \cos\phi\, v_\phi$$

and

$$2v_\theta\partial_{x_3}v_\theta = \partial_{x_3}(v_\theta^2) = \left(\cos\phi\,\partial_\rho - \frac{\sin\phi}{\rho}\partial_\phi\right)(v_\theta^2),$$

so the identity (3.61) can be equivalently written as

$$\begin{aligned} &\nabla \times \left(\Delta b - (b\cdot\nabla)b + \frac{1}{\rho}v_\theta^2\, e_\rho + \frac{\cot\phi}{\rho}v_\theta^2\, e_\phi - \partial_t b\right) \\ &= \left[\left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)\omega_\theta - b\cdot\nabla\omega_\theta + \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2) - \frac{1}{\rho^2}\partial_\phi(v_\theta^2) \right. \\ &\qquad \left. + \frac{v_\rho + \cot\phi\, v_\phi}{\rho}\omega_\theta - \partial_t\omega_\theta\right]e_\theta. \end{aligned} \tag{3.62}$$

Define

$$B = \Delta b - (b \cdot \nabla) b + \frac{1}{\rho} v_\theta^2 \, e_\rho + \frac{\cot\phi}{\rho} v_\theta^2 \, e_\phi - \partial_t b.$$

Then it follows from (3.62), (3.59) and the interior regularity of v and ω_θ that

$$\nabla \times B = 0, \quad \text{pointwise in} \quad D_m \times (0, T]. \tag{3.63}$$

By direct computation, B can be written as $B = B_\rho e_\rho + B_\phi e_\phi$, where

$$\begin{cases} B_\rho = \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right) v_\rho - b \cdot \nabla v_\rho + \frac{1}{\rho}(v_\phi^2 + v_\theta^2) - \partial_t v_\rho, \\ B_\phi = \left(\Delta - \frac{1}{\rho^2 \sin^2\phi}\right) v_\phi - b \cdot \nabla v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho - \frac{1}{\rho} v_\rho v_\phi + \frac{\cot\phi}{\rho} v_\theta^2 - \partial_t v_\phi. \end{cases} \tag{3.64}$$

Next, we discuss the regularity of B. Firstly, since $v \in L_t^2 H_x^2 \cap L_{tx}^\infty\big(D_m \times [0,T]\big)$ and $\omega_\theta = \widetilde{\omega}_\theta \in L_t^2 H_x^1\big(D_m \times [0,T]\big)$, it then follows from (3.5) and (3.14) that $\partial_t v_\theta \in L_{tx}^2\big(D_m \times [0,T]\big)$ and $\partial_t \omega_\theta \in L_t^2 H_x^{-1}\big(D_m \times [0,T]\big)$. Now we take advantage of (3.15) and (3.16) to find that both $\partial_t v_\rho$ and $\partial_t v_\phi$ belong to $L_{tx}^2\big(D_m \times [0,T]\big)$. As a consequence, $v \in H_t^1 L_x^2\big(D_m \times [0,T]\big)$ and $B \in L_{tx}^2\big(D_m \times [0,T]\big)$.

Based on formula (2.12) and equation (3.63), we have

$$\partial_\rho(\rho B_\phi) - \partial_\phi B_\rho = \rho(\nabla \times B)_\theta = 0 \quad \text{pointwise in} \quad D_m \times (0,T].$$

Since the domain D_m can be regarded as a simply connected 2D domain D'_m, defined in (2.43), on the ρ-ϕ plane, by viewing both ρB_ϕ and B_ρ as functions in ρ and ϕ in the domain D'_m, we can apply Green's theorem to find a scalar function $P \in L_t^2 H_x^1\big(D_m \times [0,T]\big)$ such that

$$\partial_\phi P = \rho B_\phi, \quad \partial_\rho P = B_\rho, \quad \text{pointwise in} \quad D'_m \times (0,T].$$

This implies that

$$B_\rho = \partial_\rho P, \quad B_\phi = \frac{1}{\rho}\partial_\phi P, \quad \text{pointwise in} \quad D_m \times (0,T]. \tag{3.65}$$

Meanwhile, without loss of generality, we can assume the average of P in the space variable on D_m is 0 for any fixed time t, that is $\int_{D_m} P(x,t)\,dx = 0$ for any t. Then it follows from Poincaré inequality that $P \in L_t^2 H_x^1\big(D_m \times [0,T]\big)$. Substituting (3.65) into (3.64) and combining with equation (3.5) for v_θ, we conclude that (v, P) satisfies the NS system (2.7) in L_{tx}^2 sense on the space-time domain $D_m \times (0,T]$. In addition, from the construction (3.5) for v_θ, and (3.15) and (3.16) for v_ρ and v_ϕ, the initial condition and the NHL boundary condition (2.19) are also satisfied. Hence, (v, P) is a strong solution such that

$$v \in E^{\sigma}_{m,T} \cap H^1_t L^2_x \cap L^2_t H^2_x \cap L^{\infty}_{tx}\big(D_m \times [0,T]\big), \quad P \in L^2_t H^1_x\big(D_m \times [0,T]\big).$$

Step 6. Uniqueness of the strong solution and preservation of the even-odd-odd symmetry.

Suppose that $(\hat{v}, \hat{P})$ is another strong solution of (2.7) with the initial data v_0 and the NHL boundary condition (2.19). Define

$$\hat{b} = \hat{v}_\rho e_\rho + \hat{v}_\phi e_\phi.$$

Then $\hat{b} \in E^{\sigma}_{m,T} \bigcap \mathrm{span}\{e_\rho, e_\phi\}$ and $\hat{b}$ is also a fixed point of the map $\mathbb{L}$ defined in Step 3. As a result,

$$\mathbb{L}(b) - \mathbb{L}(\hat{b}) = b - \hat{b}. \tag{3.66}$$

On the other hand, due to the choice (3.51) of the time T in Step 4, the map $\mathbb{L}$ is contractive so that

$$\|\mathbb{L}(b) - \mathbb{L}(\hat{b})\|_{E_{m,T}} \le \frac{1}{2}\|b - \hat{b}\|_{E_{m,T}}. \tag{3.67}$$

The combination of (3.66) and (3.67) leads to $b = \hat{b}$ in $D_m \times [0,T]$. This further implies that $\hat{v}_\theta = v_\theta$ since both of them satisfy the equation (3.5) whose solution in the energy space $E_{m,T}$ is unique. Hence, $\hat{v} = v$ in $D_m \times [0,T]$ and the uniqueness is verified.

Now we assume the initial data v_0 enjoys the even-odd-odd symmetry as in Definition 1.3. Then we will prove the unique strong solution v as constructed above also has this property. Firstly, by the characterization (2.8), we know

$$v_{0,\rho}(\rho,\phi) = v_{0,\rho}(\rho,\pi-\phi), \quad v_{0,\phi}(\rho,\phi) = -v_{0,\phi}(\rho,\pi-\phi), \quad v_{0,\theta}(\rho,\phi) = -v_{0,\theta}(\rho,\pi-\phi).$$

Then we define a new vector field $\hat{v} = \hat{v}_\rho e_\rho + \hat{v}_\phi e_\phi + \hat{v}_\theta e_\theta$ and another pressure $\hat{P}$ as

$$\begin{cases} \hat{v}_\rho(\rho,\phi,t) = v_\rho(\rho,\pi-\phi,t), \quad \hat{v}_\phi(\rho,\phi,t) = -v_\phi(\rho,\pi-\phi,t), \\ \hat{v}_\theta(\rho,\phi,t) = -v_\theta(\rho,\pi-\phi,t), \quad \hat{P}(\rho,\phi,t) = P(\rho,\pi-\phi,t). \end{cases} \tag{3.68}$$

According to this definition, one can directly check that

(1) The initial value of $\hat{v}$ matches v_0;
(2) $(\hat{v}, \hat{P})$ satisfies the equations (2.7).
(3) $\hat{v}$ satisfies the NHL boundary condition (2.18).

So $\hat{v}$ is also a strong solution, which implies $\hat{v} = v$ on $D_m \times [0,T]$ due to the uniqueness of the strong solution that we just established. Based on the definition (3.68), we deduce from the fact $\hat{v} = v$ that v has the even-odd-odd symmetry on $D_m \times [0,T]$. $\square$

Next, we aim to extend the local solution in Proposition 3.1 with a small lifespan T to be a solution with arbitrarily large lifespan. In fact, according to the proof in Step 4, the existence time T in (3.51) only depends on α, m and A_0. Noticing A_0 in (3.3) is determined by $\|v_\theta(\cdot,0)\|_{L_x^\infty(D_m)}$ and $\|\omega_\theta(\cdot,0)\|_{L_x^6(D_m)}$, and we have uniform (in time) bounds (3.7) and (3.60) on $\|v_\theta(\cdot,t)\|_{L_x^\infty(D_m)}$ and $\|\omega_\theta(\cdot,t)\|_{L_x^6(D_m)}$. As a result, the solution v constructed in Step 4 on a small time interval $[0,T]$ can be extended to arbitrary finite time. Thus, we obtain the following result.

Corollary 3.3. *Let α, m and v_0 be the same as in Proposition 3.1. Then for any time $T>0$, the problem (2.7) on $D_m\times[0,T]$ with the initial data v_0 and the NHL boundary condition (2.18) has a strong solution (v,P) such that*

$$v\in E^\sigma_{m,T}\cap H_t^1L_x^2\cap L_t^2H_x^2\cap L_{tx}^\infty\big(D_m\times[0,T]\big),\quad P\in L_t^2H_x^1(D_m\times[0,T]).$$

Moreover, if $(\hat v,\hat P)$ is another strong solution, then $\hat v$ coincides with v on $D_m\times[0,T]$. As a result, if v_0 belongs to $E^{\sigma,s}_{m,T}$, then so does v.

Remark 3.4. Although a bounded strong solution is obtained in the above corollary for any finite time T and any fixed m, the L_{tx}^∞ bound on the velocity v is neither uniform in T nor uniform in m. In the next section, after introducing some new quantities involving the vorticity (see (2.14)), we will prove that the L_{tx}^∞ norm of v on $D_m\times[0,T]$ is uniformly bounded in T and this uniform bound only depends on m through $\|v_0\|_{C^2(D_m)}$, as long as some mild restrictions on the angle α and the size of Γ_0 are imposed.

4. Uniform bounds for $\|v\|_{L_{tx}^\infty}$ on $D_m\times[0,T]$

In this section, for any fixed $m\ge 2$ and $T>0$, we consider the initial data v_0 which lies in the admissible class $\mathscr{A}_m$ with the even-odd-odd symmetry (see Definitions 1.3 and 1.4). For such initial data, we denote by v the solution in Corollary 3.3 so that $v\in E^{\sigma,s}_{m,T}\cap H_t^1L_x^2\cap L_t^2H_x^2\cap L_{tx}^\infty(D_m\times[0,T])$. Moreover, by restricting the range of α within $\big(0,\frac{\pi}{6}\big]$ and by requiring $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)}\le\frac{1}{95}$, we will deduce a uniform bound, which is independent of T and dependent on m only through $\|v_0\|_{C^2(\overline{D_m})}$, for $\|v\|_{L_{tx}^\infty(D_m\times[0,T])}$. The plan of this section, which has been outlined in the introduction, is as follows:

- Step 1: We will derive an energy inequality about v in Section 4.1. This energy inequality provides a uniform bound on $\|v\|_{E_{m,T}}$.
- Step 2: In Sections 4.2–4.4, we will take advantage of the Biot-Savart law and the condition $\alpha\in\big(0,\frac{\pi}{6}\big]$ to control the $L^2(D_m)$ norms of $\nabla(v_\rho/\rho)(\cdot,t)$ and $\nabla(v_\phi/\rho)(\cdot,t)$ by $\|\Omega(\cdot,t)\|_{L^2(D_m)}$, and control the $L^2(D_m)$ norms of $\frac{1}{\rho}\nabla(v_\rho/\rho)(\cdot,t)$ and $\frac{1}{\rho}\nabla(v_\phi/\rho)(\cdot,t)$ by $\|\nabla\Omega(\cdot,t)\|_{L^2(D_m)}$.
- Step 3: Thanks to the smallness condition $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)}\le\frac{1}{95}$, the estimates in Step 1 will be used in Section 4.5 to obtain an upper bound, which is uniform in m and T, on $\|(K,F,\Omega)\|_{L_t^\infty L_x^2(D_m\times[0,T])}$.

- Step 4: According to the uniform bound on $\|(K,F,\Omega)\|_{L_t^\infty L_x^2(D_m\times[0,T])}$, we will derive in Section 4.6 a uniform bound on $\|v/\rho\|_{L_t^\infty L_x^6(D_m\times[0,T])}$.
- Step 5: Finally in Section 4.7, we will bound $\|v\|_{L_{tx}^\infty(D_m\times[0,T])}$ in terms of $\|v_0\|_{C^2(\overline{D_m})}$, $\|v\|_{E_{m,T}}$, $\|(K,F,\Omega)\|_{L_t^\infty L_x^2(D_m\times[0,T])}$ and $\|v/\rho\|_{L_t^\infty L_x^6(D_m\times[0,T])}$. Due to the uniform estimates in Steps 1, 3 and 4, the bound on $\|v\|_{L_{tx}^\infty(D_m\times[0,T])}$ will also be uniform in m and T.

4.1. An energy inequality

In this section, we present a result on bounding the L^2 norm of ∇v by the L^2 norm of its vorticity $\nabla\times v$. This result is well-known for incompressible vector fields v with zero boundary value (see e.g. Lemma 2 in [27]), however, it may not be true if the boundary value is nonzero. For example, if $v=\frac{1}{\rho\sin\phi}\,e_\theta$, then $\nabla\times v=0$ while $\nabla v\neq 0$. But we will show in Lemma 4.1 that such an estimate still holds in D_m if the vector field satisfies the NHL boundary condition and possesses the even-odd-odd symmetry as defined in Definition 1.3.

Lemma 4.1. *Let the region D_m be as defined in (2.17) with $m\geq 2$ and the angle $\alpha\in\left(0,\frac{\pi}{6}\right]$. Let $u\in H^2(D_m)$ be an incompressible vector field. Assume further that u satisfies the NHL boundary condition (2.18) and possesses the even-odd-odd symmetry. Then*

$$\|\nabla u\|_{L^2(D_m)}\leq\sqrt{3}\|\nabla\times u\|_{L^2(D_m)}. \tag{4.1}$$

Proof. Firstly, by similar computation as that in Section A.4, we know

$$\int\limits_{D_m} u\Delta u\,dx=-\int\limits_{D_m}|\nabla\times u|^2\,dx.$$

On the other hand, it directly follows from integration by parts that

$$\int\limits_{D_m} u\Delta u\,dx=\int\limits_{\partial D_m} u\frac{\partial u}{\partial n}\,dS-\int\limits_{D_m}|\nabla u|^2\,dx.$$

As a result,

$$\int\limits_{D_m}|\nabla u|^2\,dx=\int\limits_{D_m}|\nabla\times u|^2\,dx+\underbrace{\frac{1}{2}\int\limits_{\partial D_m}\frac{\partial|u|^2}{\partial n}\,dS}_{T_1}. \tag{4.2}$$

Now we give a detailed computation of T_1 on $\partial^R D_m$ and $\partial^A D_m$ separately. For the convenience of notation, we denote $\rho_0=\frac{1}{m}$. Noticing that the normal direction on $\partial^R D_m$

is parallel to the ϕ direction, then we can take advantage of the boundary conditions in (2.20) to see that $\frac{\partial(u_\rho^2)}{\partial n} = \frac{\partial(u_\phi^2)}{\partial n} = 0$ on $\partial^R D_m$. Therefore,

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\partial^R D_m} \frac{\partial |u|^2}{\partial n}\, dS \\
&= -\pi \int\limits_{\rho_0}^{1} \frac{1}{\rho}\, \partial_\phi (u_\theta^2)\Big|_{\phi=\frac{\pi}{2}-\alpha} \rho \sin\left(\frac{\pi}{2}-\alpha\right) d\rho + \pi \int\limits_{\rho_0}^{1} \frac{1}{\rho}\, \partial_\phi (u_\theta^2)\Big|_{\phi=\frac{\pi}{2}+\alpha} \rho \sin\left(\frac{\pi}{2}+\alpha\right) d\rho \\
&= 2\pi \int\limits_{\rho_0}^{1} u_\theta^2\Big|_{\phi=\frac{\pi}{2}-\alpha} \cos\left(\frac{\pi}{2}-\alpha\right) d\rho - 2\pi \int\limits_{\rho_0}^{1} u_\theta^2\Big|_{\phi=\frac{\pi}{2}+\alpha} \cos\left(\frac{\pi}{2}+\alpha\right) d\rho.
\end{aligned}
$$

Now using the fundamental theorem of Calculus, we find

$$
\begin{aligned}
\frac{1}{2}\int\limits_{\partial^R D_m} \frac{\partial |u|^2}{\partial n}\, dS &= -2\pi \int\limits_{\rho_0}^{1} \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \partial_\phi \left[u_\theta^2(\rho,\phi)\cos\phi\right] d\phi\, d\rho \\
&= -2 \int\limits_{D_m} \frac{1}{\rho^2}\, u_\theta\, \partial_\phi u_\theta \cot\phi\, dx + \int\limits_{D_m} \frac{u_\theta^2}{\rho^2}\, dx.
\end{aligned}
\tag{4.3}
$$

Similarly, by the boundary condition in (2.21), one deduces

$$
\begin{aligned}
\frac{1}{2}\int\limits_{\partial^A D_m} \frac{\partial |u|^2}{\partial n}\, dS &= \pi \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \left[\rho^2 \partial_\rho (u_\theta^2 + u_\phi^2)\right]\Big|_{\rho=\rho_0}^{\rho=1} \sin\phi\, d\phi \\
&= -2\pi \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \left[\rho\big(u_\theta^2 + u_\phi^2\big)\right]\Big|_{\rho=\rho_0}^{\rho=1} \sin\phi\, d\phi.
\end{aligned}
$$

Then applying the fundamental theorem of Calculus,

$$
\begin{aligned}
\frac{1}{2}\int\limits_{\partial^A D_m} \frac{\partial |u|^2}{\partial n}\, dS &= -2\pi \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int\limits_{\rho_0}^{1} \partial_\rho \left[\rho(u_\theta^2 + u_\phi^2)\right] \sin\phi\, d\rho\, d\phi \\
&= -\int\limits_{D_m} \frac{1}{\rho^2}\,(u_\theta^2 + u_\phi^2)\, dx - 2\int\limits_{D_m} \frac{1}{\rho}\,(u_\theta \partial_\rho u_\theta + u_\phi \partial_\rho u_\phi)\, dx.
\end{aligned}
\tag{4.4}
$$

Thus, by adding (4.3) and (4.4),

$$T_1 = -\int\limits_{D_m} \frac{u_\phi^2}{\rho^2}\,dx - 2\int\limits_{D_m} \frac{1}{\rho^2}\,u_\theta\,\partial_\phi u_\theta \cot\phi\,dx - 2\int\limits_{D_m} \frac{1}{\rho}\,(u_\theta\partial_\rho u_\theta + u_\phi\partial_\rho u_\phi)\,dx.$$

Since $\alpha \le \frac{\pi}{6}$, this implies $0 \le \cot\phi \le \frac{1}{\sqrt{3}}$ and

$$T_1 \le \Big(-\int\limits_{D_m} \frac{u_\phi^2}{\rho^2}\,dx + 2\int\limits_{D_m} \frac{1}{\rho}\,\big|u_\phi\partial_\rho u_\phi\big|\,dx\Big) + 2\int\limits_{D_m} \frac{|u_\theta|}{\rho}\Big(\frac{1}{\sqrt{3}}\Big|\frac{1}{\rho}\partial_\phi u_\theta\Big| + |\partial_\rho u_\theta|\Big)\,dx.$$

By Cauchy-Schwarz inequality, we know

$$T_1 \le \Big(\frac{2}{3}\int\limits_{D_m} |\partial_\rho u_\phi|^2\,dx + \frac{1}{2}\int\limits_{D_m} \frac{u_\phi^2}{\rho^2}\,dx\Big) + \frac{2}{3}\int\limits_{D_m} \Big(\frac{1}{3}\Big|\frac{1}{\rho}\,\partial_\phi u_\theta\Big|^2 + |\partial_\rho u_\theta|^2\Big)\,dx + 3\int\limits_{D_m} \frac{u_\theta^2}{\rho^2}\,dx.$$

Since u satisfies the even-odd-odd symmetry assumption, both u_ϕ and u_θ are odd with respect to the plane $\{\phi = \frac{\pi}{2}\}$. Hence, it follows from the Poincaré inequality in Corollary 2.4 and the fact $\alpha \le \frac{\pi}{6}$ that

$$\begin{aligned} \int\limits_{D_m} \frac{u_\phi^2}{\rho^2}\,dx &\le \frac{2}{19}\int\limits_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\phi\Big|^2\,dx, \\ \int\limits_{D_m} \frac{u_\theta^2}{\rho^2}\,dx &\le \frac{2}{19}\int\limits_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\theta\Big|^2\,dx. \end{aligned} \tag{4.5}$$

As a result,

$$T_1 \le \frac{2}{3}\int\limits_{D_m} |\partial_\rho u_\phi|^2\,dx + \frac{1}{19}\int\limits_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\phi\Big|^2\,dx + \frac{2}{3}\int\limits_{D_m} \Big(|\partial_\rho u_\theta|^2 + \Big|\frac{1}{\rho}\,\partial_\phi u_\theta\Big|^2\Big)\,dx. \tag{4.6}$$

Next, we claim

$$\int\limits_{D_m} |\nabla u|^2\,dx \ge \int\limits_{D_m} \Big(|\partial_\rho u_\phi|^2 + |\partial_\rho u_\theta|^2 + \Big|\frac{1}{\rho}\,\partial_\phi u_\theta\Big|^2\Big)\,dx + \frac{1}{4}\int\limits_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\phi\Big|^2\,dx. \tag{4.7}$$

Assuming this claim for a moment, then it follows from (4.6) that

$$T_1 \le \frac{2}{3}\int\limits_{D_m} |\nabla u|^2\,dx.$$

Putting this estimate into (4.2) yields the desired conclusion (4.1).

Thus, it remains to verify (4.7) in the above claim. According to formula (A.8),

$$\nabla u = \begin{pmatrix} \partial_\rho u_\rho & \frac{1}{\rho}(\partial_\phi u_\rho - u_\phi) & -\frac{1}{\rho}\, u_\theta \\ \partial_\rho u_\phi & \frac{1}{\rho}(\partial_\phi u_\phi + u_\rho) & -\frac{\cot\phi}{\rho}\, u_\theta \\ \partial_\rho u_\theta & \frac{1}{\rho}\partial_\phi u_\theta & \frac{1}{\rho}(u_\rho + \cot\phi\, u_\phi) \end{pmatrix}$$

under the basis (A.7), so in order to prove (4.7), it suffices to justify the following estimate:

$$\int_{D_m} \Big(\frac{1}{\rho}\partial_\phi u_\phi + \frac{1}{\rho} u_\rho\Big)^2 + \Big(\frac{1}{\rho}u_\rho + \frac{\cot\phi}{\rho}u_\phi\Big)^2 \, dx \geq \frac{1}{4}\int_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\phi\Big|^2 \, dx. \tag{4.8}$$

Using the basic inequality that for any A, B, C in $\mathbb{R}$ and for any $0 < \lambda < 1$,

$$(A+B)^2 + (B+C)^2 \geq \frac{1}{2}(A-C)^2 \geq \frac{1}{2}\Big(\lambda A^2 - \frac{\lambda}{1-\lambda}C^2\Big),$$

we know

$$\Big(\frac{1}{\rho}\partial_\phi u_\phi + \frac{1}{\rho}u_\rho\Big)^2 + \Big(\frac{1}{\rho}u_\rho + \frac{\cot\phi}{\rho}u_\phi\Big)^2 \geq \frac{\lambda}{2}\Big(\frac{1}{\rho}\partial_\phi u_\phi\Big)^2 - \frac{\lambda}{2(1-\lambda)}\Big(\frac{\cot\phi}{\rho}u_\phi\Big)^2.$$

By choosing $\lambda = \frac{2}{3}$ and using the fact that $0 \leq \cot\phi \leq \frac{1}{\sqrt{3}}$, we find

$$\Big(\frac{1}{\rho}\partial_\phi u_\phi + \frac{1}{\rho}u_\rho\Big)^2 + \Big(\frac{1}{\rho}u_\rho + \frac{\cot\phi}{\rho}u_\phi\Big)^2 \geq \frac{1}{3}\Big(\frac{1}{\rho}\partial_\phi u_\phi\Big)^2 - \frac{1}{3}\Big(\frac{1}{\rho}u_\phi\Big)^2.$$

Integrating both sides on D_m and taking advantage of (4.5) yields

$$\int_{D_m} \Big(\frac{1}{\rho}\partial_\phi u_\phi + \frac{1}{\rho}u_\rho\Big)^2 + \Big(\frac{1}{\rho}u_\rho + \frac{\cot\phi}{\rho}u_\phi\Big)^2 \, dx \geq \Big(\frac{1}{3} - \frac{2}{57}\Big)\int_{D_m} \Big|\frac{1}{\rho}\,\partial_\phi u_\phi\Big|^2 \, dx,$$

which implies (4.8). $\square$

Remark 4.2. Let D be the original target region as defined in (1.4) or (2.4). Let $u \in H^2(D)$ be an incompressible vector field such that u satisfies the NHL boundary condition (1.5) and possesses the even-odd-odd symmetry. Then (4.1) also holds when D_m is being replaced with D. That is $\|\nabla u\|_{L^2(D)} \leq \sqrt{3}\|\nabla \times u\|_{L^2(D)}$. The proof is essentially the same as that for Lemma 4.1.

For the Cauchy problem of (1.3) involving finite energy solutions v, Leray discovered the classical energy inequality as follows.

$$\int_{\mathbb{R}^3} |v(x,T)|^2 dx + 2\int_0^T \int_{\mathbb{R}^3} |\nabla v(x,t)|^2 dx dt \le \int_{\mathbb{R}^3} |v(x,0)|^2 dx.$$

But under various boundary conditions, the above inequality may need to be modified. For example, under the NHL boundary condition (1.16), we obtain an energy inequality with a slightly different form in the following result.

Proposition 4.3. *Let the region D_m be as defined in (2.17) with $m \ge 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Let v be the solution in Corollary 3.3 on $D_m \times [0,T]$. Then*

$$\int_{D_m} |v(x,T)|^2 \, dx + 2\int_0^T \int_{D_m} |\nabla \times v(x,t)|^2 \, dx \, dt = \int_{D_m} |v(x,0)|^2 \, dx. \tag{4.9}$$

In addition,

$$\int_{D_m} |v(x,T)|^2 \, dx + \frac{2}{3}\int_0^T \int_{D_m} |\nabla v(x,t)|^2 \, dx \, dt \le \int_{D_m} |v(x,0)|^2 \, dx. \tag{4.10}$$

Proof. The proof of (4.9) is essentially the same as that in Section A.4 by replacing D with D_m. After (4.9) is established, one can combine it with Lemma 4.1 to justify (4.10). □

4.2. Modified Biot-Savart law in spherical coordinates

We first derive the relations between $\frac{v_\rho}{\rho}$, $\frac{v_\phi}{\rho}$ and Ω by taking advantage of the Biot-Savart law: $\Delta v = -\nabla \times \omega$. On the one hand, since $\operatorname{div} v = 0$, it follows from (A.12) that

$$\Delta v = \Big(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\Big) v_\rho e_\rho + \Big[\Big(\Delta - \frac{1}{\rho^2 \sin^2\phi}\Big) v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho\Big] e_\phi + \Big(\Delta - \frac{1}{\rho^2 \sin^2\phi}\Big) v_\theta e_\theta.$$

On the other hand, we know from (2.12) that

$$\nabla \times v = \frac{1}{\rho \sin\phi}\,\partial_\phi(\sin\phi\, v_\theta)\, e_\rho - \frac{1}{\rho}\,\partial_\rho(\rho v_\theta)\, e_\phi + \left(\frac{1}{\rho}\,\partial_\rho(\rho v_\phi) - \frac{1}{\rho}\,\partial_\phi v_\rho\right) e_\theta. \tag{4.11}$$

Applying the above formula (4.11) to ω gives

$$\nabla \times \omega = \frac{1}{\rho \sin\phi}\,\partial_\phi(\sin\phi\, \omega_\theta)\, e_\rho - \frac{1}{\rho}\,\partial_\rho(\rho \omega_\theta)\, e_\phi + \left(\frac{1}{\rho}\,\partial_\rho(\rho \omega_\phi) - \frac{1}{\rho}\,\partial_\phi \omega_\rho\right) e_\theta.$$

Hence, the Biot-Savart law $\Delta v = -\nabla \times \omega$ is equivalent to the following form.

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right) v_\rho = -\frac{1}{\rho \sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta), \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right) v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho = \frac{1}{\rho}\,\partial_\rho(\rho\omega_\theta), \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right) v_\theta = -\frac{1}{\rho}\,\partial_\rho(\rho\omega_\phi) + \frac{1}{\rho}\,\partial_\phi\omega_\rho. \end{cases} \tag{4.12}$$

Recalling from (2.12) that $\omega_\theta = \frac{1}{\rho}\,\partial_\rho(\rho v_\phi) - \frac{1}{\rho}\,\partial_\phi v_\rho$, so

$$\partial_\phi v_\rho = \partial_\rho(\rho v_\phi) - \rho\omega_\theta.$$

Therefore, the second equation in (4.12) can be rewritten as

$$\left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right) v_\phi + \frac{2}{\rho^2}\partial_\rho(\rho v_\phi) = \frac{1}{\rho}\,\partial_\rho(\rho\omega_\theta) + \frac{2}{\rho}\,\omega_\theta,$$

which is equivalently to

$$\left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\right) v_\phi = \frac{1}{\rho^3}\,\partial_\rho(\rho^3\omega_\theta).$$

Combining with the first equation in (4.12) and recalling $\Omega = \omega_\theta/(\rho\sin\phi)$, we obtain

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right) v_\rho = -\frac{1}{\sin\phi}\,\partial_\phi(\sin^2\phi\,\Omega), \\ \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\right) v_\phi = \frac{1}{\rho^3}\,\partial_\rho(\rho^4\sin\phi\,\Omega). \end{cases} \tag{4.13}$$

Consequently, one can get the following relations between $\frac{v_\rho}{\rho}$, $\frac{v_\phi}{\rho}$ and Ω, which we call the modified Biot-Savart law.

$$\begin{cases} \left(\Delta + \frac{4}{\rho}\,\partial_\rho + \frac{6}{\rho^2}\right)\left(\frac{v_\rho}{\rho}\right) = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin^2\phi\,\Omega), \\ \left(\Delta + \frac{4}{\rho}\partial_\rho + \frac{5-\cot^2\phi}{\rho^2}\right)\left(\frac{v_\phi}{\rho}\right) = \frac{1}{\rho^4}\,\partial_\rho(\rho^4\sin\phi\,\Omega). \end{cases} \tag{4.14}$$

In the rest of this paper, for simplicity of notation, when dealing with estimates in the domain D_m (see Fig. 3), we denote $\rho_1 = \frac{1}{m}$, $\rho_2 = 1$, $\phi_1 = \frac{\pi}{2} - \alpha$ and $\phi_2 = \frac{\pi}{2} + \alpha$. In addition, the odd symmetry of v_θ with respect to $\{\phi = \frac{\pi}{2}\}$ plays an important role in the following estimates.

4.3. Control of $\|\nabla(v_\rho/\rho)(\cdot,t)\|_{L^2}$ and $\left\|\frac{1}{\rho}\nabla(v_\rho/\rho)(\cdot,t)\right\|_{L^2}$ via $\Omega(\cdot,t)$

Firstly, recalling (3.37) in the proof of Proposition 3.1, we know for any $t > 0$,

$$\int_{\phi_1}^{\phi_2} v_\rho(\rho,\phi,t)\sin\phi\,d\phi = 0, \quad \forall\,\rho \in [\rho_1,\rho_2]. \tag{4.15}$$

Next, we will take advantage of (4.15) to estimate $\|\nabla(v_\rho/\rho)(\cdot,t)\|_{L^2(D_m)}$ and $\left\|\frac{1}{\rho}\nabla(v_\rho/\rho)(\cdot,t)\right\|_{L^2(D_m)}$ via $\Omega(\cdot,t)$.

Lemma 4.4. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Then for any $T > 0$ and for a.e. $t \in [0,T]$,*

$$\left\|\nabla\Big(\frac{v_\rho}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \leq \sqrt{3}\,\|\Omega(\cdot,t)\|_{L^2(D_m)}\,, \tag{4.16}$$

$$\left\|\frac{1}{\rho}\,\nabla\Big(\frac{v_\rho}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \leq \sqrt{44}\,\|\nabla\Omega(\cdot,t)\|_{L^2(D_m)}\,. \tag{4.17}$$

Before the proof of Lemma 4.4, we would like to point out that the (uniform in m) bound about $\|\Omega(\cdot,t)\|_{L^2(D_m)}$ and $\|\nabla\Omega(\cdot,t)\|_{L^2(D_m)}$ has not been available yet. This desired estimate is provided later in Lemma 4.7 where the (uniform in m and T) bound on

$$\sup_{t\in[0,T]} \|\Omega(\cdot,t)\|_{L^2(D_m)} \quad \text{and} \quad \|\nabla\Omega(\cdot,\cdot)\|_{L^2(D_m\times[0,T])}$$

are obtained. The reason that we put Lemma 4.4 before Lemma 4.7 is because the proof of Lemma 4.7 relies on the relation established in Lemma 4.4. After Lemma 4.7, we can go back to Lemma 4.4 to justify the (uniform in m and T) bound for

$$\sup_{t\in[0,T]} \left\|\nabla\Big(\frac{v_\rho}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \quad \text{and} \quad \left\|\frac{1}{\rho}\,\nabla\Big(\frac{v_\rho}{\rho}(\cdot,\cdot)\Big)\right\|_{L^2(D_m\times[0,T])}.$$

Proof of Lemma 4.4. Since $v \in E^{\sigma,s}_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T])$ and ρ has the lower bound $\frac{1}{m}$ on D_m, we know $\Omega \in L^2_t H^1_x(D_m \times [0,T])$. So there exists a set $S_T \subset [0,T]$ such that $[0,T] \setminus S_T$ has measure 0 and for any $t \in S_T$, $\Omega(\cdot,t)$ belongs to $H^1(D_m)$. Fixing any $t \in S_T$, it suffices to prove (4.16) and (4.17) for such t. For ease of notation, we will drop all the temporal variables in the following argument.

We first consider (4.16) and denote $f_1 = \frac{v_\rho}{\rho}$. Then it follows from (4.14) that

$$\Big(\Delta + \frac{4}{\rho}\,\partial_\rho + \frac{6}{\rho^2}\Big) f_1 = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin^2\phi\,\Omega). \tag{4.18}$$

Moreover, we see from Lemma 2.1 that

$$\begin{cases} \partial_\phi f_1 = 0,\ \Omega = 0 & \text{on} \quad \partial^R D_m; \\ f_1 = 0,\ \Omega = 0 & \text{on} \quad \partial^A D_m. \end{cases} \tag{4.19}$$

In particular, the above relations imply that

$$f_1 \partial_n f_1 = 0 \quad \text{on} \quad \partial^R D_m \cup \partial^A D_m. \tag{4.20}$$

Applying f_1 as a test function to (4.18), we deduce

$$\int_{D_m} (\Delta f_1) f_1\, dx + \int_{D_m} \frac{4}{\rho}\, (\partial_\rho f_1) f_1\, dx + \int_{D_m} \frac{6}{\rho^2}\, f_1^2\, dx = -\int_{D_m} \frac{1}{\rho \sin\phi} \partial_\phi (\sin^2\phi\, \Omega) f_1\, dx. \tag{4.21}$$

Now using integration by parts in (4.21) and taking advantage of (4.19) and (4.20), we have

$$\int_{D_m} (\Delta f_1) f_1 dx = \int_{\partial D_m} (\partial_n f_1) f_1\, dS - \int_{D_m} |\nabla f_1|^2\, dx = -\int_{D_m} |\nabla f_1|^2\, dx,$$

$$\begin{aligned}\int_{D_m} \frac{4}{\rho}\, (\partial_\rho f_1) f_1\, dx = 2\int_{D_m} \frac{1}{\rho} \partial_\rho (f_1^2)\, dx &= 4\pi \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} \partial_\rho (f_1^2)\, \rho \sin\phi\, d\rho\, d\phi \\ &= -4\pi \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f_1^2 \sin\phi\, d\rho\, d\phi \\ &= -2\int_{D_m} \frac{1}{\rho^2} f_1^2\, dx,\end{aligned}$$

and

$$\begin{aligned}-\int_{D_m} \frac{1}{\rho \sin\phi}\, \partial_\phi (\sin^2\phi\, \Omega) f_1\, dx &= -2\pi \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \rho\, \partial_\phi (\sin^2\phi\, \Omega) f_1\, d\phi\, d\rho \\ &= 2\pi \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \rho \sin^2\phi\, \Omega\, \partial_\phi f_1\, d\phi\, d\rho \\ &= \int_{D_m} \sin\phi\, \Omega\, \frac{\partial_\phi f_1}{\rho}\, dx.\end{aligned}$$

Putting the above relations into (4.21) yields

$$\int_{D_m} |\nabla f_1|^2\, dx = 4\int_{D_m} \frac{1}{\rho^2} f_1^2\, dx - \int_{D_m} \sin\phi\, \Omega\, \frac{\partial_\phi f_1}{\rho}\, dx. \tag{4.22}$$

As a result, it follows from Cauchy–Schwarz inequality that for any $\epsilon > 0$,

$$\int_{D_m} |\nabla f_1|^2\, dx \le 4\int_{D_m} \frac{1}{\rho^2} f_1^2\, dx + \epsilon \int_{D_m} \left(\frac{\partial_\phi f_1}{\rho}\right)^2 dx + \frac{1}{4\epsilon} \int_{D_m} |\Omega|^2\, dx. \tag{4.23}$$

Note that v_ρ satisfies (4.15), so

$$\int_{\phi_1}^{\phi_2} f_1 \sin\phi \, d\phi = \frac{1}{\rho} \int_{\phi_1}^{\phi_2} v_\rho \sin\phi \, d\phi = 0.$$

Then it follows from Corollary 2.4 that

$$\int_{\phi_1}^{\phi_2} f_1^2 \sin\phi \, d\phi \le C_{\alpha,A} \int_{\phi_1}^{\phi_2} (\partial_\phi f_1)^2 \sin\phi \, d\phi, \tag{4.24}$$

where $C_{\alpha,A}$ is defined as in (2.23). Hence,

$$\begin{aligned} \int_{D_m} \frac{1}{\rho^2} f_1^2 \, dx &= 2\pi \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} f_1^2 \sin\phi \, d\phi \, d\rho \\ &\le 2\pi \, C_{\alpha,A} \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} (\partial_\phi f_1)^2 \sin\phi \, d\phi \, d\rho \\ &= C_{\alpha,A} \int_{D_m} \left(\frac{\partial_\phi f_1}{\rho} \right)^2 dx. \end{aligned} \tag{4.25}$$

Putting the above inequality into (4.23) and noticing $\left|\frac{1}{\rho}\, \partial_\phi f_1\right| \le |\nabla f_1|$, we obtain

$$\Big(1 - 4C_{\alpha,A} - \epsilon\Big) \int_{D_m} |\nabla f_1|^2 \, dx \le \frac{1}{4\epsilon} \int_{D_m} |\Omega|^2 \, dx.$$

Since $C_{\alpha,A}$ is increasing in α which lies in $\big(0, \frac{\pi}{6}\big]$, it follows from (2.25) that $C_{\alpha,A} \le C_{\pi/6,A} = \frac{2}{19}$. Thus,

$$\Big(\frac{11}{19} - \epsilon\Big) \int_{D_m} |\nabla f_1|^2 \, dx \le \frac{1}{4\epsilon} \int_{D_m} |\Omega|^2 \, dx.$$

Choosing $\epsilon = \frac{11}{38}$ implies that

$$\int_{D_m} |\nabla f_1(x,t)|^2 \, dx \le 3 \int_{D_m} |\Omega(x,t)|^2 \, dx.$$

Thus, (4.16) is justified.

Next we estimate $\|\frac{1}{\rho}\nabla f_1\|_{L^2(D_m)}$. Applying $\frac{f_1}{\rho^2}$ as a test function to (4.18), we deduce

$$\int_{D_m} (\Delta f_1)\,\frac{f_1}{\rho^2}\,dx + 4\int_{D_m} \frac{f_1\partial_\rho f_1}{\rho^3}\,dx + 6\int_{D_m}\frac{f_1^2}{\rho^4}\,dx = -\int_{D_m}\frac{f_1}{\rho^3\sin\phi}\partial_\phi(\sin^2\phi\,\Omega)\,dx. \tag{4.26}$$

Since $f_1 = 0$ on $\partial^A D_m$ and $\partial_n f_1 = 0$ on $\partial^R D_m$, it follows from integration by parts that

$$\begin{aligned}\int_{D_m} (\Delta f_1)\,\frac{f_1}{\rho^2}\,dx &= -\int_{D_m} \nabla f_1\cdot\nabla\Big(\frac{f_1}{\rho^2}\Big)\,dx\\ &= -\int_{D_m}\frac{|\nabla f_1|^2}{\rho^2}\,dx + 2\int_{D_m}\frac{f_1\partial_\rho f_1}{\rho^3}\,dx.\end{aligned}$$

Plugging the above equality into (4.26) yields

$$-\int_{D_m}\frac{|\nabla f_1|^2}{\rho^2}\,dx + 6\int_{D_m}\frac{f_1\partial_\rho f_1}{\rho^3}\,dx + 6\int_{D_m}\frac{f_1^2}{\rho^4}\,dx = -\int_{D_m}\frac{f_1}{\rho^3\sin\phi}\partial_\phi(\sin^2\phi\,\Omega)\,dx. \tag{4.27}$$

Using integration by parts and noting $f_1 = 0$ on $\partial^A D_m$, we obtain

$$\begin{aligned}6\int_{D_m}\frac{f_1\partial_\rho f_1}{\rho^3}\,dx &= 6\pi\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2}\frac{1}{\rho}\,\partial_\rho(f_1^2)\sin\phi\,d\rho\,d\phi\\ &= -6\pi\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2}\partial_\rho\Big(\frac{1}{\rho}\Big)\,f_1^2\sin\phi\,d\rho\,d\phi\\ &= 3\int_{D_m}\frac{f_1^2}{\rho^4}\,dx.\end{aligned} \tag{4.28}$$

Applying integration by parts again and recalling $\Omega = 0$ on $\partial^R D_m$, we get

$$\begin{aligned}-\int_{D_m}\frac{f_1}{\rho^3\sin\phi}\,\partial_\phi(\sin^2\phi\,\Omega)\,dx &= -2\pi\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2}\frac{f_1}{\rho}\,\partial_\phi(\sin^2\phi\,\Omega)\,d\phi\,d\rho\\ &= 2\pi\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2}\frac{\partial_\phi f_1}{\rho}(\sin^2\phi\,\Omega)\,d\phi\,d\rho\\ &= \int_{D_m}\frac{\sin\phi}{\rho^3}\,(\partial_\phi f_1)\,\Omega\,dx.\end{aligned} \tag{4.29}$$

Plugging (4.28) and (4.29) into (4.27), we have

$$\int_{D_m} \frac{|\nabla f_1|^2}{\rho^2}\,dx = 9\int_{D_m} \frac{f_1^2}{\rho^4}\,dx - \int_{D_m} \frac{\sin\phi}{\rho^3}\,(\partial_\phi f_1)\,\Omega\,dx.$$

It then follows from Cauchy–Schwarz's inequality that for any $\epsilon > 0$,

$$\int_{D_m} \frac{|\nabla f_1|^2}{\rho^2}\,dx \le 9\int_{D_m} \frac{f_1^2}{\rho^4}\,dx + \epsilon\int_{D_m} \frac{1}{\rho^2}\left(\frac{\partial_\phi f_1}{\rho}\right)^2 dx + \frac{1}{4\epsilon}\int_{D_m} \frac{\Omega^2}{\rho^2}\,dx. \tag{4.30}$$

Moreover, it follows from Corollary 2.4 that

$$\begin{aligned}\int_{D_m} \frac{f_1^2}{\rho^4}\,dx &= 2\pi\int_{\rho_1}^{\rho_2} \frac{1}{\rho^2}\int_{\phi_1}^{\phi_2} f_1^2 \sin\phi\,d\phi\,d\rho \\ &\le 2\pi\,C_{\alpha,A}\int_{\rho_1}^{\rho_2} \frac{1}{\rho^2}\int_{\phi_1}^{\phi_2} (\partial_\phi f_1)^2 \sin\phi\,d\phi\,d\rho \\ &= C_{\alpha,A}\int_{D_m} \frac{1}{\rho^2}\left(\frac{\partial_\phi f_1}{\rho}\right)^2 dx.\end{aligned}$$

Putting the above inequality into (4.30) and noticing that $\left|\frac{1}{\rho}\,\partial_\phi f_1\right| \le |\nabla f_1|$, we attain

$$\big(1 - 9C_{\alpha,A} - \epsilon\big)\int_{D_m} \frac{|\nabla f_1|^2}{\rho^2}\,dx \le \frac{1}{4\epsilon}\int_{D_m} \frac{\Omega^2}{\rho^2}\,dx. \tag{4.31}$$

Again, since $C_{\alpha,A} \le \frac{2}{19}$, we choose $\epsilon = \frac{1}{38}$ and conclude

$$\int_{D_m} \frac{|\nabla f_1|^2}{\rho^2}\,dx \le 19^2\int_{D_m} \frac{\Omega^2}{\rho^2}\,dx. \tag{4.32}$$

Finally, since $\Omega = 0$ on $\partial^R D_m$, it follows from Corollary 2.6 that

$$\begin{aligned}\int_{D_m} \frac{\Omega^2}{\rho^2}\,dx &= 2\pi\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2} \Omega^2 \sin\phi\,d\phi\,d\rho \\ &\le 2\pi\,C_{\alpha,B}\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2} |\partial_\phi\Omega|^2 \sin\phi\,d\phi\,d\rho = C_{\alpha,B}\int_{D_m}\left(\frac{\partial_\phi\Omega}{\rho}\right)^2 dx.\end{aligned} \tag{4.33}$$

Combining (4.32), (4.33) and the fact that $C_{\alpha,B} \leq C_{\pi/6,B} = \frac{3}{25}$, we get

$$\int_{D_m} \frac{1}{\rho^2} |\nabla f_1(x,t)|^2 dx \leq 44 \int_{D_m} \left(\frac{\partial_\phi \Omega(x,t)}{\rho} \right)^2 dx \leq 44 \int_{D_m} |\nabla \Omega(x,t)|^2 \, dx.$$

This completes the proof of (4.17). □

4.4. *Control of* $\|\nabla(v_\phi/\rho)(\cdot,t)\|_{L^2}$ *and* $\left\|\frac{1}{\rho}\nabla(v_\phi/\rho)(\cdot,t)\right\|_{L^2}$ *via* $\Omega(\cdot,t)$

Lemma 4.5. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Then for any $T > 0$ and for a.e. $t \in [0,T]$,*

$$\left\| \nabla \Big(\frac{v_\phi}{\rho}(\cdot,t) \Big) \right\|_{L^2(D_m)} \leq \sqrt{3} \, \|\Omega(\cdot,t)\|_{L^2(D_m)} \,, \tag{4.34}$$

$$\left\| \frac{1}{\rho} \, \nabla \Big(\frac{v_\phi}{\rho}(\cdot,t) \Big) \right\|_{L^2(D_m)} \leq 20 \, \|\nabla\Omega(\cdot,t)\|_{L^2(D_m)} \,. \tag{4.35}$$

Proof. Since $v \in E^{\sigma,s}_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T])$ and ρ has the lower bound $\frac{1}{m}$ on D_m, we know $\Omega \in L^2_t H^1_x(D_m \times [0,T])$. So there exists a set $S_T \subset [0,T]$ such that $[0,T] \setminus S_T$ has measure 0 and for any $t \in S_T$, $\Omega(\cdot,t)$ belongs to $H^1(D_m)$. Fixing any $t \in S_T$, it suffices to prove (4.16) and (4.17) for such t. For simplicity of notation, we will drop all the temporal variables in the following proof.

We first focus on (4.34) and define $f_2 = \frac{v_\phi}{\rho}$. Then it follows from the second equation in (4.14) that

$$\Big(\Delta + \frac{4}{\rho} \partial_\rho + \frac{5 - \cot^2\phi}{\rho^2} \Big) f_2 = \frac{1}{\rho^4} \, \partial_\rho(\rho^4 \sin\phi \, \Omega). \tag{4.36}$$

On the boundary portion $\partial^R D_m$, owing to $v_\phi = \partial_\rho v_\phi = 0$, one concludes that

$$f_2 = \partial_\rho f_2 = 0, \quad \text{on} \quad \partial^R D_m. \tag{4.37}$$

Meanwhile, since $\partial_\rho(\rho v_\phi) = 0$ on the boundary portion $\partial^A D_m$, one deduces that

$$\partial_\rho f_2 + \frac{2}{\rho} \, f_2 = 0, \quad \text{on} \quad \partial^A D_m. \tag{4.38}$$

Multiplying (4.36) by f_2 and integrating on domain D_m, one derives that

$$\underbrace{\int_{D_m} f_2 \, \Delta f_2 \, dx}_{I_1} + 4 \int_{D_m} \frac{1}{\rho} f_2 \, \partial_\rho f_2 \, dx + \int_{D_m} (5 - \cot^2\phi) \frac{f_2^2}{\rho^2} \, dx = \underbrace{\int_{D_m} \frac{f_2}{\rho^4} \, \partial_\rho(\rho^4 \sin\phi \, \Omega) \, dx}_{I_2} . \tag{4.39}$$

Using integration by parts,

$$I_1 = -\int\limits_{D_m} |\nabla f_2|^2 dx + \underbrace{\int\limits_{\partial D_m} f_2\,\partial_n f_2\,dS}_{I_{11}}. \tag{4.40}$$

By boundary conditions (4.37) and (4.38), I_{11} satisfies

$$I_{11} = -\int\limits_{A_{1,m}} f_2\,\partial_\rho f_2\,dS + \int\limits_{A_{2,m}} f_2\,\partial_\rho f_2\,dS = \int\limits_{A_{1,m}} \frac{2f_2^2}{\rho}\,dS - \int\limits_{A_{2,m}} \frac{2f_2^2}{\rho}\,dS,$$

where the meaning of $A_{1,m}$ and $A_{2,m}$ can be found in Fig. 3. By the fundamental theorem of calculus, we further notice that

$$\begin{aligned} I_{11} &= 4\pi\int\limits_{\phi_1}^{\phi_2} \rho_1 \sin\phi\, f_2^2\,d\phi - 4\pi\int\limits_{\phi_1}^{\phi_2} \rho_2 \sin\phi\, f_2^2\,d\phi \\ &= -4\pi\int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho_1}^{\rho_2} \partial_\rho(\rho\sin\phi\, f_2^2)\,d\rho\,d\phi \\ &= -4\int\limits_{D_m} \frac{f_2}{\rho}\,\partial_\rho f_2\,dx - 2\int\limits_{D_m} \frac{f_2^2}{\rho^2}\,dx. \end{aligned}$$

Plugging the above expression of I_{11} in (4.40), one has

$$I_1 = -\int\limits_{D_m} |\nabla f_2|^2 dx - 2\int\limits_{D_m} \frac{f_2^2}{\rho^2}\,dx - 4\int\limits_{D_m} \frac{f_2}{\rho}\,\partial_\rho f_2\,dx. \tag{4.41}$$

For I_2 which can be rewritten as

$$I_2 = 2\pi\int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho_1}^{\rho_2} \frac{\sin\phi}{\rho^2}\, f_2\,\partial_\rho(\rho^4\sin\phi\,\Omega)\,d\rho\,d\phi,$$

we use integration by parts and the fact that $\Omega = 0$ on $\partial^A D_m$ to obtain

$$\begin{aligned} I_2 &= -2\pi\int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho_1}^{\rho_2} \partial_\rho\Big(\frac{\sin\phi}{\rho^2}\, f_2\Big)\,\rho^4\sin\phi\,\Omega\,d\rho\,d\phi \\ &= 2\int\limits_{D_m} \frac{\sin\phi}{\rho}\, f_2\,\Omega\,dx - \int\limits_{D_m} \sin\phi\,(\partial_\rho f_2)\,\Omega\,dx. \end{aligned} \tag{4.42}$$

Plugging (4.41) and (4.42) into (4.39) yields

$$-\int_{D_m} |\nabla f_2|^2\,dx + \int_{D_m} (3-\cot^2\phi)\frac{f_2^2}{\rho^2}\,dx = 2\int_{D_m} \frac{\sin\phi}{\rho}\,f_2\,\Omega\,dx - \int_{D_m} \sin\phi\,(\partial_\rho f_2)\,\Omega\,dx.$$

As a result,

$$\int_{D_m} |\nabla f_2|^2\,dx \le 3\int_{D_m} \frac{f_2^2}{\rho^2}\,dx + 2\int_{D_m} \frac{\sin\phi}{\rho}\,|f_2\,\Omega|\,dx + \int_{D_m} \sin\phi\,|(\partial_\rho f_2)\,\Omega|\,dx.$$

By Cauchy-Schwarz inequality, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\int_{D_m} |\nabla f_2|^2\,dx \le (3+\epsilon_1)\int_{D_m} \frac{f_2^2}{\rho^2}\,dx + \epsilon_2\int_{D_m} |\partial_\rho f_2|^2\,dx + \Big(\frac{1}{\epsilon_1} + \frac{1}{4\epsilon_2}\Big)\int_{D_m} \Omega^2\,dx. \tag{4.43}$$

Since $f_2 = 0$ on $\partial^R D_m$, then by a similar derivation as that in (4.33), we get

$$\int_{D_m} \frac{f_2^2}{\rho^2}\,dx \le C_{\alpha,B}\int_{D_m} \left(\frac{\partial_\phi f_2}{\rho}\right)^2 dx.$$

Putting the above estimate into (4.43) and recalling the estimate $C_{\alpha,B} \le \frac{3}{25}$ in (2.25), we obtain

$$\int_{D_m} |\nabla f_2|^2\,dx \le \frac{3(3+\epsilon_1)}{25}\int_{D_m} \left(\frac{\partial_\phi f_2}{\rho}\right)^2 dx + \epsilon_2\int_{D_m} |\partial_\rho f_2|^2\,dx + \Big(\frac{1}{\epsilon_1} + \frac{1}{4\epsilon_2}\Big)\int_{D_m} \Omega^2\,dx.$$

By choosing $\epsilon_1 = 2$ and choosing $\epsilon_2 = \frac{3(3+\epsilon_1)}{25} = \frac{3}{5}$, we find

$$\int_{D_m} |\nabla f_2|^2\,dx \le \frac{3}{5}\int_{D_m} |\nabla f_2|^2\,dx + \int_{D_m} \Omega^2\,dx.$$

This implies that

$$\int_{D_m} |\nabla f_2(x,t)|^2\,dx \le 3\int_{D_m} \Omega^2(x,t)\,dx,$$

which proves (4.34).

Next, we are going to prove (4.35). Multiplying (4.36) by $\frac{f_2}{\rho^2}$ and integrating on D_m yields

$$\underbrace{\int\limits_{D_m} \frac{f_2}{\rho^2}\,\Delta f_2\,dx}_{J_1} + 4\int\limits_{D_m} \frac{1}{\rho^3}\,f_2\partial_\rho f_2\,dx + \int\limits_{D_m} \frac{5-\cot^2\phi}{\rho^4}\,f_2^2\,dx = \underbrace{\int\limits_{D_m} \frac{f_2}{\rho^6}\,\partial_\rho(\rho^4\sin\phi\,\Omega)\,dx}_{J_2}\,. \tag{4.44}$$

Using integration by parts,

$$\begin{aligned} J_1 &= -\int\limits_{D_m} \nabla\Big(\frac{f_2}{\rho^2}\Big)\cdot\nabla f_2\,dx + \int\limits_{\partial D_m} \frac{f_2}{\rho^2}\,\partial_n f_2\,dS \\ &= -\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2\,dx + 2\int\limits_{D_m} \frac{f_2}{\rho^3}\,\partial_\rho f_2\,dx + \underbrace{\int\limits_{\partial D_m} \frac{f_2}{\rho^2}\,\partial_n f_2\,dS}_{J_{11}}\,. \end{aligned}$$

Similar to the computation of I_{11} above, we find

$$J_{11} = 2\int\limits_{D_m} \frac{f_2^2}{\rho^4}\,dx - 4\int\limits_{D_m} \frac{f_2}{\rho^3}\,\partial_\rho f_2\,dx.$$

So

$$J_1 = -\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2\,dx + 2\int\limits_{D_m} \frac{f_2^2}{\rho^4}\,dx - 2\int\limits_{D_m} \frac{f_2}{\rho^3}\,\partial_\rho f_2\,dx. \tag{4.45}$$

Next, by direct computation,

$$J_2 = 4\int\limits_{D_m} \frac{\sin\phi}{\rho^3}\,f_2\,\Omega\,dx + \int\limits_{D_m} \frac{\sin\phi}{\rho^2}\,f_2\,\partial_\rho\Omega\,dx.$$

Substituting the above expression for J_2 and (4.45) for J_1 into (4.44), one deduces

$$\begin{aligned} \int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx = &\int\limits_{D_m} \frac{7-\cot^2\phi}{\rho^4}\,f_2^2\,dx + 2\int\limits_{D_m} \frac{f_2}{\rho^3}\,\partial_\rho f_2\,dx \\ &- 4\int\limits_{D_m} \frac{\sin\phi}{\rho^3}\,f_2\,\Omega\,dx - \int\limits_{D_m} \frac{\sin\phi}{\rho^2}\,f_2\,\partial_\rho\Omega\,dx. \end{aligned}$$

Thus,

$$\begin{aligned} \int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx \le 7&\int\limits_{D_m} \frac{f_2^2}{\rho^4}\,dx + 2\int\limits_{D_m} \frac{1}{\rho^3}\,|f_2\,\partial_\rho f_2|\,dx \\ &+ 4\int\limits_{D_m} \frac{1}{\rho^3}\,|f_2\,\Omega|\,dx + \int\limits_{D_m} \frac{1}{\rho^2}\,|f_2\,\partial_\rho\Omega|\,dx. \end{aligned} \tag{4.46}$$

By Cauchy-Schwarz inequality, for any constants ϵ_1, ϵ_2, $\epsilon_3 > 0$, one has

$$\begin{aligned}\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx \leq \Big(7 + \frac{1}{\epsilon_1} + \epsilon_2 + \epsilon_3\Big) \int\limits_{D_m} \frac{f_2^2}{\rho^4}\, dx + \epsilon_1 \int\limits_{D_m} \frac{1}{\rho^2}\, |\partial_\rho f_2|^2\, dx \\ + \frac{4}{\epsilon_2} \int\limits_{D_m} \frac{\Omega^2}{\rho^2}\, dx + \frac{1}{4\epsilon_3} \int\limits_{D_m} |\partial_\rho \Omega|^2\, dx.\end{aligned} \tag{4.47}$$

Now since $f_2 = \Omega = 0$ on $\partial^R D_m$, then similar to the derivation of (4.33), we obtain

$$\int\limits_{D_m} \frac{f_2^2}{\rho^4}\, dx \leq C_{\alpha,B} \int\limits_{D_m} \frac{1}{\rho^2}\Big(\frac{\partial_\phi f_2}{\rho}\Big)^2 dx, \tag{4.48}$$

$$\int\limits_{D_m} \frac{\Omega^2}{\rho^2}\, dx \leq C_{\alpha,B} \int\limits_{D_m} \Big(\frac{\partial_\phi \Omega}{\rho}\Big)^2 dx. \tag{4.49}$$

Plugging (4.48) and (4.49) into (4.47) and recalling $C_{\alpha,B} \leq \frac{3}{25}$, one deduces

$$\begin{aligned}\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx \leq \frac{3}{25}\Big(7 + \frac{1}{\epsilon_1} + \epsilon_2 + \epsilon_3\Big) \int\limits_{D_m} \frac{1}{\rho^2}\Big(\frac{\partial_\phi f_2}{\rho}\Big)^2 dx + \epsilon_1 \int\limits_{D_m} \frac{1}{\rho^2}\, |\partial_\rho f_2|^2\, dx \\ + \frac{12}{25\epsilon_2} \int\limits_{D_m} \Big(\frac{\partial_\phi \Omega}{\rho}\Big)^2 dx + \frac{1}{4\epsilon_3} \int\limits_{D_m} |\partial_\rho \Omega|^2\, dx.\end{aligned}$$

By choosing $\epsilon_1 = \frac{20}{21}$, $\epsilon_2 = \frac{1}{20}$ and $\epsilon_3 = \frac{1}{40}$, we have

$$\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx \leq \frac{39}{40} \int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2|^2 dx + 10 \int\limits_{D_m} |\nabla \Omega|^2\, dx.$$

This implies

$$\int\limits_{D_m} \frac{1}{\rho^2}|\nabla f_2(x,t)|^2 dx \leq 400 \int\limits_{D_m} |\nabla \Omega(x,t)|^2\, dx,$$

completing the proof of (4.35) and Lemma 4.5. $\square$

4.5. Uniform bounds for $\|(K, F, \Omega)\|_{L_t^\infty L_x^2}$ *and* $\|(\nabla K, \nabla F, \nabla \Omega)\|_{L_{tx}^2}$

In this subsection, we will derive some energy estimates for K, F and Ω.

Lemma 4.6. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Let K, F and Ω be defined as in (2.14). Then for any $T > 0$, the following three energy identities (4.50)–(4.52) hold.*

$$\begin{aligned}
&\frac{1}{2}\int_{D_m} K^2(x,T)\,dx - \frac{1}{2}\int_{D_m} K^2(x,0)\,dx + \int_0^T\int_{D_m} |\nabla K|^2\,dx\,dt \\
&= 3\int_0^T\int_{D_m} \frac{K^2}{\rho^2}\,dx\,dt - 2\int_0^T\int_{D_m} \frac{K}{\rho}\,\partial_\rho K\,dx\,dt \\
&\quad + \int_0^T\int_{D_m} \frac{v_\theta}{\rho}\left[\partial_\phi\left(\frac{v_\rho}{\rho}\right)\partial_\rho K - \partial_\rho\left(\frac{v_\rho}{\rho}\right)\partial_\phi K\right] dx\,dt.
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
&\frac{1}{2}\int_{D_m} F^2(x,T)\,dx - \frac{1}{2}\int_{D_m} F^2(x,0)\,dx + \int_0^T\int_{D_m} |\nabla F|^2\,dx\,dt \\
&= \int_0^T\int_{D_m} \frac{1-\cot^2\phi}{\rho^2} F^2\,dx\,dt - 2\int_0^T\int_{D_m} \frac{\cot\phi}{\rho^2} F\partial_\phi F\,dx\,dt + 2\int_0^T\int_{D_m} \frac{(\partial_\phi K)F}{\rho^2}\,dx\,dt \\
&\quad + \int_0^T\int_{D_m} \frac{v_\theta}{\rho}\left[\partial_\phi\left(\frac{v_\phi}{\rho}\right)\partial_\rho F - \partial_\rho\left(\frac{v_\phi}{\rho}\right)\partial_\phi F\right] dx\,dt.
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
&\frac{1}{2}\int_{D_m} \Omega^2(x,T)\,dx - \frac{1}{2}\int_{D_m} \Omega^2(x,0)\,dx + \int_0^T\int_{D_m} |\nabla \Omega|^2\,dx\,dt \\
&= -2\int_0^T\int_{D_m} \frac{v_\theta}{\rho\sin\phi} K\Omega\,dx\,dt - 2\int_0^T\int_{D_m} \frac{v_\theta\cos\phi}{\rho\sin^2\phi} F\Omega\,dx\,dt.
\end{aligned} \tag{4.52}$$

Proof. Firstly, since $v \in E^{\sigma,s}_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T])$ and ρ has the lower bound $\frac{1}{m}$ on D_m, all of K, F and Ω are in $L^2_t H^1_x(D_m \times [0,T])$. Meanwhile, all the integrals in (4.50)–(4.52) are well-defined. In addition, K is even, and F and Ω are odd symmetric with respect to the plane $\{\phi = \frac{\pi}{2}\}$. (4.50), (4.51) and (4.52) can be justified by testing $(2.15)_1$, $(2.15)_2$ and $(2.15)_3$ by K, F and Ω respectively. The derivations for these three energy identities are similar, so we will only show details for (4.51) which is relatively the most complicated one.

Multiplying $(2.15)_2$ by F and integrating on $D_m \times [0,T]$ yields

$$\frac{1}{2}\int_{D_m} F^2(x,T)\,dx - \frac{1}{2}\int_{D_m} F^2(x,0)\,dx = L_1 - L_2 + L_3 + L_4, \tag{4.53}$$

where

$$L_1 = \int_0^T \int_{D_m} \left[\left(\Delta + \frac{2}{\rho} \partial_\rho + \frac{1 - \cot^2 \phi}{\rho^2} \right) F \right] F \, dx \, dt, \tag{4.54}$$

$$L_2 = \int_0^T \int_{D_m} (b \cdot \nabla F) F \, dx \, dt, \tag{4.55}$$

$$L_3 = \int_0^T \int_{D_m} \frac{2}{\rho^2} (\partial_\phi K) F \, dx \, dt, \tag{4.56}$$

$$L_4 = \int_0^T \int_{D_m} \left(\omega \cdot \nabla \left(\frac{v_\phi}{\rho} \right) \right) F \, dx \, dt. \tag{4.57}$$

We will first compute L_1. Using integration by parts,

$$\int_0^T \int_{D_m} (\Delta F) F \, dx \, dt = \int_0^T \int_{\partial D_m} (\partial_n F) F \, dS \, dt - \int_0^T \int_{D_m} |\nabla F|^2 \, dx \, dt. \tag{4.58}$$

According to Lemma 2.1, $F = 0$ on $\partial^A D_m$ and $\partial_\phi F = - \cot \phi \, F$ on $\partial^R D_m$, so

$$\begin{aligned} \int_{\partial D_m} (\partial_n F) F \, dS &= \int_{\partial^R D_m} (\partial_n F) F \, dS \\ &= - \int_{R_{1,m}} \left(\frac{1}{\rho} \partial_\phi F \right) F \, dS + \int_{R_{2,m}} \left(\frac{1}{\rho} \partial_\phi F \right) F \, dS \\ &= \int_{R_{1,m}} \frac{\cot \phi_1}{\rho} F^2 \, dS - \int_{R_{2,m}} \frac{\cot \phi_2}{\rho} F^2 \, dS, \end{aligned}$$

where the definition of the boundaries $R_{1,m}$ and $R_{2,m}$ can be found in Fig. 3. Noticing $dS = 2\pi \rho \sin \phi \, d\rho$, we find

$$\int_{\partial D_m} (\partial_n F) F \, dS = 2\pi \int_{\rho_1}^{\rho_2} \cos \phi_1 \, F^2 \, d\rho - 2\pi \int_{\rho_1}^{\rho_2} \cos \phi_2 \, F^2 \, d\rho.$$

Now applying the fundamental theorem of Calculus yields

$$\begin{aligned}\int\limits_{\partial D_m} (\partial_n F)F\, dS &= -2\pi \int\limits_{\rho_1}^{\rho_2}\int\limits_{\phi_1}^{\phi_2} \partial_\phi(\cos\phi\, F^2)\, d\phi\, d\rho \\ &= 2\pi \int\limits_{\rho_1}^{\rho_2}\int\limits_{\phi_1}^{\phi_2} \sin\phi\, F^2\, d\phi\, d\rho - 4\pi \int\limits_{\rho_1}^{\rho_2}\int\limits_{\phi_1}^{\phi_2} \cos\phi\, F\partial_\phi F\, d\phi\, d\rho \\ &= \int\limits_{D_m} \frac{F^2}{\rho^2}\, dx - 2\int\limits_{D_m} \frac{\cot\phi}{\rho^2}\, F\partial_\phi F\, dx.\end{aligned}$$

Substituting the above identity into (4.54) gives

$$\begin{aligned}\int\limits_0^T\int\limits_{D_m} (\Delta F)F\, dx\, dt = -\int\limits_0^T\int\limits_{D_m} |\nabla F|^2\, dx\, dt + \int\limits_0^T\int\limits_{D_m} \frac{F^2}{\rho^2}\, dx\, dt \\ - 2\int\limits_0^T\int\limits_{D_m} \frac{\cot\phi}{\rho^2}\, F\partial_\phi F\, dx\, dt.\end{aligned} \tag{4.59}$$

We continue to deal with the first-order term in (4.54).

$$\int\limits_{D_m} \frac{2}{\rho}\, (\partial_\rho F)F\, dx = 2\pi \int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho_1}^{\rho_2} \rho\sin\phi\, \partial_\rho(F^2)\, d\rho\, d\phi.$$

Recalling $F = 0$ on $\partial^A D_m$, so we apply integration by parts to obtain

$$\int\limits_{D_m} \frac{2}{\rho}\, (\partial_\rho F)F\, dx = -2\pi \int\limits_{\phi_1}^{\phi_2}\int\limits_{\rho_1}^{\rho_2} \sin\phi\, F^2\, d\rho\, d\phi = -\int\limits_{D_m} \frac{F^2}{\rho^2}\, dx. \tag{4.60}$$

Plugging (4.59) and (4.60) into (4.54) shows

$$L_1 = -\int\limits_0^T\int\limits_{D_m} |\nabla F|^2\, dx\, dt + \int\limits_0^T\int\limits_{D_m} \frac{1-\cot^2\phi}{\rho^2}\, F^2\, dx\, dt - 2\int\limits_0^T\int\limits_{D_m} \frac{\cot\phi}{\rho^2}\, F\partial_\phi F\, dx\, dt. \tag{4.61}$$

Next, we calculate L_2. By divergence theorem, we have

$$L_2 = \frac{1}{2}\int_0^T \int_{D_m} b\cdot\nabla(F^2)\,dx\,dt$$
$$= \frac{1}{2}\int_0^T \int_{\partial D_m} (b\cdot n)F^2\,dS\,dt - \frac{1}{2}\int_0^T \int_{D_m} (\nabla\cdot b)F^2\,dx\,dt.$$

Noticing that $b\cdot n = v\cdot n = 0$ on ∂D_m and $\nabla\cdot b = \nabla\cdot v = 0$ in D_m, so

$$L_2 = 0. \tag{4.62}$$

Finally, the term L_4 will be treated. Based on the formula (2.12) for ω,

$$\omega\cdot\nabla\Big(\frac{v_\phi}{\rho}\Big) = \omega_\rho\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big) + \omega_\phi\,\frac{1}{\rho}\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)$$
$$= \frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,v_\theta)\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big) - \frac{1}{\rho^2}\,\partial_\rho(\rho v_\theta)\,\partial_\phi\Big(\frac{v_\phi}{\rho}\Big).$$

Thus,

$$L_4 = L_{41} - L_{42}, \tag{4.63}$$

where

$$L_{41} = \int_0^T \int_{D_m} \frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,v_\theta)\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F\,dx\,dt,$$
$$= 2\pi\int_0^T\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2} \rho\,\partial_\phi(\sin\phi\,v_\theta)\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F\,d\phi\,d\rho\,dt,$$

and

$$L_{42} = \int_0^T \int_{D_m} \frac{1}{\rho^2}\,\partial_\rho(\rho v_\theta)\,\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F\,dx\,dt,$$
$$= 2\pi\int_0^T\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} \sin\phi\,\partial_\rho(\rho v_\theta)\,\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F\,d\rho\,d\phi\,dt.$$

For L_{41}, since $v_\phi = 0$ on $\partial^R D_m$, then $\partial_\rho v_\phi = 0$ on $\partial^R D_m$, which further implies $\partial_\rho\big(\frac{v_\phi}{\rho}\big) = 0$ on $\partial^R D_m$. This enables one to do integration by parts with vanishing boundary terms to get

$$\begin{aligned}\int_{\phi_1}^{\phi_2} \rho\,\partial_\phi(\sin\phi\, v_\theta)\,\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F\,d\phi &= -\int_{\phi_1}^{\phi_2} \rho\sin\phi\, v_\theta\,\partial_\phi\Big[\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F\Big]\,d\phi \\ &= -\int_{\phi_1}^{\phi_2} \rho\sin\phi\, v_\theta\,\Big[\partial_\phi\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\phi F\Big]\,d\phi.\end{aligned}$$

Hence,

$$\begin{aligned}L_{41} &= -2\pi\int_0^T\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2} \rho\sin\phi\, v_\theta\Big[\partial_\phi\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\phi F\Big]\,d\phi\,d\rho\,dt \\ &= -\int_0^T\int_{D_m} \frac{v_\theta}{\rho}\,\Big[\partial_\phi\partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\phi F\Big]\,dx\,dt.\end{aligned} \tag{4.64}$$

For L_{42}, by taking advantage of the fact that $F = 0$ on $\partial^A D_m$, we can again apply integration by parts with vanishing boundary terms to obtain

$$\begin{aligned}\int_{\rho_1}^{\rho_2} \sin\phi\,\partial_\rho(\rho v_\theta)\,\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F\,d\rho &= -\int_{\rho_1}^{\rho_2} \sin\phi\,\rho v_\theta\,\partial_\rho\Big[\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F\Big]\,d\rho \\ &= -\int_{\rho_1}^{\rho_2} \rho\sin\phi\, v_\theta\,\Big[\partial_\rho\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\rho F\Big]\,d\rho.\end{aligned}$$

Hence,

$$\begin{aligned}L_{42} &= -2\pi\int_0^T\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} \rho\sin\phi\, v_\theta\,\Big[\partial_\rho\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\rho F\Big]\,d\rho\,d\phi\,dt \\ &= -\int_0^T\int_{D_m} \frac{v_\theta}{\rho}\,\Big[\partial_\rho\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,F + \partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\rho F\Big]\,dx\,dt.\end{aligned} \tag{4.65}$$

By substituting (4.64) and (4.65) into (4.63), we see that the super-critical terms containing $\partial_\rho\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)$ are canceled out and we find

$$L_4 = \int_0^T\int_{D_m} \frac{v_\theta}{\rho}\,\Big[\partial_\phi\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\rho F - \partial_\rho\Big(\frac{v_\phi}{\rho}\Big)\,\partial_\phi F\Big]\,dx\,dt. \tag{4.66}$$

Finally, putting (4.61), (4.62), (4.56) and (4.66) into (4.53) justifies (4.51). □

In the next lemma, we close the energy estimate for K, F and Ω, which is the key result in this paper. Here, we would like to explain the reason of choosing $\frac{1}{95}$ in (4.67) instead of $\frac{1}{100}$ which is the upper bound for $\Gamma(\cdot,0) := rv_{0,\theta}$ in (1.17) in Theorem 1.5. Actually, in order to prove Theorem 1.5, we need to first choose a sequence of initial data $v_0^{(m)}$ on the approximating domains D_m which converges to v_0, see Page 72 in Section 6. Then the size of $rv_{0,\theta}^{(m)}$ may not be bounded by $\frac{1}{100}$ any more, so we need a slightly larger bound, say $\frac{1}{95}$, to bound $\|rv_{0,\theta}^m\|$ on D_m, see (6.3).

Lemma 4.7. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Let Γ, K, F and Ω be defined as in (2.9) and (2.14). Assume*

$$\|\Gamma(\cdot,0)\|_{L^\infty(D_m)} \leq \frac{1}{95}. \tag{4.67}$$

Then for any $T > 0$,

$$\begin{aligned} &\int_{D_m} (K^2 + F^2 + \Omega^2)(x,T)\,dx + \frac{1}{10}\int_0^T\int_{D_m} \left(|\nabla K|^2 + |\nabla F|^2 + |\nabla\Omega|^2\right) dx\,dt \\ &\leq \int_{D_m} (K^2 + F^2 + \Omega^2)(x,0)\,dx \qquad (4.68) \\ &\leq C\|v_0\|^2_{H^2(D_m)}, \qquad (4.69) \end{aligned}$$

where $C = C(\alpha)$.

Proof. We add (4.50), (4.51) and (4.52) together to obtain

$$\begin{aligned} &\frac{1}{2}\int_{D_m} (K^2 + F^2 + \Omega^2)(x,T)\,dx - \frac{1}{2}\int_{D_m} (K^2 + F^2 + \Omega^2)(x,0)\,dx \\ &\quad + \int_0^T\int_{D_m} |\nabla K|^2 + |\nabla F|^2 + |\nabla\Omega|^2\,dx\,dt = S_1 + S_2 + S_3, \end{aligned} \tag{4.70}$$

where

$$S_1 = 3\int_0^T\int_{D_m} \frac{K^2}{\rho^2}\,dx\,dt - 2\int_0^T\int_{D_m} \frac{K}{\rho}\,\partial_\rho K\,dx\,dt,$$

$$S_2 = \int_0^T\int_{D_m} \frac{1-\cot^2\phi}{\rho^2}\,F^2\,dx\,dt - 2\int_0^T\int_{D_m} \frac{\cot\phi}{\rho^2}\,F\partial_\phi F\,dx\,dt + 2\int_0^T\int_{D_m} \frac{(\partial_\phi K)F}{\rho^2}\,dx\,dt$$

and

$$S_3 = \int_0^T \int_{D_m} \frac{v_\theta}{\rho} \left[\partial_\phi \left(\frac{v_\rho}{\rho} \right) \partial_\rho K - \partial_\rho \left(\frac{v_\rho}{\rho} \right) \partial_\phi K + \partial_\phi \left(\frac{v_\phi}{\rho} \right) \partial_\rho F - \partial_\rho \left(\frac{v_\phi}{\rho} \right) \partial_\phi F \right.$$
$$\left. - \frac{2}{\sin\phi} K\Omega - \frac{2\cos\phi}{\sin^2\phi} F\Omega \right] dx\, dt. \tag{4.71}$$

We first estimate S_1 and S_2. By Cauchy Schwarz inequality, for any $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$, we have

$$S_1 \le \epsilon_1 \int_0^T \int_{D_m} (\partial_\rho K)^2 \, dx\, dt + \left(3 + \frac{1}{\epsilon_1}\right) \int_0^T \int_{D_m} \frac{K^2}{\rho^2} \, dx\, dt, \tag{4.72}$$

and

$$S_2 \le \epsilon_2 \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho} \right)^2 dx\, dt + \epsilon_3 \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho} \right)^2 dx\, dt$$
$$+ \left(1 + \frac{1}{\epsilon_2}\right) \int_0^T \int_{D_m} \frac{F^2}{\rho^2} \, dx\, dt + \left(\frac{1}{\epsilon_3} - 1\right) \int_0^T \int_{D_m} \frac{\cot^2\phi}{\rho^2} F^2 \, dx\, dt.$$

Since $\alpha \in \left(0, \frac{\pi}{6}\right]$, then $\phi \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ and $\cot^2\phi \le \frac{1}{3}$. Therefore,

$$S_2 \le \epsilon_2 \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho} \right)^2 dx\, dt + \epsilon_3 \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho} \right)^2 dx\, dt$$
$$+ \left(\frac{2}{3} + \frac{1}{\epsilon_2} + \frac{1}{3\epsilon_3}\right) \int_0^T \int_{D_m} \frac{F^2}{\rho^2} \, dx\, dt. \tag{4.73}$$

Adding (4.72) and (4.73) together leads to

$$S_1 + S_2 \le \epsilon_1 \int_0^T \int_{D_m} (\partial_\rho K)^2 \, dx\, dt + \epsilon_2 \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho} \right)^2 dx\, dt + \epsilon_3 \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho} \right)^2 dx\, dt$$
$$+ \left(3 + \frac{1}{\epsilon_1}\right) \int_0^T \int_{D_m} \frac{K^2}{\rho^2} \, dx\, dt + \left(\frac{2}{3} + \frac{1}{\epsilon_2} + \frac{1}{3\epsilon_3}\right) \int_0^T \int_{D_m} \frac{F^2}{\rho^2} \, dx\, dt. \tag{4.74}$$

According to Lemma 2.1, $K = 0$ on $\partial^R D_m$, so similar to the derivation in (4.33), we deduce

$$\int_{D_m} \frac{K^2}{\rho^2}\,dx \le C_{\alpha,B} \int_{D_m} \left(\frac{\partial_\phi K}{\rho}\right)^2 dx. \tag{4.75}$$

On the other hand, since v_θ is odd with respect to $\{\phi = \frac{\pi}{2}\}$, then we can derive from (2.12) that ω_ϕ is also odd with respect to $\{\phi = \frac{\pi}{2}\}$. Thus, F is odd with respect to $\{\phi = \frac{\pi}{2}\}$, which implies

$$\int_{\phi_1}^{\phi_2} F \sin\phi\, d\phi = 0.$$

Then analogous to the estimate in (4.25), we get

$$\int_{D_m} \frac{F^2}{\rho^2}\,dx \le C_{\alpha,A} \int_{D_m} \left(\frac{\partial_\phi F}{\rho}\right)^2 dx. \tag{4.76}$$

Putting (4.75) and (4.76) into (4.74), and recalling $C_{\alpha,A} \le \frac{2}{19}$ and $C_{\alpha,B} \le \frac{3}{25}$, we obtain

$$\begin{aligned} S_1 + S_2 \le{}& \epsilon_1 \int_0^T \int_{D_m} (\partial_\rho K)^2\,dx\,dt + \epsilon_2 \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho}\right)^2 dx\,dt + \epsilon_3 \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho}\right)^2 dx\,dt \\ &+ \frac{3}{25}\left(3 + \frac{1}{\epsilon_1}\right) \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho}\right)^2 dx\,dt \\ &+ \frac{2}{19}\left(\frac{2}{3} + \frac{1}{\epsilon_2} + \frac{1}{3\epsilon_3}\right) \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho}\right)^2 dx\,dt. \end{aligned} \tag{4.77}$$

By choosing $\epsilon_1 = \frac{9}{10}$, $\epsilon_2 = \frac{1}{3}$ and $\epsilon_3 = \frac{1}{5}$, we conclude

$$\begin{aligned} S_1 + S_2 &\le \frac{9}{10} \int_0^T \int_{D_m} (\partial_\rho K)^2 + \left(\frac{\partial_\phi K}{\rho}\right)^2 + \left(\frac{\partial_\phi F}{\rho}\right)^2 dx\,dt \\ &\le \frac{9}{10} \int_0^T \int_{D_m} |\nabla K|^2 + |\nabla F|^2\,dx\,dt. \end{aligned} \tag{4.78}$$

Next, we estimate S_3. Denote $C^* = \frac{1}{95}$, then the assumption on Γ becomes $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)} \le C^*$. This enables one to derive from Lemma 2.10 that

$$\|\Gamma\|_{L^\infty(D_m\times(0,T))} \le C^*.$$

Recalling $\Gamma = \rho \sin\phi \, v_\theta$ and $\alpha \in \left(0, \frac{\pi}{6}\right]$, so $\phi \in \left[2\pi/3, 4\pi/3\right]$ and

$$|v_\theta| \le \frac{C^*}{\rho \sin\phi} \le \frac{2}{\sqrt{3}} \frac{C^*}{\rho}. \tag{4.79}$$

Combining (4.79) with (4.71) yields

$$|S_3| \le \frac{2C^*}{\sqrt{3}} \left(S_{31} + S_{32} + S_{33}\right), \tag{4.80}$$

where

$$S_{31} = \int_0^T \int_{D_m} \frac{1}{\rho} \left| \frac{1}{\rho} \partial_\phi \left(\frac{v_\rho}{\rho} \right) \partial_\rho K \right| + \frac{1}{\rho} \left| \partial_\rho \left(\frac{v_\rho}{\rho} \right) \frac{1}{\rho} \partial_\phi K \right| dx\, dt,$$

$$S_{32} = \int_0^T \int_{D_m} \frac{1}{\rho} \left| \frac{1}{\rho} \partial_\phi \left(\frac{v_\phi}{\rho} \right) \partial_\rho F \right| + \frac{1}{\rho} \left| \partial_\rho \left(\frac{v_\phi}{\rho} \right) \frac{1}{\rho} \partial_\phi F \right| dx\, dt,$$

$$S_{33} = \frac{4}{\sqrt{3}} \int_0^T \int_{D_m} \frac{1}{\rho^2} |K\Omega| \, dx\, dt + \frac{4}{3} \int_0^T \int_{D_m} \frac{1}{\rho^2} |F\Omega| \, dx\, dt.$$

By using Cauchy-Schwarz inequality,

$$\begin{aligned} S_{31} &\le \int_0^T \int_{D_m} \left| \frac{1}{\rho} \nabla \left(\frac{v_\rho}{\rho} \right) \right| |\nabla K| \, dx\, dt \\ &\le \int_0^T \left\| \frac{1}{\rho} \nabla \left(\frac{v_\rho}{\rho} (\cdot, t) \right) \right\|_{L^2(D_m)} \|\nabla K(\cdot, t)\|_{L^2(D_m)} \, dt. \end{aligned}$$

Now applying Lemma 4.4 and Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} S_{31} &\le \sqrt{44} \int_0^T \|\nabla \Omega(\cdot, t)\|_{L^2(D_m)} \|\nabla K(\cdot, t)\|_{L^2(D_m)} \, dt \\ &\le \epsilon_4 \int_0^T \int_{D_m} |\nabla K|^2 \, dx\, dt + \frac{11}{\epsilon_4} \int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx\, dt, \end{aligned} \tag{4.81}$$

where ϵ_4 is any positive number. In a similar manner by using Cauchy-Schwarz inequality and Lemma 4.5, we have

$$S_{32} \le \epsilon_5 \int_0^T \int_{D_m} |\nabla F|^2 \, dx \, dt + \frac{100}{\epsilon_5} \int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx \, dt, \tag{4.82}$$

where ϵ_5 is any positive number. For S_{33}, it directly follows from Cauchy-Schwarz inequality that

$$S_{33} \le \int_0^T \int_{D_m} \frac{K^2}{\rho^2} \, dx \, dt + \int_0^T \int_{D_m} \frac{F^2}{\rho^2} \, dx \, dt + \frac{16}{9} \int_0^T \int_{D_m} \frac{\Omega^2}{\rho^2} \, dx \, dt. \tag{4.83}$$

Based on (4.75), we know

$$\int_0^T \int_{D_m} \frac{K^2}{\rho^2} \, dx \, dt \le C_{\alpha,B} \int_0^T \int_{D_m} \left(\frac{\partial_\phi K}{\rho} \right)^2 dx \, dt \le \frac{3}{25} \int_0^T \int_{D_m} |\nabla K|^2 \, dx \, dt.$$

Similarly,

$$\int_0^T \int_{D_m} \frac{\Omega^2}{\rho^2} \, dx \, dt \le \frac{3}{25} \int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx \, dt.$$

On the other hand, according to (4.76) and the assumption that $\alpha \in \big(0, \frac{\pi}{6}\big]$, we attain

$$\int_0^T \int_{D_m} \frac{F^2}{\rho^2} \, dx \, dt \le C_{\alpha,A} \int_0^T \int_{D_m} \left(\frac{\partial_\phi F}{\rho} \right)^2 dx \, dt \le \frac{2}{19} \int_0^T \int_{D_m} |\nabla F|^2 \, dx \, dt.$$

Plugging the above estimates into (4.83) yields

$$S_{33} \le \frac{3}{25} \int_0^T \int_{D_m} |\nabla K|^2 \, dx \, dt + \frac{2}{19} \int_0^T \int_{D_m} |\nabla F|^2 \, dx \, dt + \frac{16}{75} \int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx \, dt. \tag{4.84}$$

Putting (4.81), (4.82) and (4.84) into (4.80) leads to

$$\begin{aligned} |S_3| \le \frac{2C^*}{\sqrt{3}} \Bigg[& \left(\epsilon_4 + \frac{3}{25} \right) \int_0^T \int_{D_m} |\nabla K|^2 \, dx \, dt + \left(\epsilon_5 + \frac{2}{19} \right) \int_0^T \int_{D_m} |\nabla F|^2 \, dx \, dt \\ & + \left(\frac{11}{\epsilon_4} + \frac{100}{\epsilon_5} + \frac{16}{75} \right) \int_0^T \int_{D_m} |\nabla \Omega|^2 \, dx \, dt \Bigg]. \end{aligned}$$

By choosing $\epsilon_4 = \epsilon_5 = 3$, we derive from the above inequality that

$$|S_3| \le \frac{2C^*}{\sqrt{3}}\Big(4\int_0^T\int_{D_m} |\nabla K|^2 + |\nabla F|^2\,dx\,dt + 40\int_0^T\int_{D_m} |\nabla\Omega|^2\,dx\,dt\Big).$$

Recalling $C^* = \frac{1}{95}$, so the above estimate implies that

$$|S_3| \le \frac{1}{20}\int_0^T\int_{D_m} |\nabla K|^2 + |\nabla F|^2\,dx\,dt + \frac{1}{2}\int_0^T\int_{D_m} |\nabla\Omega|^2\,dx\,dt. \tag{4.85}$$

Finally, by plugging (4.78) and (4.85) into (4.70), we conclude that

$$\begin{aligned}&\frac{1}{2}\int_{D_m}(K^2+F^2+\Omega^2)(x,T)\,dx - \frac{1}{2}\int_{D_m}(K^2+F^2+\Omega^2)(x,0)\,dx\\ &+\frac{1}{20}\int_0^T\int_{D_m}|\nabla K|^2+|\nabla F|^2+|\nabla\Omega|^2\,dx\,dt \le 0,\end{aligned}$$

which implies (4.68).

Now it remains to justify (4.69), we first use the Poincaré inequality in Corollary 2.6 and the fact that $\omega_\rho = 0$ on $\partial^R D_m$ to establish

$$\int_{D_m} K^2(x,0)\,dx = \int_{D_m}\frac{\omega_{0,\rho}^2}{\rho^2}\,dx \le C\int_{D_m}\Big|\frac{1}{\rho}\partial_\phi\omega_{0,\rho}\Big|^2\,dx \le C\|v_0\|^2_{H^2(D_m)},$$

where $C = C(\alpha)$. Then the term $\int_{D_m}\Omega^2(x,0)\,dx$ can be handled in the same way. In order to treat F, we take advantage of the property that ω_ϕ is odd with respect to $\{\phi = \frac{\pi}{2}\}$ and then use the Poincaré inequality in Corollary 2.4 to obtain

$$\int_{D_m} F^2(x,0)\,dx = \int_{D_m}\frac{\omega_{0,\phi}^2}{\rho^2}\,dx \le C\int_{D_m}\Big|\frac{1}{\rho}\partial_\phi\omega_{0,\phi}\Big|^2\,dx \le C\|v_0\|^2_{H^2(D_m)}.$$

Hence, (4.69) is verified. □

4.6. A uniform bound for $\|v/\rho\|_{L_t^\infty L_x^6}$

In the previous Section 4.3 and Section 4.4, we have used the first two relations in the Biot-Savart law (4.12) to obtain estimates on some norms about v_ρ and v_ϕ in Lemma 4.4 and Lemma 4.5 via ω_θ. Now we will use the third relation in (4.12) to deduce similar estimates about v_θ via ω_ρ and ω_ϕ.

Lemma 4.8. *Let the region D_m be as defined in (2.17) with $m \ge 2$ and the angle $\alpha \in (0, \frac{\pi}{6}]$. Then for any $T > 0$ and for a.e. $t \in [0,T]$,*

$$\left\|\nabla\Big(\frac{v_\theta}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \le 5\left(\|K(\cdot,t)\|_{L^2(D_m)} + \|F(\cdot,t)\|_{L^2(D_m)}\right), \tag{4.86}$$

$$\left\|\frac{1}{\rho}\nabla\Big(\frac{v_\theta}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \le 2\sqrt{3}\left(\|\nabla K(\cdot,t)\|_{L^2(D_m)} + \|\nabla F(\cdot,t)\|_{L^2(D_m)}\right). \tag{4.87}$$

Proof. Since $v \in E^{\sigma,s}_{m,T} \cap L^2_t H^2_x \cap L^\infty_{tx}(D_m \times [0,T])$ and ρ has the lower bound $\frac{1}{m}$ on D_m, we know $\omega_\rho, \omega_\phi, K, F \in L^2_t H^1_x(D_m \times [0,T])$. So there exists a set $S_T \subset [0,T]$ such that $[0,T] \setminus S_T$ has measure 0 and for any $t \in S_T$, all of $\omega_\rho, \omega_\phi, K, F$ belong to $H^1(D_m)$. Fixing any $t \in S_T$, it suffices to prove (4.86) for such t. For convenience of notation, we will drop all the temporal variables in the following proof.

Recall that the third equation in the Biot-Savart law (4.12) reads

$$\left(\Delta - \frac{1}{\rho^2 \sin^2\phi}\right) v_\theta = -\frac{1}{\rho}\partial_\rho(\rho\omega_\phi) + \frac{1}{\rho}\partial_\phi\omega_\rho \quad \text{in} \quad D_m.$$

Denote $g = \frac{v_\theta}{\rho}$. Then it follows from the above equation and the boundary condition for v_θ in Lemma 2.1 that

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\right) g = -\frac{1}{\rho^2}\partial_\rho(\rho^2 F) + \frac{1}{\rho}\partial_\phi K \quad \text{in} \quad D_m, \\ \partial_\phi g = -(\cot\phi) g \quad \text{on} \quad \partial^R D_m, \quad \partial_\rho g = -\frac{2}{\rho} g \quad \text{on} \quad \partial^A D_m. \end{cases} \tag{4.88}$$

Testing (4.88) by g on D_m, then it follows from the integration by parts and the previous trick of converting boundary integrals into interior integrals that

$$\begin{aligned} &\int_{D_m} |\nabla g|^2 dx + \int_{D_m} \frac{\cot^2\phi}{\rho^2} g^2\, dx \\ &= -2\int_{D_m} \frac{1}{\rho} g\partial_\rho g\, dx - 2\int_{D_m} \frac{\cot\phi}{\rho^2} g\partial_\phi g\, dx + \int_{D_m} \left(\frac{K}{\rho}\partial_\phi g + \frac{K\cot\phi}{\rho} g - F\partial_\rho g\right) dx. \end{aligned} \tag{4.89}$$

By Cauchy-Schwarz inequality, for any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$,

$$\begin{aligned} \text{RHS of (4.89)} \le\ & \epsilon_1 \int_{D_m} (\partial_\rho g)^2\, dx + \frac{1}{\epsilon_1}\int_{D_m} \frac{g^2}{\rho^2}\, dx + \epsilon_2 \int_{D_m} \left(\frac{1}{\rho}\partial_\phi g\right)^2 dx \\ & + \frac{1}{\epsilon_2}\int_{D_m} \frac{\cot^2\phi}{\rho^2} g^2\, dx + \epsilon_3 \int_{D_m} \left[\left(\frac{1}{\rho}\partial_\phi g\right)^2 + \frac{\cot^2\phi}{\rho^2} g^2 + (\partial_\rho g)^2\right] dx \\ & + \frac{1}{\epsilon_3}\int_{D_m} \left(\frac{1}{2}K^2 + \frac{1}{4}F^2\right) dx. \end{aligned}$$

Since g is odd with respect to the plane $\{\phi = \frac{\pi}{2}\}$, $\int_{\pi/2-\alpha}^{\pi/2+\alpha} g(\rho, \phi)\, d\phi = 0$ for any ρ. As a consequence, it follows from the weighted Poincaré inequality in Corollary 2.4 that for any $0 < \alpha \le \frac{\pi}{6}$,

$$\int_{D_m} \frac{g^2}{\rho^2}\, dx \le \frac{2}{19} \int_{D_m} \left(\frac{1}{\rho}\, \partial_\phi g\right)^2 dx$$

$$\int_{D_m} \frac{\cot^2\phi}{\rho^2} g^2\, dx \le \tan^2\alpha \int_{D_m} \frac{g^2}{\rho^2} dx \le \frac{2}{57} \int_{D_m} \left(\frac{1}{\rho}\partial_\phi g\right)^2 dx.$$

Therefore.

$$\begin{aligned}\text{RHS of (4.89)}\ &\le (\epsilon_1 + \epsilon_3) \int_{D_m} (\partial_\rho g)^2\, dx + \left(\frac{2}{19\epsilon_1} + \epsilon_2 + \frac{2}{57\epsilon_2} + \frac{59}{57}\epsilon_3\right) \int_{D_m} \left(\frac{1}{\rho}\partial_\phi g\right)^2 dx \\ &+ \frac{1}{\epsilon_3} \int_{D_m} \left(\frac{1}{2}K^2 + \frac{1}{4}F^2\right) dx.\end{aligned}$$

Choosing $\epsilon_1 = \frac{3}{5}, \epsilon_2 = \frac{1}{5}$ and $\epsilon_3 = \frac{1}{10}$. Then

$$\text{RHS of (4.89)}\ \le 0.7 \int_{D_m} |\nabla g|^2 dx + 5 \int_{D_m} (K^2 + F^2)\, dx.$$

As a result,

$$\int_{D_m} |\nabla g|^2\, dx \le \frac{50}{3} \int_{D_m} (K^2 + F^2)\, dx,$$

which implies (4.86).

Next, we will verify (4.87). Testing (4.88) by $\frac{1}{\rho^2} g$ on D_m, then it follows from the integration by parts and the previous trick of converting boundary integrals into interior integrals that

$$\begin{aligned}\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx &= \int_{D_m} \frac{4 - \cot^2\phi}{\rho^4} g^2 dx - 2 \int_{D_m} \frac{\cot\phi}{\rho^4} g\partial_\phi g dx \\ &+ \int_{D_m} \left(\frac{1}{\rho}\partial_\rho g - \frac{2g}{\rho^2}\right) \frac{F}{\rho}\, dx + \int_{D_m} \left(\frac{1}{\rho^2}\partial_\phi g + \frac{\cot\phi}{\rho^2} g\right) \frac{K}{\rho}\, dx.\end{aligned}$$

Then for any $\epsilon_1, \epsilon_2 \in (0, 1)$, we apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\int_{D_m}\left|\frac{1}{\rho}\nabla g\right|^2 dx \le 4\int_{D_m}\frac{g^2}{\rho^4}\,dx + \left(\frac{1}{\epsilon_1}-1\right)\int_{D_m}\frac{\cot^2\phi}{\rho^4}g^2\,dx + \epsilon_1\int_{D_m}\left(\frac{1}{\rho^2}\partial_\phi g\right)^2 dx \\ + \epsilon_2\int_{D_m}\left[\left(\frac{1}{\rho}\partial_\rho g - \frac{2g}{\rho^2}\right)^2 + \left(\frac{1}{\rho^2}\partial_\phi g + \frac{\cot\phi}{\rho}g\right)^2\right] dx \\ + \frac{1}{4\epsilon_2}\int_{D_m}\frac{F^2+K^2}{\rho^2}\,dx.\end{aligned} \tag{4.90}$$

By Cauchy-Schwarz inequality again,

$$\begin{aligned}&\epsilon_2\int_{D_m}\left[\left(\frac{1}{\rho}\partial_\rho g - \frac{2g}{\rho^2}\right)^2 + \left(\frac{1}{\rho^2}\partial_\phi g + \frac{\cot\phi}{\rho}g\right)^2\right] dx \\ &\le 2\epsilon_2\int_{D_m}\left[\left(\frac{1}{\rho}\partial_\rho g\right)^2 + \frac{4g^2}{\rho^4} + \left(\frac{1}{\rho^2}\partial_\phi g\right)^2 + \frac{\cot^2\phi}{\rho^4}g^2\right] dx.\end{aligned} \tag{4.91}$$

Plugging (4.91) into (4.90) yields

$$\begin{aligned}\int_{D_m}\left|\frac{1}{\rho}\nabla g\right|^2 dx \le 2\epsilon_2\int_{D_m}\left(\frac{1}{\rho}\partial_\rho g\right)^2 dx + (\epsilon_1+2\epsilon_2)\int_{D_m}\left(\frac{1}{\rho^2}\partial_\phi g\right)^2 dx + (4+8\epsilon_2)\int_{D_m}\frac{g^2}{\rho^4}\,dx \\ + \left(\frac{1}{\epsilon_1}-1+2\epsilon_2\right)\int_{D_m}\frac{\cot^2\phi}{\rho^4}g^2\,dx + \frac{1}{4\epsilon_2}\int_{D_m}\frac{F^2+K^2}{\rho^2}\,dx.\end{aligned} \tag{4.92}$$

By choosing $\epsilon_1 = \frac{1}{5}$ and $\epsilon_2 = \frac{1}{40}$, and noticing $\cot^2\phi \le \frac{1}{3}$, we obtain

$$\begin{aligned}\int_{D_m}\left|\frac{1}{\rho}\nabla g\right|^2 dx \le \frac{1}{20}\int_{D_m}\left(\frac{1}{\rho}\partial_\rho g\right)^2 dx + \frac{1}{4}\int_{D_m}\left(\frac{1}{\rho^2}\partial_\phi g\right)^2 dx + 6\int_{D_m}\frac{g^2}{\rho^4}\,dx \\ + 10\int_{D_m}\frac{F^2+K^2}{\rho^2}\,dx.\end{aligned} \tag{4.93}$$

Since v_θ is odd with respect to $\{\phi = \frac{\pi}{2}\}$, it then follows from the Poincaré inequality in Lemma 2.3 that

$$\int_{D_m}\frac{g^2}{\rho^4}\,dx \le \frac{2}{19}\int_{D_m}\left(\frac{1}{\rho^2}\partial_\phi g\right)^2 dx. \tag{4.94}$$

Putting (4.94) into (4.93) leads to

$$\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx \le \frac{9}{10}\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx + 10\int_{D_m} \frac{F^2+K^2}{\rho^2}\, dx,$$

which implies

$$\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx \le 100\int_{D_m} \frac{F^2+K^2}{\rho^2}\, dx.$$

Combining this estimate with Poincaré inequalities in Lemma 2.3 and Lemma 2.5, we find

$$\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx \le 100\left[\frac{2}{19}\int_{D_m} \left(\frac{1}{\rho}\partial_\phi F\right)^2 dx + \frac{3}{25}\int_{D_m} \left(\frac{1}{\rho}\partial_\phi K\right)^2 dx\right].$$

So

$$\int_{D_m} \left|\frac{1}{\rho}\nabla g\right|^2 dx \le 12\Big(\int_{D_m} |\nabla F|^2\, dx + \int_{D_m} |\nabla K|^2\, dx\Big),$$

which results in (4.87). $\square$

Before estimating the $L_t^\infty L_x^6$ norms of v_ρ/ρ, v_ϕ/ρ and v_θ/ρ, we need a uniform Sobolev embedding on regions $\{D_m\}_{m\ge 2}$. The key point here is that the embedding constant s_0 in (4.95) is independent of m. Since the regions $\{D_m\}_{m\ge 2}$ are Lipschitz and their limiting region, as $m\to\infty$, is also Lipschitz, the embedding result is essentially known. But for completeness, we still give a short illustration based on [1].

Lemma 4.9. *Let D_m be the region in (2.17) with $m \ge 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Then there exist two constants s_0 and s_1, which depend on α but are independent of m, such that the following two estimates hold.*

(a) For any $f \in W^{1,2}(D_m)$, $\|f\|_{L^6(D_m)} \le s_0\|f\|_{H^1(D_m)}$.
(b) For any $f \in W^{1,2}(D_m)$ such that either $f = 0$ on $\partial^R D_m$ or $\int_{\pi/2-\alpha}^{\pi/2+\alpha} f(\rho,\phi)\sin\phi\, d\phi = 0$ for any $\rho \in \left(\frac{1}{m}, 1\right)$,

$$\|f\|_{L^6(D_m)} \le s_1\|\nabla f\|_{L^2(D_m)}. \tag{4.95}$$

Proof. Recall the *cone condition* in Definition 4.6 on Page 82 in [1]: a domain Ω satisfies the *cone condition* if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

Based on the above definition, it is readily seen that for any $m \ge 2$, D_m satisfies the cone condition. Moreover, the cone C in the cone condition for D_m can be chosen as a uniform one (i.e. independent of m) since all D_m share the same angle α.

Now we recall Theorem 4.12 (Part I, Case C) on Page 85 in [1] which implies that if $\Omega \in \mathbb{R}^3$ satisfies the cone condition, then $H^1(\Omega)$ is embedded in $L^6(\Omega)$, where the embedding constant only depends on the dimensions of the cone C in the cone condition.

Thanks to this theorem and the fact that the cone C in the cone condition for D_m is uniform, we can find a constant s_0, which only depends on α, such that

$$\|f\|_{L^6(D_m)} \le s_0 \|f\|_{H^1(D_m)}. \tag{4.96}$$

This justifies part (a).

For part (b), due to the extra assumption (i) or (ii) and the restriction $\alpha \le \pi/6$, we are allowed to apply Poincaré inequality in the ϕ direction to conclude

$$\begin{aligned}
\int\limits_{D_m} f^2(x)\,dx &= 2\pi \int\limits_{1/m}^{1} \rho^2 \int\limits_{\pi/2-\alpha}^{\pi/2+\alpha} f^2(\rho,\phi)\sin\phi\,d\rho\,d\phi \\
&\le 2\pi\lambda_1 \int\limits_{1/m}^{1} \rho^2 \int\limits_{\pi/2-\alpha}^{\pi/2+\alpha} (\partial_\phi f)^2(\rho,\phi)\sin\phi\,d\rho\,d\phi \\
&\le \lambda_1 \int\limits_{D_m} |\nabla f(x)|^2\,dx,
\end{aligned}$$

where λ_1 is a constant that only depends on α. Combining this inequality with (4.96) leads to (4.95). $\square$

Now we can take advantage of the above Sobolev embedding to control the $L_t^\infty L_x^6$ norms of v_ρ/ρ, v_ϕ/ρ and v_θ/ρ.

Lemma 4.10. *Let the region D_m be as defined in (2.17) with $m \ge 2$ and the angle $\alpha \in \big(0, \frac{\pi}{6}\big]$. Then there exists some constant $C = C(\alpha)$ such that for any $T > 0$,*

$$\left\| \frac{|v_\rho| + |v_\phi| + |v_\theta|}{\rho} \right\|_{L_t^\infty L_x^6(D_m\times[0,T])} \le C \big\| |K| + |F| + |\Omega| \big\|_{L_t^\infty L_x^2(D_m\times[0,T])}. \tag{4.97}$$

Proof. Firstly, it follows from Lemma 4.4, Lemma 4.5 and Lemma 4.8 that for a.e. $t \in [0,T]$,

$$\begin{aligned}
&\left\|\nabla\Big(\frac{v_\rho}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} + \left\|\nabla\Big(\frac{v_\phi}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} + \left\|\nabla\Big(\frac{v_\theta}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)} \\
&\le 5\left(\|\Omega(\cdot,t)\|_{L^2(D_m)} + \|K(\cdot,t)\|_{L^2(D_m)} + \|F(\cdot,t)\|_{L^2(D_m)}\right).
\end{aligned}$$

Next, due to the property (4.15) of v_ρ, the boundary condition of v_ϕ on $\partial^R D_m$, and the odd-symmetry of v_θ with respect to $\{\phi = \frac{\pi}{2}\}$, we can apply part (b) in Lemma 4.9 to conclude

$$\begin{aligned}&\left\|\frac{v_\rho}{\rho}(\cdot,t)\right\|_{L^6(D_m)}+\left\|\frac{v_\phi}{\rho}(\cdot,t)\right\|_{L^6(D_m)}+\left\|\frac{v_\theta}{\rho}(\cdot,t)\right\|_{L^6(D_m)}\\ \le{}& C\left(\left\|\nabla\Big(\frac{v_\rho}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)}+\left\|\nabla\Big(\frac{v_\phi}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)}+\left\|\nabla\Big(\frac{v_\theta}{\rho}(\cdot,t)\Big)\right\|_{L^2(D_m)}\right).\end{aligned}$$

Combining the above two estimates leads to (4.97). □

4.7. Uniform bounds for $\|v\|_{L^\infty_{tx}}$ and $\|\omega_\theta\|_{L^\infty_{tx}}$

The goal of this subsection is to obtain uniform bounds on $\|v\|_{L^\infty_{tx}}$ and $\|\omega_\theta\|_{L^\infty_{tx}}$ which are independent of the time T and only dependent on α and the initial value.

4.7.1. L^∞ boundedness of v_θ

We first derive an upper bound for the supremum norm of v_θ.

Proposition 4.11. *Let the region D_m be as defined in (2.17) with $m \ge 2$ and the angle $\alpha \in \big(0, \frac{\pi}{6}\big]$. Then for any $T > 0$,*

$$\|v_\theta\|_{L^\infty_{tx}(D_m\times[0,T])} \le CC_*^5\Big(\|v_0\|_{L^2(D_m)} + \|v_{0,\theta}\|_{L^\infty(D_m)} + 1\Big), \tag{4.98}$$

where $C = C(\alpha)$ and

$$C_* = 2 + \left\|\frac{|v_\theta| + |v_\rho| + |v_\phi|}{\rho}\right\|_{L^\infty_t L^6_x(D_m\times[0,T])}. \tag{4.99}$$

Proof. Fix any $T > 0$ and let $\eta : [0,T] \to [0,1]$ be a smooth function in the time-variable. The specific choice of η will be determined later. For any rational number $q \ge 1$ in the form of $(2k-1)/(2l-1)$, where k and l are positive integers, denote

$$f = v_\theta^q.$$

Based on equation (3.5) for v_θ, we know f solves the following problem:

$$\begin{cases}\Delta f - q(q-1)v_\theta^{q-2}\,|\nabla v_\theta|^2 - \frac{q}{\rho^2\sin^2\phi}f - b\cdot\nabla f - \frac{q}{\rho}(v_\rho + \cot\phi\, v_\phi)f - \partial_t f = 0\\ \quad \text{in } D_m\times(0,T];\\ \partial_\phi f = -q\cot\phi\, f \ \text{ on } \partial^R D_m\times(0,T], \quad \partial_\rho f = -\frac{q}{\rho}f \ \text{ on } \partial^A D_m\times(0,T];\\ f(x,0) = v_{0,\theta}^q(x), \quad x\in D_m.\end{cases} \tag{4.100}$$

For any $t \in (0,T]$, we test (4.100) by $\eta^2 f$ on $D_m\times[0,t]$. By using integration by parts and then converting the boundary integral into the interior integral, we find

$$\int_0^t \eta^2 \int_{D_m} f\Delta f\,dx\,d\tau = -\int_0^t \eta^2 \int_{D_m} |\nabla f|^2\,dx\,d\tau - 2q\int_0^t \eta^2 \int_{D_m} \frac{1}{\rho} f(\partial_\rho f)\,dx\,d\tau$$
$$- 2q\int_0^t \eta^2 \int_{D_m} \frac{\cot\phi}{\rho^2} f(\partial_\phi f)\,dx\,d\tau.$$

As a result, we obtain

$$\begin{aligned} &\frac{2q-1}{q}\int_0^t \int_{D_m} |(\nabla f)\eta|^2\,dx\,d\tau + q\int_0^t \int_{D_m} \frac{1}{\rho^2\sin^2\phi} f^2\eta^2\,dx\,d\tau + \frac{1}{2}\eta^2(t)\int_{D_m} f^2(x,t)\,dx \\ &= \underbrace{-2q\int_0^t \int_{D_m} \frac{1}{\rho} f(\partial_\rho f)\eta^2\,dx\,d\tau - 2q\int_0^t \int_{D_m} \frac{\cot\phi}{\rho^2} f(\partial_\phi f)\eta^2\,dx\,d\tau}_{R_1} \\ &\quad - q\int_0^t \int_{D_m} \frac{v_\rho + \cot\phi\, v_\phi}{\rho} f^2\eta^2\,dx\,d\tau + \frac{1}{2}\eta^2(0)\int_{D_m} f^2(x,0)\,dx + \int_0^t \int_{D_m} f^2\eta\eta'\,dx\,d\tau. \end{aligned} \tag{4.101}$$

Note when deriving the above equation, we used the fact that $\int_{D_m} (b\cdot\nabla f)f\,dx = 0$ due to the incompressibility and the boundary condition of b. Using Cauchy-Schwarz inequality, we find

$$\begin{aligned} |R_1| &\le 2q\int_0^t \int_{D_m} \left|\frac{1}{\rho} f\partial_\rho f\right| \eta^2 dx\,d\tau + 2q\int_0^t \int_{D_m} \left|\frac{\cot\phi}{\rho^2} f\partial_\phi f\right| \eta^2 dx\,d\tau \\ &\le \frac{1}{2}\int_0^t \int_{D_m} |(\nabla f)\eta|^2 dx\,d\tau + 2q^2\int_0^t \int_{D_m} \left(\frac{f^2\eta^2}{\rho^2} + \frac{\cot^2\phi}{\rho^2} f^2\eta^2\right) dx\,d\tau. \end{aligned}$$

When $\alpha \in \left(0, \frac{\pi}{6}\right]$, $\cot^2\phi \le \tan^2\alpha \le \frac{1}{3}$, so

$$|R_1| \le \frac{1}{2}\int_0^t \int_{D_m} |(\nabla f)\eta|^2\,dx\,d\tau + \frac{8}{3}q^2\int_0^t \int_{D_m} \frac{1}{\rho^2} f^2\eta^2\,dx\,d\tau.$$

Combining with (4.101) and noticing $\frac{2q-1}{q} \ge 1$, we deduce

$$\frac{1}{2}\int_0^t \int_{D_m} |(\nabla f)\eta|^2\,dx\,d\tau + \frac{1}{2}\eta^2(t)\int_{D_m} f^2(x,t)\,dx$$

$$\leq \frac{8}{3}q^2 \int_0^t \int_{D_m} \frac{1}{\rho^2} f^2 \eta^2 dx\, d\tau + q \int_0^t \int_{D_m} \frac{|v_\rho| + |v_\phi|}{\rho} f^2 \eta^2 dx\, d\tau$$

$$+ \frac{1}{2}\eta^2(0) \int_{D_m} f^2(x,0)dx + \int_0^t \int_{D_m} f^2 \left|\eta\eta'\right| dx\, d\tau.$$

Taking supremum norm with respect to $t \in [0,T]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_{D_m} |(\nabla f)\eta|^2 \, dx\, d\tau + \frac{1}{2} \sup_{t\in[0,T]} \int_{D_m} f^2(x,t)\eta^2(t)\, dx \\
&\leq \frac{16}{3}q^2 \int_0^T \int_{D_m} \frac{1}{\rho^2} f^2 \eta^2 dx\, d\tau + 2q \int_0^T \int_{D_m} \frac{|v_\rho| + |v_\phi|}{\rho} f^2 \eta^2 dx\, d\tau \\
&\quad + \eta^2(0) \int_{D_m} f^2(x,0)dx + 2\int_0^T \int_{D_m} f^2 \left|\eta\eta'\right| dx\, d\tau.
\end{aligned} \tag{4.102}$$

Since v_θ is odd with respect to $\{\phi = \frac{\pi}{2}\}$ and q is in the form of $(2k-1)/(2l-1)$, where k and l are positive integers, we know f is also odd with respect to $\{\phi = \frac{\pi}{2}\}$. Therefore, it follows from part (b) in Lemma 4.9 that

$$\|f(\cdot,\tau)\|_{L^6_x(D_m)} \leq s_1 \left\|\nabla\big(f(\cdot,\tau)\big)\right\|_{L^2_x(D_m)}, \quad \forall\, \tau \in [0,T],$$

where s_1 is some constant that only depends on α. Hence, it follows from (4.102) that

$$\begin{aligned}
&\frac{1}{s_1^2}\|f\eta\|^2_{L^2_t L^6_x(D_m\times[0,T])} + \|f\eta\|^2_{L^\infty_t L^2_x(D_m\times[0,T])} \\
&\leq \frac{32}{3}\, q^2 \left\|\frac{f\eta}{\rho}\right\|^2_{L^2_{tx}(D_m\times[0,T])} + 4q \left\|\frac{|v_\rho| + |v_\phi|}{\rho} f^2\eta^2\right\|_{L^1_{tx}(D_m\times[0,T])} \\
&\quad + 2\eta^2(0)\, \|f(\cdot,0)\|^2_{L^2(D_m)} + 4\|f^2\eta\eta'\|_{L^1_{tx}(D_m\times[0,T])}.
\end{aligned} \tag{4.103}$$

Denote C_* as in (4.99) and define h as

$$h = |v_\theta|^q \vee 1 = (|v_\theta| \vee 1)^q, \tag{4.104}$$

where "$\vee$" means "max". Then

$$\begin{aligned}
\left\|\frac{f\eta}{\rho}\right\|_{L^2_{tx}(D_m\times[0,T])} &= \left\|\frac{v_\theta}{\rho}\cdot v_\theta^{q-1}\eta\right\|_{L^2_{tx}(D_m\times[0,T])} \\
&\leq \left\|\frac{v_\theta}{\rho}\right\|_{L^\infty_t L^6_x(D_m\times[0,T])} \left\|v_\theta^{q-1}\eta\right\|_{L^2_t L^3_x(D_m\times[0,T])}
\end{aligned}$$

$$\leq C_* \|h\eta\|_{L_t^2 L_x^3(D_m\times[0,T])},$$

and

$$\begin{aligned}\left\|\frac{|v_\rho|+|v_\phi|}{\rho} f^2\eta^2\right\|_{L^1_{tx}(D_m\times[0,T])} &\leq \left\|\frac{|v_\rho|+|v_\phi|}{\rho}\right\|_{L_t^\infty L_x^6(D_m\times[0,T])} \|f\eta\|^2_{L_t^2 L_x^{12/5}(D_m\times[0,T])} \\ &\leq 2C_* \|f\eta\|^2_{L_t^2 L_x^{12/5}(D_m\times[0,T])}.\end{aligned}$$

Plugging the above estimates into (4.103) yields

$$\begin{aligned}&\frac{1}{s_1^2}\|f\eta\|^2_{L_t^2 L_x^6(D_m\times[0,T])} + \|f\eta\|^2_{L_t^\infty L_x^2(D_m\times[0,T])} \\ &\leq 12C_*^2 q^2 \|h\eta\|^2_{L_t^2 L_x^3(D_m\times[0,T])} + 8C_* q \|f\eta\|^2_{L_t^2 L_x^{12/5}(D_m\times[0,T])} \\ &\quad + 2\eta^2(0)\, \|f(\cdot,0)\|^2_{L^2(D_m)} + 4\|f^2\eta\eta'\|_{L^1_{tx}(D_m\times[0,T])},\end{aligned} \tag{4.105}$$

where $f = v_\theta^q$ and $h = (|v_\theta| \vee 1)^q$. Next, we have two cases to deal with.

Case 1: $T \leq 2$. In this case, we take $\eta \equiv 1$ on $[0,T]$. Putting this η into (4.105), we have

$$\begin{aligned}&\frac{1}{s_1^2}\|f\|^2_{L_t^2 L_x^6(D_m\times[0,T])} + \|f\|^2_{L_t^\infty L_x^2(D_m\times[0,T])} \\ &\leq 12C_*^2 q^2 \|h\|^2_{L_t^2 L_x^3(D_m\times[0,T])} + 8C_* q \|f\|^2_{L_t^2 L_x^{12/5}(D_m\times[0,T])} + 2\|f(\cdot,0)\|^2_{L^2(D_m)}.\end{aligned}$$

Recalling $h = |f| \vee 1$, so there exists a constant $C = C(\alpha)$ such that

$$\begin{aligned}&\|h\|^2_{L_t^2 L_x^6(D_m\times[0,T])} + \|h\|^2_{L_t^\infty L_x^2(D_m\times[0,T])} \\ &\leq C\Big(C_*^2 q^2 \|h\|^2_{L_t^2 L_x^3(D_m\times[0,T])} + C_* q \|h\|^2_{L_t^2 L_x^{12/5}(D_m\times[0,T])} + \|h(\cdot,0)\|^2_{L^2(D_m)}\Big).\end{aligned} \tag{4.106}$$

In order to estimate the right-hand side of (4.106), we interpolate $L_t^2 L_x^3$ and $L_t^2 L_x^{12/5}$ between $L_t^2 L_x^6$ and $L_t^2 L_x^2$, and then apply the Young's inequality. Consequently, it follows from (4.106) that

$$\|h\|^2_{L_t^2 L_x^6(D_m\times[0,T])} + \|h\|^2_{L_t^\infty L_x^2(D_m\times[0,T])} \leq C\Big(C_*^4 q^4 \|h\|^2_{L_t^2 L_x^2(D_m\times[0,T])} + \|h(\cdot,0)\|^2_{L^2(D_m)}\Big).$$

Again, by applying interpolation to the left-hand side of the above estimate, we obtain

$$\|h\|_{L_{tx}^{10/3}(D_m\times[0,T])} \leq C\Big(C_*^2 q^2 \|h\|_{L_{tx}^2(D_m\times[0,T])} + \|h(\cdot,0)\|_{L^2(D_m)}\Big). \tag{4.107}$$

Since $h = \psi^q$, where $\psi := |v_\theta| \vee 1$, then it follows from the above relation that

$$
\begin{aligned}
\|\psi\|^q_{L^{10q/3}_{tx}(D_m\times[0,T])} &\le CC_*^2q^2\|\psi\|^q_{L^{2q}_{tx}(D_m\times[0,T]}+C\|\psi(\cdot,0)\|^q_{L^{2q}(D_m)}\\
&\le CC_*^2q^2\|\psi\|^q_{L^{2q}_{tx}(D_m\times[0,T])}+C\,|D_m|^{\frac12}\,\|\psi(\cdot,0)\|^q_{L^\infty(D_m)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Big(\|\psi\|_{L^{10q/3}_{tx}(D_m\times[0,T])}\vee\|\psi(\cdot,0)\|_{L^\infty(D_m)}\Big)\\
\le&(CC_*^2)^{\frac1q}q^{\frac2q}\Big(\|\psi\|_{L^{2q}_{tx}(D_m\times[0,T])}\vee\|\psi(\cdot,0)\|_{L^\infty D_m)}\Big).
\end{aligned}
\tag{4.108}
$$

By choosing $q=q_k=\left(\frac53\right)^k$ for $k=0,1,2,\cdots$ in (4.108), and applying Moser's iteration, we find

$$
\Big(\|\psi\|_{L^\infty_{tx}(D_m\times[0,T])}\vee\|\psi(\cdot,0)\|_{L^\infty(D_m)}\Big)\le CC_*^5\Big(\|\psi\|_{L^2_{tx}(D_m\times[0,T]}\vee\|\psi(\cdot,0)\|_{L^\infty(D_m)}\Big).
$$

Since $\psi=|v_\theta|\vee 1$ and $T\le 2$, we deduce that

$$
\|v_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}\le CC_*^5\big(\|v_\theta\|_{L^2_{tx}(D_m\times[0,T])}+\|v_\theta(\cdot,0)\|_{L^\infty(D_m)}+1\big).
\tag{4.109}
$$

Finally, thanks to the energy estimate (4.9), the above inequality implies that

$$
\|v_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}\le CC_*^5\big(\|v_0\|_{L^2(D_m)}+\|v_{0,\theta}\|_{L^\infty(D_m)}+1\big),\quad \forall\, 0<T\le 2.
\tag{4.110}
$$

Case 2: $T>2$. In this case, we take $\eta\in C^\infty([0,T])$ such that $0\le\eta\le1$ and

$$
\eta(t)=\begin{cases}0, & \text{if}\quad 0\le t\le T-2,\\ 1, & \text{if}\quad T-1\le t\le T.\end{cases}
$$

Putting this η into (4.105), we know

$$
\begin{aligned}
&\frac{1}{s_1^2}\|f\eta\|^2_{L^2_tL^6_x(D_m\times[T-2,T])}+\|f\eta\|^2_{L^\infty_tL^2_x(D_m\times[T-2,T])}\\
\le\ &12C_*^2q^2\|h\eta\|^2_{L^2_tL^3_x(D_m\times[T-2,T])}+8C_*q\|f\eta\|^2_{L^2_tL^{12/5}_x(D_m\times[T-2,T])}\\
&+4\|f^2\eta\eta'\|_{L^1_{tx}(D_m\times[T-2,T])}.
\end{aligned}
$$

Then similar to the derivation of (4.107), we know there exists some constant $C=C(\alpha)$ such that

$$
\|h\eta\|^2_{L^{10/3}_{tx}(D_m\times[T-2,T])}\le C\Big(C_*^4q^4\|h\eta\|^2_{L^2_{tx}(D_m\times[T-2,T])}+\|h^2\eta\eta'\|_{L^1_{tx}(D_m\times[T-2,T])}\Big).
$$

Recalling $h=\psi^q$, where $\psi=|v_\theta|\vee 1$, so

$$\|\psi^q\eta\|^2_{L^{10/3}_{tx}(D_m\times[T-2,T])} \le C\Big(C_*^4q^4\|\psi^q\eta\|^2_{L^2_{tx}(D_m\times[T-2,T])} + \|\psi^{2q}\eta\eta'\|_{L^1_{tx}(D_m\times[T-2,T])}\Big). \tag{4.111}$$

For $k = 0,1,2,\cdots$, we denote $q_k = \left(\frac{5}{3}\right)^k$, $T_k = T-1-2^{-k}$. Meanwhile, we define $\eta_k \in C^\infty([0,T])$ such that $0\le\eta_k\le 1$,

$$\eta_k(t) = \begin{cases} 0, & \text{if} \quad 0\le t\le T_k. \\ 1, & \text{if} \quad T_{k+1}\le t\le T, \end{cases}$$

and $\sup\limits_{t\in[0,T]} |\eta_k'(t)| \le 2^{k+2}$. Plugging $q = q_k$ and $\eta = \eta_k$ into (4.111), we find

$$\|\psi^{q_k}\|^2_{L^{10/3}_{tx}(D_m\times[T_{k+1},T])} \le C\Big(C_*^4q_k^4\|\psi^{q_k}\|^2_{L^2_{tx}(D_m\times[T_k,T])} + 2^{k+2}\|\psi^{2q_k}\|_{L^1_{tx}(D_m\times[T_k,T])}\Big).$$

Therefore,

$$\|\psi\|^{2q_k}_{L^{2q_{k+1}}_{tx}(D_m\times[T_{k+1},T])} \le CC_*^4q_k^4\|\psi\|^{2q_k}_{L^{2q_k}_{tx}(D_m\times[T_k,T])}.$$

Now we can apply Moser's iteration to obtain

$$\|\psi\|_{L^\infty_{tx}(D_m\times[T-1,T])} \le CC_*^5\|\psi\|_{L^2_{tx}(D_m\times[T-2,T])}. \tag{4.112}$$

This implies that

$$\|v_\theta\|_{L^\infty_{tx}(D_m\times[T-1,T])} \le CC_*^5\big(\|v_\theta\|_{L^2_{tx}(D_m\times[T-2,T])} + \|1\|_{L^2_{tx}(D_m\times[T-2,T])}\big). \tag{4.113}$$

Taking advantage of the energy estimate (4.9) again, we deduce from (4.113) that

$$\|v_\theta\|_{L^\infty_{tx}(D_m\times[T-1,T])} \le CC_*^5\big(\|v_0\|_{L^2(D_m)} + 1\big), \quad \forall T>2. \tag{4.114}$$

Finally, by combining (4.110) in Case 1 and (4.114) in Case 2 together, we have justified (4.98). $\square$

4.7.2. L^∞ boundedness of ω_θ

In this subsection, we will prove the L^∞ bound of ω_θ which is needed to establish the L^∞ bounds of v_ρ and v_ϕ in the next subsection.

Proposition 4.12. *Let the region D_m be as defined in (2.17) with $m\ge 2$ and $\alpha\in(0,\frac{\pi}{6}]$. Then for any $T>0$,*

$$\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[0,T])} \le CC_*^{10}\left(\|v_0\|_{L^2(D_m)} + \|\omega_{0,\theta}\|_{L^\infty(D_m)} + 1\right), \tag{4.115}$$

where $C = C(\alpha)$ and

$$C_* = \max\left\{\left\|\frac{|v_\rho|+|v_\phi|}{\rho}\right\|_{L^\infty_t L^6_x(D_m\times[0,T])}, \|v_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}, \big\||K|+|F|\big\|_{L^\infty_t L^2_x(D_m\times[0,T])}, 2\right\}. \tag{4.116}$$

Proof. Recall ω_θ satisfies (3.59), i.e.,

$$\begin{cases} \Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)\omega_\theta - b\cdot\nabla\omega_\theta + \frac{1}{\rho}\left(v_\rho + \cot\phi\, v_\phi\right)\omega_\theta \\ \qquad\qquad -\frac{1}{\rho^2}\partial_\phi\big(v_\theta^2\big) + \frac{\cot\phi}{\rho}\partial_\rho\big(v_\theta^2\big) - \partial_t\omega_\theta = 0, \quad \text{in} \quad D_m\times(0,T], \\ \omega_\theta = 0, \quad \text{on} \quad \partial D_m\times(0,T], \\ \omega_\theta(x,0) = \omega_{0,\theta}(x), \quad x\in D_m. \end{cases}$$

Noticing

$$-\frac{1}{\rho^2}\partial_\phi\big(v_\theta^2\big) + \frac{\cot\phi}{\rho}\partial_\rho\big(v_\theta^2\big) = -\frac{2v_\theta}{\rho}(\omega_\rho + \cot\phi\,\omega_\phi) = -2v_\theta(K + \cot\phi\,F),$$

so the above equation about ω_θ can be written as

$$\begin{cases} \Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)\omega_\theta - b\cdot\nabla\omega_\theta + \frac{1}{\rho}\left(v_\rho + \cot\phi\, v_\phi\right)\omega_\theta \\ \qquad\qquad -2v_\theta(K+\cot\phi\,F) - \partial_t\omega_\theta = 0, \quad \text{in} \quad D_m\times(0,T], \\ \omega_\theta = 0, \quad \text{on} \quad \partial D_m\times(0,T], \\ \omega_\theta(x,0) = \omega_{0,\theta}(x), \quad x\in D_m. \end{cases} \tag{4.117}$$

Let $\eta : [0,T] \longrightarrow [0,1]$ be a smooth function in the time variable. The specific choice of η will be determined later. For any rational number $q \geq 1$ in the form of $(2k-1)/(2l-1)$, where k and l are positive integers. Denote $f = \omega_\theta^q$. Then for any $t\in(0,T]$, we test (4.117) by $q\omega_\theta^{2q-1}\eta^2$ on $D_m\times(0,t]$ to find

$$\begin{aligned} &\frac{2q-1}{q}\int_0^t\int_{D_m}|(\nabla f)\eta|^2\,dx\,d\tau + q\int_0^t\int_{D_m}\frac{1}{\rho^2\sin^2\phi}f^2\eta^2\,dx\,d\tau + \frac{1}{2}\eta^2(t)\int_{D_m}f^2(x,t)\,dx \\ &= q\int_0^t\int_{D_m}\frac{1}{\rho}(v_\rho + \cot\phi\,v_\phi)f^2\eta^2\,dx\,d\tau - 2q\int_0^t\int_{D_m}v_\theta(K+\cot\phi\,F)\omega_\theta^{2q-1}\eta^2\,dx\,d\tau \\ &\quad + \int_0^t\int_{D_m}f^2\eta\eta'\,dx\,d\tau + \frac{1}{2}\eta^2(0)\int_{D_m}f^2(x,0)\,dx. \end{aligned}$$

As a consequence,

$$\begin{aligned} &\int_0^t\int_{D_m}|(\nabla f)\eta|^2\,dx\,d\tau + \frac{1}{2}\eta^2(t)\int_{D_m}f^2(x,t)\,dx \\ &\leq q\int_0^t\int_{D_m}\frac{|v_\rho|+|v_\phi|}{\rho}f^2\eta^2\,dx\,d\tau + 2q\int_0^t\int_{D_m}|v_\theta|\big(|K|+|F|\big)|\omega_\theta|^{2q-1}\eta^2\,dx\,d\tau \end{aligned}$$

$$+\int_0^t \int_{D_m} f^2\eta|\eta'|\,dx\,d\tau + \frac{1}{2}\eta^2(0)\int_{D_m} f^2(x,0)\,dx.$$

Taking supremum with respect to t on $[0,T]$, then

$$\begin{aligned}&\int_0^T \int_{D_m} |(\nabla f)\eta|^2\,dx\,d\tau + \frac{1}{2}\sup_{t\in[0,T]}\int_{D_m} f^2(x,t)\eta^2(t)\,dx\\ &\le 2q\int_0^T\int_{D_m} \frac{|v_\rho|+|v_\phi|}{\rho}f^2\eta^2\,dx\,d\tau + 4q\int_0^T\int_{D_m}|v_\theta|\big(|K|+|F|\big)|\omega_\theta|^{2q-1}\eta^2\,dx\,d\tau\\ &\quad + 2\int_0^T\int_{D_m} f^2\eta|\eta'|\,dx\,d\tau + \eta^2(0)\int_{D_m} f^2(x,0)\,dx.\end{aligned}\tag{4.118}$$

Since $f=0$ on ∂D_m, it follows from Lemma 4.9 that

$$\|f(\cdot,\tau)\eta(\tau)\|_{L^6_x(D_m)} \le s_1\big\|\nabla\big(f(\cdot,\tau)\eta(\tau)\big)\big\|_{L^2_x(D_m)}, \quad \forall\,\tau\in[0,T],$$

where $s_1 = s_1(\alpha)$. Thus, it follows from (4.118) that

$$\begin{aligned}&\frac{1}{s_1^2}\|f\eta\|^2_{L^2_tL^6_x(D_m\times[0,T])} + \frac{1}{2}\|f\eta\|^2_{L^\infty_tL^2_x(D_m\times[0,T])}\\ &\le 2q\left\|\frac{|v_\rho|+|v_\phi|}{\rho}f^2\eta^2\right\|_{L^1_{tx}(D_m\times[0,T])} + 4q\big\|v_\theta(|K|+|F|)\omega_\theta^{2q-1}\eta^2\big\|_{L^1_{tx}(D_m\times[0,T])}\\ &\quad + 2\|f^2\eta\eta'\|_{L^1_{tx}(D_m\times[0,T])} + \eta^2(0)\|f(\cdot,0)\|^2_{L^2(D_m)}.\end{aligned}\tag{4.119}$$

Denote C_* as in (4.116) and define h as

$$h = |\omega_\theta|^q \vee 1 = (|\omega_\theta|\vee 1)^q,$$

where "$\vee$" means "max". Then

$$\begin{aligned}\left\|\frac{|v_\rho|+|v_\phi|}{\rho}f^2\eta^2\right\|_{L^1_{tx}(D_m\times[0,T])} &\le \left\|\frac{|v_\rho|+|v_\phi|}{\rho}\right\|_{L^\infty_tL^6_x(D_m\times[0,T])}\|f\eta\|^2_{L^2_tL^{12/5}_x(D_m\times[0,T])}\\ &\le C_*\|f\eta\|^2_{L^2_tL^{12/5}_x(D_m\times[0,T])},\end{aligned}$$

and

$$\begin{aligned}&\big\|v_\theta(|K|+|F|)\omega_\theta^{2q-1}\eta^2\big\|_{L^1_{tx}(D_m\times[0,T])}\\ &\le \|v_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}\big\||K|+|F|\big\|_{L^\infty_tL^2_x(D_m\times[0,T])}\|h\eta\|^2_{L^2_tL^4_x(D_m\times[0,T])}\\ &\le C_*^2\|h\eta\|^2_{L^2_tL^4_x(D_m\times[0,T])}.\end{aligned}$$

Plugging the above estimates into (4.119) yields

$$\begin{aligned}&\frac{1}{s_1^2}\|f\eta\|^2_{L^2_tL^6_x(D_m\times[0,T])}+\frac{1}{2}\|f\eta\|^2_{L^\infty_tL^2_x(D_m\times[0,T])}\\ &\le 2qC_*\|f\eta\|^2_{L^2_tL^{12/5}_x(D_m\times[0,T])}+4qC_*^2\|h\eta\|^2_{L^2_tL^4_x(D_m\times[0,T])}\\ &\quad+2\|f^2\eta\eta'\|_{L^1_{tx}(D_m\times[0,T])}+\eta^2(0)\|f(\cdot,0)\|^2_{L^2(D_m)},\end{aligned}\tag{4.120}$$

where $f=\omega_\theta^q$ and $h=(|\omega_\theta|\vee 1)^q$. Then there are two cases to be dealt with.

Case 1: $T\le 2$. In this case, we follow the argument for (4.109) in Case 1 in the proof of Proposition 4.11 to obtain

$$\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}\le CC_*^{10}\big(\|\omega_\theta\|_{L^2_{tx}(D_m\times[0,T])}+\|\omega_\theta(\cdot,0)\|_{L^\infty(D_m)}+1\big).$$

Actually, the zero boundary condition of ω_θ makes the argument simpler. Combining with the energy estimate (4.9), we find

$$\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[0,T])}\le CC_*^{10}\big(\|v_0\|_{L^2(D_m)}+\|\omega_{0,\theta}\|_{L^\infty(D_m)}+1\big).\tag{4.121}$$

Case 2: $T>2$. In this case, we follow the argument for (4.113) in Case 2 in the proof of Proposition 4.11 to find

$$\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[T-1,T])}\le CC_*^{10}\big(\|\omega_\theta\|_{L^2_{tx}(D_m\times[T-2,T])}+\|1\|_{L^2_{tx}(D_m\times[T-2,T])}\big).$$

Then due to the energy estimate (4.9) again, we conclude

$$\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[T-1,T])}\le CC_*^{10}\big(\|v_0\|_{L^2(D_m)}+1\big).\tag{4.122}$$

Finally, by combining (4.121) in Case 1 and (4.122) in Case 2 together, (4.115) is justified. □

4.7.3. L^∞ boundedness of v_ρ and v_ϕ

Proposition 4.13. *Let the region D_m be as defined in (2.17) with $m\ge 2$ and the angle $\alpha\in\big(0,\frac{\pi}{6}\big]$. Then for any $T>0$,*

$$\|v_\rho\|_{L^\infty_{tx}(D_m\times[0,T])}+\|v_\phi\|_{L^\infty_{tx}(D_m\times[0,T])}\le CC_*^3(\|v_0\|_{L^2(D_m)}+1),\tag{4.123}$$

where $C=C(\alpha)$ and

$$C_*=\max\left\{\left\|\frac{|v_\rho|+|v_\phi|}{\rho}\right\|_{L^\infty_tL^6_x(D_m\times[0,T])},\|\omega_\theta\|_{L^\infty_{tx}(D_m\times[0,T])},2\right\}.\tag{4.124}$$

Proof. Fix any $t \in [0,T]$. The following proof will be derived based on this fixed t and we will drop the temporal variable within the proof for simplicity.

We first estimate $\|v_\rho\|_{L^\infty_{tx}(D_m\times[0,T])}$. According to the Biot-Savart law (4.13) and the boundary conditions in Lemma 2.1, v_ρ satisfies the following equations.

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)v_\rho = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta), \quad \text{in} \quad D_m; \\ \partial_\phi v_\rho = 0 \quad \text{on} \quad \partial^R D_m, \qquad v_\rho = 0 \quad \text{on} \quad \partial^A D_m. \end{cases}$$

For any integer $q \geq 1$, we denote v_ρ^q by f. Then f satisfies the equations below.

$$\begin{cases} \Delta f - q(q-1)v_\rho^{q-2}|\nabla v_\rho|^2 + \frac{2}{\rho}\partial_\rho f + \frac{2q}{\rho^2} f = -\frac{q v_\rho^{q-1}}{\rho\sin\phi}\partial_\phi(\sin\phi\,\omega_\theta), \quad \text{in} \quad D_m; \\ \partial_\phi f = 0 \quad \text{on} \quad \partial^R D_m, \qquad f = 0 \quad \text{on} \quad \partial^A D_m. \end{cases}$$

Testing the above problem by f on D_m yields

$$-\frac{2q-1}{q}\int\limits_{D_m} |\nabla f|^2\,dx + 2\int\limits_{D_m} \frac{1}{\rho} f\partial_\rho f\,dx + 2q\int\limits_{D_m} \frac{1}{\rho^2} f^2\,dx = -q\int\limits_{D_m} \frac{v_\rho^{2q-1}}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta)\,dx. \tag{4.125}$$

By converting the integrals into the form of spherical coordinates, and then using integration by parts, we have

$$\begin{aligned} \int\limits_{D_m} \frac{2}{\rho} f\partial_\rho f\,dx &= 2\pi \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int\limits_{\frac{1}{m}}^{1} \rho\sin\phi\,\partial_\rho(f^2)\,d\rho\,d\phi \\ &= -2\pi \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int\limits_{\frac{1}{m}}^{1} \sin\phi\, f^2\,d\rho\,d\phi = -\int\limits_{D_m} \frac{f^2}{\rho^2}\,dx; \end{aligned}$$

and

$$\begin{aligned} -q\int\limits_{D_m} \frac{v_\rho^{2q-1}}{\rho\sin\phi}\partial_\phi(\sin\phi\,\omega_\theta)\,dx &= -2\pi q \int\limits_{\frac{1}{m}}^{1} \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \rho v_\rho^{2q-1}\partial_\phi(\sin\phi\,\omega_\theta)\,d\phi\,d\rho \\ &= 2\pi q(2q-1) \int\limits_{\frac{1}{m}}^{1} \int\limits_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \rho\sin\phi\, v_\rho^{2q-2}(\partial_\phi v_\rho)\omega_\theta\,d\phi\,d\rho \\ &= (2q-1)\int\limits_{D_m} \frac{\partial_\phi f}{\rho} v_\rho^{q-1}\omega_\theta\,dx. \end{aligned}$$

Putting the above estimates into (4.125) and then multiplying the equation by $\frac{q}{2q-1}$, one deduces

$$\int_{D_m} |\nabla f|^2 \, dx = q \int_{D_m} \frac{f^2}{\rho^2} \, dx - q \int_{D_m} \frac{\partial_\phi f}{\rho} v_\rho^{q-1} \omega_\theta \, dx := I_1 + I_2. \tag{4.126}$$

For I_1, Hölder's inequality shows that

$$\begin{aligned} |I_1| = q \left\| \frac{v_\rho}{\rho} \, v_\rho^{q-1} \right\|_{L^2(D_m)}^2 &\leq q \left\| \frac{v_\rho}{\rho} \right\|_{L^6(D_m)}^2 \left\| v_\rho^{q-1} \right\|_{L^3(D_m)}^2 \\ &\leq q \left\| \frac{v_\rho}{\rho} \right\|_{L^6(D_m)}^2 \left\| (|v_\rho| \vee 1)^q \right\|_{L^3(D_m)}^2. \end{aligned} \tag{4.127}$$

For I_2, applying Hölder inequality and Young's inequality, we have

$$\begin{aligned} |I_2| &\leq q \left\| \frac{\partial_\phi f}{\rho} \right\|_{L^2(D_m)} \left\| v_\rho^{q-1} \right\|_{L^2(D_m)} \|\omega_\theta\|_{L^\infty(D_m)} \\ &\leq \frac{1}{4} \|\nabla f\|_{L^2(D_m)}^2 + q^2 \left\| (|v_\rho| \vee 1)^q \right\|_{L^2(D_m)}^2 \|\omega_\theta\|_{L^\infty(D_m)}^2 . \end{aligned} \tag{4.128}$$

Plugging (4.127) and (4.128) into (4.126), we know

$$\begin{aligned} \frac{3}{4} \|\nabla f\|_{L^2(D_m)}^2 &\leq q \left\| \frac{v_\rho}{\rho} \right\|_{L^6(D_m)}^2 \|h\|_{L^3(D_m)}^2 + q^2 \|h\|_{L^2(D_m)}^2 \|\omega_\theta\|_{L^\infty(D_m)}^2 , \\ &\leq C_*^2 \Big(q \|h\|_{L^3(D_m)}^2 + q^2 \|h\|_{L^2(D_m)}^2 \Big), \end{aligned} \tag{4.129}$$

where $h := (|v_\rho| \vee 1)^q$ and C_* is as defined in (4.124). Then it follows from Lemma 4.9 that there exists some constant s_0, which only depends on α, such that

$$\|f\|_{L^6(D_m)} \leq s_0 \|f\|_{H^1(D_m)}.$$

So (4.129) implies that

$$\|f\|_{L^6(D_m)}^2 \leq C C_*^2 \Big(q \|h\|_{L^3(D_m)}^2 + q^2 \|h\|_{L^2(D_m)}^2 \Big) + C \|f\|_{L^2(D_m)}^2. \tag{4.130}$$

In addition, since $f = v_\rho^q$ and $h = |f| \vee 1$, we derive from (4.130) that

$$\|h\|_{L^6(D_m)}^2 \leq C C_*^2 \Big(q \|h\|_{L^3(D_m)}^2 + q^2 \|h\|_{L^2(D_m)}^2 \Big). \tag{4.131}$$

Now we interpolate $\|h\|_{L^3}$ between $\|h\|_{L^6}$ and $\|h\|_{L^2}$ to get

$$C C_*^2 q \|h\|_{L^3(D_m)}^2 \leq \frac{1}{4} \|h\|_{L^6(D_m)}^2 + C^2 C_*^4 q^2 \|h\|_{L^2(D_m)}^2.$$

Therefore, it follows from (4.131) that

$$\|h\|_{L^6(D_m)}^2 \le CC_*^4 q^2 \|h\|_{L^2(D_m)}^2.$$

By writing $h = \psi^q$, where $\psi = |v_\rho| \vee 1$, the above estimate is converted into

$$\|\psi\|_{L^{6q}(D_m)} \le (CC_*^4)^{\frac{1}{2q}} q^{\frac{1}{q}} \|\psi\|_{L^{2q}(D_m)}. \tag{4.132}$$

Now we choose $q = q_k = 3^k$ in (4.132), where $k = 0, 1, 2, \cdots$, then by iterative estimates, we obtain

$$\|\psi\|_{L^\infty(D_m)} \le CC_*^3 \|\psi\|_{L^2(D_m)},$$

where $C = C(\alpha)$. This result yields

$$\|v_\rho(\cdot,t)\|_{L^\infty(D_m)} \le CC_*^3 (\|v_\rho(\cdot,t)\|_{L^2(D_m)} + 1), \quad \forall\, t \in [0,T].$$

Taking advantage of the energy estimate (4.9) and taking supremum with respect to t, we conclude

$$\|v_\rho\|_{L_{tx}^\infty(D_m\times[0,T])} \le CC_*^3 (\|v_0\|_{L^2(D_m)} + 1). \tag{4.133}$$

Next, we use the similar method as above to estimate $\|v_\phi\|_{L_{tx}^\infty(D_m\times[0,T])}$. Based on the Biot-Savart law (4.13) and the boundary conditions in Lemma 2.1, v_ϕ satisfies the following equations.

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\right) v_\phi = \frac{1}{\rho^3}\,\partial_\rho(\rho^3\omega_\theta), \quad \text{in} \ \ D_m; \\ v_\phi = 0 \ \ \text{on} \ \ \partial^R D_m, \quad \partial_\rho v_\phi = -\frac{1}{\rho} v_\phi \ \ \text{on} \ \ \partial^A D_m. \end{cases}$$

For any integer $q \ge 1$, we denote v_ϕ^q by g. Then g satisfies the equations below.

$$\begin{cases} \Delta g - q(q-1) v_\phi^{q-2} |\nabla v_\phi|^2 + \frac{2}{\rho}\partial_\rho g + \frac{q(1-\cot^2\phi)}{\rho^2} g = \frac{q}{\rho^3} v_\phi^{q-1} \partial_\rho(\rho^3\omega_\theta), \quad \text{in} \ \ D_m; \\ g = 0 \ \ \text{on} \ \ \partial^R D_m, \quad \partial_\rho g = -\frac{q}{\rho}\, g \ \ \text{on} \ \ \partial^A D_m. \end{cases}$$

Testing this problem by g on D_m, we obtain

$$\begin{aligned} &\underbrace{\int\limits_{D_m} g\Delta g\, dx}_{G_1} - \frac{q-1}{q} \int\limits_{D_m} |\nabla g|^2\, dx + \int\limits_{D_m} \frac{2}{\rho} g \partial_\rho g\, dx + q \int\limits_{D_m} \frac{1-\cot^2\phi}{\rho^2} g^2\, dx \\ &= q \int\limits_{D_m} \frac{v_\phi^{2q-1}}{\rho^3} \partial_\rho(\rho^3\omega_\theta)\, dx. \end{aligned} \tag{4.134}$$

Using integration by parts and then converting the boundary integral to the interior integral, we see

$$G_1 = -\int_{D_m} |\nabla g|^2\, dx - 2q \int_{D_m} \frac{g}{\rho} \partial_\rho g\, dx - q \int_{D_m} \frac{g^2}{\rho^2}\, dx.$$

Substituting this identity into (4.134) leads to

$$\begin{aligned} &\frac{2q-1}{q} \int_{D_m} |\nabla g|^2\, dx + q \int_{D_m} \frac{\cot^2 \phi}{\rho^2} g^2\, dx \\ =\ & -(2q-2) \int_{D_m} \frac{g}{\rho} \partial_\rho g\, dx - q \int_{D_m} \frac{v_\phi^{2q-1}}{\rho^3} \partial_\rho (\rho^3 \omega_\theta)\, dx. \end{aligned}$$

This implies

$$\int_{D_m} |\nabla g|^2 dx \leq \underbrace{2(q-1)\Big| \int_{D_m} \frac{g}{\rho} \partial_\rho g\, dx \Big|}_{G_2} + \underbrace{q \Big| \int_{D_m} \frac{v_\phi^{2q-1}}{\rho^3} \partial_\rho (\rho^3 \omega_\theta)\, dx \Big|}_{G_3}. \tag{4.135}$$

Moreover, by applying Hölder's inequality, we have

$$\begin{aligned} G_2 &\leq 2q \|\nabla g\|_{L^2(D_m)} \left\| \frac{v_\phi}{\rho} \right\|_{L^6(D_m)} \|v_\phi^{q-1}\|_{L^3(D_m)} \\ &\leq \frac{1}{4} \|\nabla g\|^2_{L^2(D_m)} + 4q^2 \left\| \frac{v_\phi}{\rho} \right\|^2_{L^6(D_m)} \|(|v_\phi| \vee 1)^q\|^2_{L^3(D_m)} \\ &\leq \frac{1}{4} \|\nabla g\|^2_{L^2(D_m)} + 4C_*^2 q^2 \|h_1\|^2_{L^3(D_m)}, \end{aligned} \tag{4.136}$$

where C_* is as defined in (4.124) and $h_1 := (|v_\phi| \vee 1)^q$. In order to estimate G_3, we first use spherical coordinates and integration by parts to find

$$\begin{aligned} G_3 &= 2\pi q \Big| \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{1}{m}}^{1} \frac{v_\phi^{2q-1}}{\rho} \partial_\rho (\rho^3 \omega_\theta) \sin\phi\, d\rho\, d\phi \Big| \\ &= 2\pi q \Big| \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \int_{\frac{1}{m}}^{1} \Big(-\frac{v_\phi^{2q-1}}{\rho^2} + \frac{2q-1}{\rho} v_\phi^{2q-2} \partial_\rho v_\phi \Big) \rho^3 \omega_\theta \sin\phi\, d\phi\, d\rho \Big| \\ &\leq q \int_{D_m} |\omega_\theta| \left| \frac{v_\phi}{\rho} \right| |v_\phi|^{2q-2}\, dx + (2q-1) \int_{D_m} |\omega_\theta| |\partial_\rho g| |v_\phi|^{q-1}\, dx. \end{aligned}$$

Using Hölder's inequality,

$$G_3 \le q\|\omega_\theta\|_{L^6(D_m)}\Big\|\frac{v_\phi}{\rho}\Big\|_{L^6(D_m)}\|h_1\|^2_{L^3(D_m)} + 2q\|\omega_\theta\|_{L^\infty(D_m)}\|\nabla g\|_{L^2(D_m)}\|h_1\|_{L^2(D_m)}$$
$$\le C_*^2 q\|h_1\|^2_{L^3(D_m)} + 2C_* q\|\nabla g\|_{L^2(D_m)}\|h_1\|_{L^2(D_m)}.$$

Applying Cauchy-Schwarz inequality,

$$G_3 \le \frac{1}{4}\|\nabla g\|^2_{L^2(D_m)} + 4C_*^2 q^2\|h_1\|^2_{L^2(D_m)} + C_*^2 q\|h_1\|^2_{L^3(D_m)}. \tag{4.137}$$

Substituting (4.136) and (4.137) in (4.135), one finds

$$\|\nabla g\|^2_{L^2(D_m)} \le CC_*^2 q^2\big(\|h_1\|^2_{L^3(D_m)} + \|h_1\|^2_{L^2(D_m)}\big), \tag{4.138}$$

where C is a numerical constant. Since $g = v_\phi^q = 0$ on $\partial^R D_m$, it follows from Lemma 4.9 that there exists some constant s_1, which only depends on α, such that

$$\|g\|_{L^6(D_m)} \le s_1\|\nabla g\|_{L^2(D_m)}.$$

Moreover, noticing $h_1 = |g| \vee 1$, so it follows from the above embedding and (4.138) that

$$\|h_1\|_{L^6(D_m)} \le CC_*^2 q^2\big(\|h_1\|^2_{L^3(D_m)} + \|h_1\|^2_{L^2(D_m)}\big).$$

This estimate is a parallel result to (4.131), so the remaining proof is similar to that for (4.133). Thus, we obtain

$$\|v_\phi\|_{L^\infty_{tx}(D_m\times[0,T])} \le CC_*^3(\|v_0\|_{L^2(D_m)} + 1). \tag{4.139}$$

The combination of (4.133) and (4.139) justifies (4.123). □

By tracing the constants in Lemma 4.7, Lemma 4.10, Propositions 4.11, 4.12 and 4.13, we can obtain the following corollary. The key is that both $\|v_0\|_{H^2(D_m)}$ and $\|\omega_{0,\theta}\|_{L^\infty(D_m)}$ are controlled by $\|v_0\|_{C^2(\overline{D_m})}$.

Corollary 4.14. *Let the region D_m be as defined in (2.17) with $m \ge 2$ and the angle $\alpha \in \big(0, \frac{\pi}{6}\big]$. Assume (4.67), that is $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)} \le \frac{1}{95}$. Then for any $T > 0$,*

$$\|v\|_{L^\infty_{tx}(D_m\times[0,T])} + \|\omega_\theta\|_{L^\infty_{tx}(D_m\times[0,T])} \le C_0^*, \tag{4.140}$$

where C_0^* is a constant which only depends on α and $\|v_0\|_{C^2(\overline{D_m})}$.

5. Uniform bounds for $\|v\|_{L_t^2 H_x^2}$ and $\|v\|_{H_t^1 L_x^2}$ on $D_m \times [0, T]$

The basic setup of this section is the same as that in the beginning of Section 4. More precisely, for any fixed $m \geq 2$ and $T > 0$, we consider the initial data v_0 which lies in the admissible class $\mathscr{A}_m$ with the even-odd-odd symmetry. For such initial data, we denote by v the solution in Corollary 3.3 so that $v \in E_{m,T}^{\sigma,s} \cap H_t^1 L_x^2 \cap L_t^2 H_x^2 \cap L_{tx}^\infty (D_m \times [0, T])$. Moreover, we restrict the range of α within $\left(0, \frac{\pi}{6}\right]$ and require $\|\Gamma(\cdot, 0)\|_{L^\infty(D_m)} \leq \frac{1}{95}$. Then by taking advantage of the results in Section 4, in particular Lemma 4.7 and Corollary 4.14, we will obtain uniform bounds, which are independent of T and dependent on m only via $\|v_0\|_{C^2(\overline{D_m})}$, for $\|v\|_{L_t^2 H_x^2 (D_m \times [0,T])}$ and $\|v\|_{H_t^1 L_x^2 (D_m \times [0,T])}$. The strategy is as follows:

- Step 1: Based on the uniform boundedness of $\|\nabla K\|_{L_{tx}^2}$ and $\|\nabla F\|_{L_{tx}^2}$ on $D_m \times [0, T]$, we will derive a uniform bound for $\|v_\theta\|_{L_t^2 H_x^2 (D_m \times [0,T])}$. Then the uniform bound of $\|\partial_t v_\theta\|_{L_{tx}^2 (D_m \times [0,T])}$ can be obtained via the equation of v_θ.
- Step 2: Thanks to the Biot-Savart law and the uniform boundedness of $\|\nabla \Omega\|_{L_{tx}^2}$, we manage to derive uniform bounds for $\|v_\rho\|_{L_t^2 H_x^2 (D_m \times [0,T])}$ and $\|v_\phi\|_{L_t^2 H_x^2 (D_m \times [0,T])}$.
- Step 3: The uniform boundedness of $\|\partial_t \omega_\theta\|_{L_{tx}^2}$ can be verified by studying the equation of ω_θ.
- Step 4: By taking advantage of the Biot-Savart law again and also utilizing the uniform boundedness of $\|\partial_t \omega_\theta\|_{L_{tx}^2}$, we are able to justify both $\|\partial_t v_\rho\|_{L_{tx}^2}$ and $\|\partial_t v_\phi\|_{L_{tx}^2}$ are uniformly bounded.

We first summarize some pertinent results from earlier sections with minor extensions which will be needed in the later proof.

Proposition 5.1. *Let the region D_m be as defined in (2.17) with $m \geq 2$ and the angle $\alpha \in \left(0, \frac{\pi}{6}\right]$. Assume $\|\Gamma(\cdot, 0)\|_{L^\infty(D_m)} \leq \frac{1}{95}$. Then there exists a constant C, which only depends on α and $\|v_0\|_{C^2(\overline{D_m})}$ such that for any $T > 0$,*

$$\left\| \left(|K| + |F| + |\Omega| \right) \right\|_{L_t^\infty L_x^2} + \left\| \left(|\nabla K| + |\nabla F| + |\nabla \Omega| \right) \right\|_{L_{tx}^2} \leq C, \quad (\textit{cf. Lemma 4.7}) \tag{5.1}$$

$$\|v\|_{L_{tx}^\infty} + \left\| \frac{1}{\rho} \nabla v \right\|_{L_t^\infty L_x^2} + \left\| \frac{1}{\rho^2} \nabla v \right\|_{L_{tx}^2} + \left\| \frac{1}{\rho^3} v \right\|_{L_{tx}^2} \leq C, \tag{5.2}$$

$$\|\omega_\theta\|_{L_{tx}^\infty} + \left\| \frac{1}{\rho} \nabla \omega_\theta \right\|_{L_{tx}^2} + \left\| \frac{1}{\rho^2} \omega_\theta \right\|_{L_{tx}^2} \leq C, \tag{5.3}$$

where all the above space-time norms are taken on $D_m \times [0, T]$.

Proof. Only (5.2) and (5.3) are required to be verified. We start with the estimate (5.2). Firstly, the uniform boundedness of v is due to Corollary 4.14. Then from Lemmas 4.4, 4.5 and 4.8, we have

$$\Big\|\nabla\Big(\frac{v_\rho}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}+\Big\|\nabla\Big(\frac{v_\phi}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}+\Big\|\nabla\Big(\frac{v_\theta}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}\le 5\big\|\big(|K|+|F|+|\Omega|\big)\big\|_{L_t^\infty L_x^2},\tag{5.4}$$

$$\Big\|\frac{1}{\rho}\nabla\Big(\frac{v_\rho}{\rho}\Big)\Big\|_{L_{tx}^2}+\frac{1}{\rho}\Big\|\nabla\Big(\frac{v_\phi}{\rho}\Big)\Big\|_{L_{tx}^2}+\Big\|\frac{1}{\rho}\nabla\Big(\frac{v_\theta}{\rho}\Big)\Big\|_{L_{tx}^2}\le 40\big\|\big(|\nabla K|+|\nabla F|+|\nabla\Omega|\big)\big\|_{L_{tx}^2}.\tag{5.5}$$

Since $\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} v_\rho(\rho,\phi)\sin\phi\, d\phi=0$ for any $\rho\in\big(\frac{1}{m},1\big)$, it follows from the Poincaré inequality in Lemma 2.3 that

$$\Big\|\frac{1}{\rho^2}v_\rho(\cdot,t)\Big\|_{L_x^2}\le\sqrt{\frac{2}{19}}\Big\|\frac{1}{\rho^2}\partial_\phi v_\rho(\cdot,t)\Big\|_{L_x^2}\le\frac{1}{3}\Big\|\frac{1}{\rho^2}\partial_\phi v_\rho(\cdot,t)\Big\|_{L_x^2},\quad \forall\, t\in[0,T].$$

In addition, since

$$\nabla\Big(\frac{v_\rho}{\rho}\Big)=\frac{1}{\rho}\nabla v_\rho-\Big(\frac{1}{\rho^2}v_\rho\Big)e_\rho,$$

we know that

$$\Big\|\frac{1}{\rho}\nabla v_\rho\Big\|_{L_t^\infty L_x^2}\le\frac{4}{3}\Big\|\nabla\Big(\frac{v_\rho}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}.\tag{5.6}$$

Similarly,

$$\Big\|\frac{1}{\rho}\nabla v_\phi\Big\|_{L_t^\infty L_x^2}\le\frac{4}{3}\Big\|\nabla\Big(\frac{v_\phi}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}\quad\text{and}\quad\Big\|\frac{1}{\rho}\nabla v_\theta\Big\|_{L_t^\infty L_x^2}\le\frac{4}{3}\Big\|\nabla\Big(\frac{v_\theta}{\rho}\Big)\Big\|_{L_t^\infty L_x^2}.\tag{5.7}$$

Plugging (5.6) and (5.7) into (5.4), and then using (5.1), we obtain $\big\|\frac{1}{\rho}\nabla v\big\|_{L_t^\infty L_x^2}\le C$. By an analogous argument, we can take advantage of (5.5) to show $\big\|\frac{1}{\rho^2}\nabla v\big\|_{L_{tx}^2}\le C$, which further implies $\big\|\frac{1}{\rho^3}v\big\|_{L_t^\infty L_x^2}\le C$. Thus, (5.2) is justified.

We next investigate the estimate (5.3). Firstly, the uniform boundedness of ω_θ is due to Corollary 4.14. Then by direct computation, we find

$$\nabla\Omega=\nabla\Big(\frac{\omega_\theta}{\rho\sin\phi}\Big)=\frac{1}{\rho\sin\phi}\nabla\omega_\theta-\frac{\omega_\theta}{\rho^2\sin\phi}(e_\rho+\cot\phi\, e_\phi).\tag{5.8}$$

Since $0\le\cot\phi\le\cot\alpha\le 1/\sqrt{3}$, then for any $t\in[0,T]$,

$$\Big\|\frac{\omega_\theta(\cdot,t)}{\rho^2\sin\phi}(e_\rho+\cot\phi\, e_\phi)\Big\|_{L_x^2(D_m)}^2\le\frac{4}{3}\Big\|\frac{\omega_\theta(\cdot,t)}{\rho^2\sin\phi}\Big\|_{L_x^2(D_m)}^2.\tag{5.9}$$

Thanks to the restriction that $\alpha\in\big(0,\frac{\pi}{6}\big]$, we know $\sqrt{3}/2\le\sin\phi\le 1$ and therefore,

$$\Big\|\frac{\omega_\theta(\cdot,t)}{\rho^2\sin\phi}\Big\|_{L_x^2(D_m)}^2=2\pi\int_{\frac{1}{m}}^{1}\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\frac{\omega_\theta^2}{\rho^2\sin\phi}\,d\phi\,d\rho\le 2\pi\frac{2}{\sqrt{3}}\int_{\frac{1}{m}}^{1}\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\frac{\omega_\theta^2}{\rho^2}\,d\phi\,d\rho.$$

Since ω_θ vanishes on the boundary of D_m, we apply the Poincaré inequality in Lemma 2.5 to the right-hand side of the above inequality to obtain

$$\begin{aligned}\left\|\frac{\omega_\theta(\cdot,t)}{\rho^2\sin\phi}\right\|^2_{L^2_x(D_m)} &\le 2\pi\,\frac{2}{\sqrt{3}}\,\frac{3}{25}\int_{\frac{1}{m}}^{1}\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\frac{1}{\rho^2}(\partial_\phi\omega_\theta)^2\,d\phi\,d\rho \\ &\le \frac{6}{25\sqrt{3}}\left\|\frac{1}{\rho\sin\phi}\nabla\omega_\theta(\cdot,t)\right\|^2_{L^2_x(D_m)}.\end{aligned} \tag{5.10}$$

The combination of (5.9) and (5.10) yields

$$\left\|\frac{\omega_\theta(\cdot,t)}{\rho^2\sin\phi}(e_\rho+\cot\phi\,e_\phi)\right\|_{L^2_x(D_m)} \le \frac{1}{2}\left\|\frac{1}{\rho\sin\phi}\nabla\omega_\theta(\cdot,t)\right\|_{L^2_x(D_m)}. \tag{5.11}$$

Based on (5.11), it then follows from (5.8) that

$$\left\|\frac{1}{\rho\sin\phi}\nabla\omega_\theta\right\|_{L^2_{tx}} \le 2\|\nabla\Omega\|_{L^2_{tx}}.$$

Hence, we conclude $\left\|\frac{1}{\rho}\nabla\omega_\theta\right\|_{L^2_{tx}} \le C$, which further implies that $\left\|\frac{1}{\rho^2}\omega_\theta\right\|_{L^2_{tx}} \le C$. Thus, (5.3) is established. □

Based on Proposition 5.1, we will prove the main result of this section shown as below.

Proposition 5.2. *Let the region D_m be defined as in (2.17) with $m \ge 2$ and the angle $\alpha \in \left(0,\frac{\pi}{6}\right]$. Assume $\|\Gamma(\cdot,0)\|_{L^\infty(D_m)} \le \frac{1}{95}$. Then there exists a constant C, which only depends on α and $\|v_0\|_{C^2(\overline{D_m})}$, such that for any $T>0$,*

$$\|\nabla^2 v\|_{L^2_{tx}(D_m\times[0,T])} + \|\partial_t v\|_{L^2_{tx}(D_m\times[0,T])} \le C. \tag{5.12}$$

Proof. In the proof, C denotes constants which are independent of T, but may be dependent on α and $\|v_0\|_{C^2(\overline{D_m})}$. On the other hand, unless stated otherwise, all the norms in this proof are taken on the space-time domain $D_m\times[0,T]$.

Step 1: Uniform bounds on $\|\nabla^2 v_\theta\|_{L^2_{tx}}$ and $\|\partial_t v_\theta\|_{L^2_{tx}}$.

Firstly, since $\nabla v_\theta = (\partial_\rho v_\theta)e_\rho + \left(\frac{1}{\rho}\partial_\phi v_\theta\right)e_\phi$, it then follows from the formula (A.8) that under the basis (A.7),

$$\nabla^2 v_\theta = \begin{pmatrix} \partial_\rho^2 v_\theta & \frac{1}{\rho}\partial_\phi\partial_\rho v_\theta - \frac{1}{\rho^2}\partial_\phi v_\theta & 0 \\ \frac{1}{\rho}\partial_\rho\partial_\phi v_\theta - \frac{1}{\rho^2}\partial_\phi v_\theta & \frac{1}{\rho^2}\partial_\phi^2 v_\theta + \frac{1}{\rho}\partial_\rho v_\theta & 0 \\ 0 & 0 & \frac{1}{\rho}\partial_\rho v_\theta + \frac{\cot\phi}{\rho^2}\partial_\phi v_\theta \end{pmatrix}. \tag{5.13}$$

Thanks to (5.2) in Proposition 5.1, we infer from (5.13) that

$$\|\nabla^2 v_\theta\|_{L^2_{tx}} \leq 2\Big(\big\|\partial_\rho^2 v_\theta\big\|_{L^2_{tx}} + \Big\|\frac{1}{\rho^2}\partial_\phi^2 v_\theta\Big\|_{L^2_{tx}} + \Big\|\frac{1}{\rho}\partial_\rho\partial_\phi v_\theta\Big\|_{L^2_{tx}}\Big) + C. \tag{5.14}$$

Recall

$$\nabla K = \nabla\Big(\frac{\omega_\rho}{\rho}\Big) = K_1 e_\rho + K_2 e_\phi,$$

where

$$\begin{cases} K_1 = \dfrac{1}{\rho^2}\partial_\rho\partial_\phi v_\theta - \dfrac{2}{\rho^3}\partial_\phi v_\theta + \dfrac{\cot\phi}{\rho^2}\partial_\rho v_\theta - \dfrac{2\cot\phi}{\rho^3}v_\theta, \\ K_2 = \dfrac{1}{\rho^3}\partial_\phi^2 v_\theta + \dfrac{\cot\phi}{\rho^3}\partial_\phi v_\theta - \dfrac{1}{\rho^3\sin^2\phi}v_\theta. \end{cases}$$

Equivalently,

$$\begin{cases} \dfrac{1}{\rho^2}\partial_\rho\partial_\phi v_\theta = K_1 + \dfrac{2}{\rho^3}\partial_\phi v_\theta - \dfrac{\cot\phi}{\rho^2}\partial_\rho v_\theta + \dfrac{2\cot\phi}{\rho^3}v_\theta, \\ \quad \dfrac{1}{\rho^3}\partial_\phi^2 v_\theta = K_2 - \dfrac{\cot\phi}{\rho^3}\partial_\phi v_\theta + \dfrac{1}{\rho^3\sin^2\phi}v_\theta. \end{cases}$$

Based on the above expressions, we obtain from (5.1) and (5.2) that

$$\Big\|\frac{1}{\rho^2}\partial_\rho\partial_\phi v_\theta\Big\|_{L^2_{tx}} + \Big\|\frac{1}{\rho^3}\partial_\phi^2 v_\theta\Big\|_{L^2_{tx}} \leq C. \tag{5.15}$$

In a similar way, by analyzing $\nabla F = \nabla\Big(\frac{\omega_\phi}{\rho}\Big) = -\nabla\Big(\frac{1}{\rho}\partial_\rho v_\theta + \frac{1}{\rho^2}v_\theta\Big)$, we find

$$\Big\|\frac{1}{\rho}\partial_\rho^2 v_\theta\Big\|_{L^2_{tx}} \leq C. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16) together leads to

$$\|\nabla^2 v_\theta\|_{L^2_{tx}} \leq C. \tag{5.17}$$

Next, we rewrite the equation (3.5) for v_θ as

$$\partial_t v_\theta = \Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)v_\theta - b\cdot\nabla v_\theta - \frac{1}{\rho}\big(v_\rho + \cot\phi\, v_\phi\big)v_\theta.$$

Then we deduce from (5.17) and Proposition 5.1 that

$$\|\partial_t v_\theta\|_{L^2_{tx}} \leq C. \tag{5.18}$$

Step 2: Uniform bounds on $\|\nabla^2 v_\rho\|_{L^2_{tx}}$ and $\|\nabla^2 v_\phi\|_{L^2_{tx}}$.

According to the Biot-Savart law (4.12), we know

$$\left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)v_\rho = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta).$$

By writing Δv_ρ into spherical coordinates, it holds that

$$\partial_\rho^2 v_\rho + \frac{1}{\rho^2}\partial_\phi^2 v_\rho = R_1, \tag{5.19}$$

where

$$R_1 = -\frac{4}{\rho}\partial_\rho v_\rho - \frac{\cot\phi}{\rho^2}\partial_\phi v_\rho - \frac{2}{\rho^2}v_\rho - \frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta).$$

It is readily seen that $\|R_1\|_{L^2_{tx}} \le C$ due to Proposition 5.1. Then we take $L^2(D_m\times[0,T])$ norm on both sides of (5.19) to obtain, after rearranging terms,

$$\int_0^T\int_{D_m}\left[\left(\partial_\rho^2 v_\rho\right)^2 + \frac{1}{\rho^4}\left(\partial_\phi^2 v_\rho\right)^2\right]dx\,dt + 2\underbrace{\int_0^T\int_{D_m}\frac{1}{\rho^2}\,\partial_\rho^2 v_\rho\,\partial_\phi^2 v_\rho\,dx\,dt}_{I_1} \le C. \tag{5.20}$$

By using spherical coordinates,

$$I_1 = 2\pi\int_0^T\int_{\frac{1}{m}}^1\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\sin\phi\,\partial_\rho^2 v_\rho\,\partial_\phi^2 v_\rho\,d\phi\,d\rho.$$

Recalling the boundary conditions in Lemma 2.1 for v_ρ: $\partial_\phi v_\rho = 0$ on $\partial^R D_m$ and $v_\rho = 0$ on $\partial^A D_m$, we further deduce that $\partial_\phi v_\rho = 0$ on $\partial^R D_m \cup \partial^A D_m$. Consequently, one can use integration by parts to find

$$\begin{aligned} I_1 =& -2\pi\int_0^T\int_{\frac{1}{m}}^1\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\cos\phi\,\partial_\rho^2 v_\rho\,\partial_\phi v_\rho\,d\phi\,d\rho \\ & -2\pi\int_0^T\int_{\frac{1}{m}}^1\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}\sin\phi\,\partial_\phi\partial_\rho^2 v_\rho\,\partial_\phi v_\rho\,d\phi\,d\rho \\ :=& I_{11} + I_{12}. \end{aligned} \tag{5.21}$$

For I_{11}, we change back to Euclidean coordinates to get

$$I_{11} = -\int_0^T \int_{D_m} \frac{\cot\phi}{\rho^2}\, \partial_\rho^2 v_\rho\, \partial_\phi v_\rho\, dx\, dt.$$

For I_{12}, we apply integration by parts again to infer that

$$\begin{aligned} I_{12} &= -2\pi \int_0^T \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi \int_{\frac{1}{m}}^{1} \partial_\rho(\partial_\phi \partial_\rho v_\rho)\, \partial_\phi v_\rho\, d\rho\, d\phi\, dt \\ &= 2\pi \int_0^T \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sin\phi \int_{\frac{1}{m}}^{1} (\partial_\phi \partial_\rho v_\rho)^2\, d\rho\, d\phi\, dt \\ &= \int_0^T \int_{D_m} \frac{1}{\rho^2} (\partial_\phi \partial_\rho v_\rho)^2\, dx\, dt. \end{aligned}$$

Plugging the above expressions for I_{11} and I_{12} into (5.21), it then follows from (5.20) that

$$\begin{aligned} &\int_0^T \int_{D_m} \left[\left(\partial_\rho^2 v_\rho\right)^2 + \frac{1}{\rho^4} \left(\partial_\phi^2 v_\rho\right)^2 \right] dx\, dt + 2 \int_0^T \int_{D_m} \frac{1}{\rho^2} (\partial_\phi \partial_\rho v_\rho)^2\, dx\, dt \\ &\le C + 2 \int_0^T \int_{D_m} \frac{\cot\phi}{\rho^2}\, \partial_\rho^2 v_\rho\, \partial_\phi v_\rho\, dx\, dt. \end{aligned}$$

Using Cauchy-Schwarz inequality, the above estimate further implies that

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_{D_m} \left[\left(\partial_\rho^2 v_\rho\right)^2 + \frac{1}{\rho^4} \left(\partial_\phi^2 v_\rho\right)^2 \right] dx\, dt + 2 \int_0^T \int_{D_m} \frac{1}{\rho^2} (\partial_\phi \partial_\rho v_\rho)^2\, dx\, dt \\ &\le C + 2 \int_0^T \int_{D_m} \frac{\cot^2\phi}{\rho^4}\, (\partial_\phi v_\rho)^2\, dx\, dt. \end{aligned} \tag{5.22}$$

Thanks to Proposition 5.1 and the fact that $0 \le \cot\phi \le 1/\sqrt{3}$,

$$\int_0^T \int_{D_m} \frac{\cot^2\phi}{\rho^4} (\partial_\phi v_\rho)^2\, dx\, dt \le \frac{1}{3} \left\| \frac{1}{\rho} \nabla v_\rho \right\|_{L^2_{tx}} \le C.$$

Putting this estimate into (5.22) and using Proposition 5.1 again, we conclude

$$\|\nabla^2 v_\rho\|_{L^2_{tx}} \leq C. \tag{5.23}$$

In order to estimate $\|\nabla^2 v_\phi\|_{L^2_{tx}}$, we make use of the incompressible condition $\nabla \cdot v = 0$, which can be written as

$$\frac{1}{\rho}\partial_\phi v_\phi = -\partial_\rho v_\rho - \frac{2}{\rho}v_\rho - \frac{\cot\phi}{\rho}v_\phi. \tag{5.24}$$

By taking derivatives ∂_ρ and $\frac{1}{\rho}\partial_\phi$ of (5.24), it follows from (5.23) and Proposition 5.1 that

$$\left\|\frac{1}{\rho}\partial_\rho\partial_\phi v_\phi\right\|_{L^2_{tx}} + \left\|\frac{1}{\rho^2}\partial_\phi^2 v_\phi\right\|_{L^2_{tx}} \leq C. \tag{5.25}$$

Then according to the Biot-Savart law (4.12) again,

$$\left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho = \frac{1}{\rho}\,\partial_\rho(\rho\omega_\theta).$$

Writing Δv_ϕ in spherical coordinates, we know

$$\partial_\rho^2 v_\phi = -\frac{1}{\rho^2}\partial_\phi^2 v_\phi - \frac{2}{\rho}\partial_\rho v_\phi - \frac{\cot\phi}{\rho^2}\partial_\phi v_\phi + \frac{1}{\rho^2\sin^2\phi}v_\phi - \frac{2}{\rho^2}\partial_\phi v_\rho + \frac{1}{\rho}\partial_\rho(\rho\omega_\theta).$$

Based on (5.25) and Proposition 5.1, we infer from the above expression that

$$\|\partial_\rho^2 v_\phi\|_{L^2_{tx}} \leq C. \tag{5.26}$$

Combining (5.25), (5.26) with Proposition 5.1 yields

$$\|\nabla^2 v_\phi\|_{L^2_{tx}} \leq C. \tag{5.27}$$

Step 3: Uniform bound on $\|\partial_t\omega_\theta\|_{L^2_{tx}}$.

By rearranging the equation (3.59) for ω_θ, we have

$$\begin{cases} \Delta\omega_\theta - \partial_t\omega_\theta = R_2, & \text{in} \quad D_m \times (0,T]; \\ \omega_\theta = 0, & \text{on} \quad \partial D_m \times (0,T]; \\ \omega_\theta(x,0) = \omega_{0,\theta}(x), & \text{in} \quad D_m, \end{cases} \tag{5.28}$$

where

$$R_2 = \frac{1}{\rho^2\sin^2\phi}\omega_\theta + b\cdot\nabla\omega_\theta - \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta + \frac{1}{\rho^2}\partial_\phi(v_\theta^2) - \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2).$$

Firstly, we deduce from the above expression and Proposition 5.1 that $\|R_2\|_{L^2_{tx}} \leq C$. Then according to (5.28),

$$\int_0^T \int_{D_m} (\Delta\omega_\theta - \partial_t\omega_\theta)^2 \, dx \, dt = \int_0^T \int_{D_m} R_2^2 \, dx \, dt \leq C.$$

Equivalently,

$$\int_0^T \int_{D_m} (\Delta\omega_\theta)^2 + (\partial_t\omega_\theta)^2 \, dx \, dt \leq C + 2\int_0^T \int_{D_m} (\Delta\omega_\theta)(\partial_t\omega_\theta) \, dx \, dt. \tag{5.29}$$

Since $\omega_\theta = 0$ on ∂D_m, which implies $\partial_t\omega_\theta = 0$ on ∂D_m, we can apply integration by parts to obtain

$$\begin{aligned}\text{RHS of } (5.29) &= C - 2\int_0^T \int_{D_m} (\nabla\omega_\theta) \cdot \partial_t(\nabla\omega_\theta) \, dx \, dt \\ &= C - \int_{D_m} \int_0^T \partial_t\left(|\nabla\omega_\theta|^2\right) dt \, dx.\end{aligned}$$

By fundamental theorem of Calculus, the above relation further yields

$$\text{RHS of } (5.29) \leq C + \int_{D_m} |\nabla\omega_{0,\theta}|^2 \, dx \leq C + C\|v_0\|^2_{H^2(D_m)} \leq C.$$

Then we infer from (5.29) that

$$\|\partial_t\omega_\theta\|_{L^2_{tx}} \leq C. \tag{5.30}$$

Step 4: Uniform bounds on $\|\partial_t v_\rho\|_{L^2_{tx}}$ and $\|\partial_t v_\phi\|_{L^2_{tx}}$.

We first estimate $\|\partial_t v_\rho\|_{L^2_{tx}}$. Recall by the Biot-Savart law, v_ρ solves the following problem:

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\right) v_\rho = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\omega_\theta) \quad \text{in} \quad D_m, \\ \partial_\phi v_\rho = 0 \quad \text{on} \quad \partial^R D_m, \quad v_\rho = 0 \quad \text{on} \quad \partial^A D_m. \end{cases}$$

By taking derivative with respect to t, we find $\partial_t v_\rho$ satisfies the equations below:

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\right)(\partial_t v_\rho) = -\frac{1}{\rho\sin\phi}\,\partial_\phi(\sin\phi\,\partial_t\omega_\theta) \quad \text{in} \quad D_m, \\ \partial_\phi(\partial_t v_\rho) = 0 \quad \text{on} \quad \partial^R D_m, \quad \partial_t v_\rho = 0 \quad \text{on} \quad \partial^A D_m. \end{cases}$$

We emphasize that the above equation can be made rigorously by firstly considering the finite difference in time or the Steklov average of v_ρ and ω_θ instead of $\partial_t v_\rho$ and $\partial_t\omega_\theta$, and then taking the limit. Meanwhile, we have

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} (\partial_t v_\rho) \sin\phi \, d\phi = \partial_t \left(\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} v_\rho \sin\phi \, d\phi \right) = 0. \tag{5.31}$$

Then we can argue in an analogous way as that for the proof of Lemma 4.4 to deduce

$$\|\nabla \partial_t v_\rho\|_{L^2_{tx}} \le C \|\partial_t \omega_\theta\|_{L^2_{tx}} \le C,$$

where the last inequality is due to (5.30). Now, by taking advantage of (5.31), it follows from the Poincaré inequality in Lemma 2.3 that

$$\|\partial_t v_\rho\|_{L^2_{tx}} \le C \|\nabla \partial_t v_\rho\|_{L^2_{tx}} \le C. \tag{5.32}$$

Similar to the above argument, we are also able to justify

$$\|\partial_t v_\phi\|_{L^2_{tx}} \le C. \tag{5.33}$$

After establishing the previous Step 1-4, the desired estimate (5.12) will be readily proved. Firstly, the uniform bound $\|\partial_t v\|_{L^2_{tx}} \le C$ follows from (5.18), (5.32) and (5.33). Secondly, since

$$\|\nabla^2 v\|_{L^2_{tx}} \le C \left(\|\nabla^2 v_\rho\|_{L^2_{tx}} + \|\nabla^2 v_\phi\|_{L^2_{tx}} + \|\nabla^2 v_\theta\|_{L^2_{tx}} + \left\| \frac{1}{\rho} \nabla v \right\|_{L^2_{tx}} + \left\| \frac{1}{\rho^2} v \right\|_{L^2_{tx}} \right),$$

the uniform bound $\|\nabla^2 v\|_{L^2_{tx}} \le C$ follows from (5.17), (5.23), (5.27) and Proposition 5.1. □

6. Completion of the proof of Theorem 1.5: existence and uniqueness of strong solutions

In this section, we will establish the main result of this paper by utilizing the uniform bounds derived in the previous Section 4 and Section 5.

Proof of Theorem 1.5. We first show the existence of a strong solution (v, P) which has the even-odd-odd symmetry and satisfies (1.18) and (1.19). Pick any v_0 in the admissible class $\mathscr{A}$ that satisfies the properties (i) and (ii) in Theorem 1.5. By Definition 1.4, there exists a sequence $\{v_0^{(m)}\}_{m\ge 2}$ such that $v_0^{(m)} \in \mathscr{A}_m$ and

$$\lim_{m\to\infty} \|v_0 - v_0^{(m)}\|_{C^2(\overline{D_m})} = 0. \tag{6.1}$$

Since v_0 has the even-odd-odd symmetry due to property (i), we can modify $v_0^{(m)}$ so that it enjoys the same symmetry as well. In fact, by setting

$$\tilde{v}_0^{(m)} = \tilde{v}_{0,\rho}^{(m)} e_\rho + \tilde{v}_{0,\phi}^{(m)} e_\phi + \tilde{v}_{0,\theta}^{(m)} e_\theta,$$

where

$$\begin{cases} \tilde{v}_{0,\rho}^{(m)}(\rho,\phi) = \left[v_{0,\rho}^{(m)}(\rho,\phi) + v_{0,\rho}^{(m)}(\rho,\pi-\phi)\right]/2, \\ \tilde{v}_{0,\phi}^{(m)}(\rho,\phi) = \left[v_{0,\phi}^{(m)}(\rho,\phi) - v_{0,\phi}^{(m)}(\rho,\pi-\phi)\right]/2, \\ \tilde{v}_{0,\theta}^{(m)}(\rho,\phi) = \left[v_{0,\theta}^{(m)}(\rho,\phi) - v_{0,\theta}^{(m)}(\rho,\pi-\phi)\right]/2, \end{cases}$$

then one can directly check that $\tilde{v}_0^{(m)}$ possesses the even-odd-odd symmetry and $\tilde{v}_0^{(m)} \in \mathscr{A}_m$. In addition, the convergence (6.1) is still valid by replacing $v_0^{(m)}$ with $\tilde{v}_0^{(m)}$. For ease of notation, we still denote $\tilde{v}_0^{(m)}$ to be $v_0^{(m)}$. On the other hand, due to the convergence (6.1) and the fact that $\|rv_{0,\theta}\|_{L^\infty(D)} \le \frac{1}{100}$ due to property (ii), there exists some m_0 such that for any $m \ge m_0$,

$$\|v_0^{(m)}\|_{C^2(\overline{D_m})} \le \|v_0\|_{C^2(\overline{D})} + 1, \tag{6.2}$$

$$\|rv_{0,\theta}^{(m)}\|_{L^\infty(D_m)} \le \frac{1}{95}. \tag{6.3}$$

In the following, we will only consider those $v_0^{(m)}$ for $m \ge m_0$.

Now we fix any time $T > 0$. According to Corollary 3.3, for each m, there exists a strong solution $(v^m, P^{(m)})$ of (2.7) on $D_m \times [0,T]$ with the initial data $v_0^{(m)}$ and the NHL boundary condition (2.18). In addition, $v^{(m)}$ is bounded and has the even-odd-odd symmetry. On the other hand, we can assume

$$\int\limits_{D_m} P^{(m)}(x,t)\,dx = 0, \quad \forall\, t \in [0,T]. \tag{6.4}$$

Actually, if we define $\widetilde{P}^{(m)}(x,t) = P^{(m)}(x,t) - \frac{1}{|D_m|}\int_{D_m} P^{(m)}(x,t)\,dx$, then $\widetilde{P}^{(m)}$ satisfies (6.4) and $(v^m, \widetilde{P}^{(m)})$ is also a strong solution.

Next, according to Proposition 5.1 and Proposition 5.2, there exists some constant C_m, only depending on α and $\|v_0^{(m)}\|_{C^2(\overline{D_m})}$ such that

$$\|v^{(m)}\|_{L^\infty_{tx}(D_m\times[0,T])} + \|v^{(m)}\|_{H^1_t L^2_x(D_m\times[0,T])} + \|v^{(m)}\|_{L^2_t H^2_x(D_m\times[0,T])} \le C_m. \tag{6.5}$$

Meanwhile, since (v^m, P^m) is a strong solution of (2.7), then

$$\begin{cases} \partial_\rho P = \left(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2}{\rho^2}\right)v_\rho - b\cdot\nabla v_\rho + \frac{1}{\rho}(v_\phi^2 + v_\theta^2) - \partial_t v_\rho, \\ \frac{1}{\rho}\partial_\phi P = \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)v_\phi - b\cdot\nabla v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho - \frac{1}{\rho}v_\rho v_\phi + \frac{\cot\phi}{\rho}v_\theta^2 - \partial_t v_\phi. \end{cases}$$

By applying Proposition 5.1 and 5.2 again, we find that

$$\|\nabla P^{(m)}\|_{L^2_{tx}(D_m\times[0,T])} \le C_m.$$

Thanks to (6.4), the above estimate further implies that

$$\|P^{(m)}\|_{L_t^2 H_x^1(D_m\times[0,T])} \le C_m. \tag{6.6}$$

Now taking advantage of the uniform bound (6.2), we infer from (6.5) and (6.6) that

$$\|v^{(m)}\|_{L^\infty_{tx}(D_m\times[0,T])} + \|v^{(m)}\|_{H_t^1 L_x^2(D_m\times[0,T])} + \|v^{(m)}\|_{L_t^2 H_x^2(D_m\times[0,T])} + \|P^{(m)}\|_{L_t^2 H_x^1(D_m\times[0,T])} \le C, \tag{6.7}$$

where C is a constant that only depends on α and $\|v_0\|_{C^2(\overline{D})}$. Meanwhile, recalling Proposition 4.3, the following energy inequality for $v^{(m)}$ holds:

$$\int_{D_m} |v^{(m)}(x,T)|^2\,dx + \frac{2}{3}\int_0^T\int_{D_m} |\nabla v^{(m)}(x,t)|^2\,dx\,dt \le \int_{D_m} |v_0^{(m)}(x)|^2\,dx. \tag{6.8}$$

Thanks to the uniform bound (6.7) and the fact that D_m is increasing to D with respect to containment, we can extract a subsequence, still denoted as $\{(v^{(m)}, P^{(m)})\}$, and a vector field $v \in L^\infty_{tx} \cap H_t^1 L_x^2 \cap L_t^2 H_x^2(D\times[0,T])$ and a pressure term $P \in L_t^2 H_x^1(D\times[0,T])$ such that

$$v^{(m)} \to v \text{ pointwisely on } D\times[0,T], \tag{6.9}$$

$$\begin{aligned} &v^{(m)} \to v \text{ weakly in } H_t^1 L_x^2 \cap L_t^2 H_x^2(D\times[0,T]),\\ &P^{(m)} \to P \text{ weakly in } L_t^2 H_x^1(D\times[0,T]), \end{aligned} \tag{6.10}$$

$$v^{(m)} \to v,\ \nabla v^{(m)} \to \nabla v,\ \nabla\times v^{(m)} \to \nabla\times v \text{ weakly in } L^2_{tx}(\partial D\times[0,T]), \tag{6.11}$$

$$v^{(m)}(\cdot,t) \to v(\cdot,0),\ v^{(m)}(\cdot,T) \to v(\cdot,T) \text{ weakly in } L^2(D). \tag{6.12}$$

Based on (6.9), (6.10), we infer from (6.7) that

$$\|v\|_{L^\infty_{tx}(D\times[0,T])} + \|v\|_{H_t^1 L_x^2(D\times[0,T])} + \|v\|_{L_t^2 H_x^2(D\times[0,T])} + \|P\|_{L_t^2 H_x^1(D\times[0,T])} \le C, \tag{6.13}$$

where C is a constant that only depends on α and $\|v_0\|_{C^2(\overline{D})}$. Meanwhile, we can also deduce from (6.9) that v enjoys the even-odd-odd symmetry.

Since $(v^{(m)}, P^{(m)})$ satisfies (2.7) in L^2_{tx} sense on $D_m\times[0,T]$ with initial data v_0 and the NHL boundary condition (2.18), by changing back to Euclidean coordinates, we know that $(v^{(m)}, P^{(m)})$ solves (1.3) in L^2_{tx} sense on $D_m\times[0,T]$ with initial data v_0 and the NHL boundary condition (1.16). More precisely,

$$\Delta v^{(m)} - (v^{(m)}\cdot\nabla)v^{(m)} - \nabla P^{(m)} - \partial_t v^{(m)} = 0, \quad \text{in } L^2\big(D_m\times(0,T]\big), \tag{6.14}$$

$$\nabla\cdot v^{(m)} = 0, \quad \text{in } L^2\big(D_m\times[0,T]\big), \tag{6.15}$$

$$v^{(m)}(\cdot,0) = v_0^{(m)}, \quad \text{in } L^2(D_m), \tag{6.16}$$

$$v^{(m)}\cdot n = 0, \quad (\nabla\times v^{(m)})\times n = 0, \quad \text{on } L^2\big(\partial D_m\times[0,T]\big). \tag{6.17}$$

According to (6.9) and (6.10), we have

$$\big[\Delta v^{(m)} - (v^{(m)}\cdot\nabla)v^{(m)} - \nabla P^{(m)} - \partial_t v^{(m)}\big] \to \big[\Delta v - (v\cdot\nabla)v - \nabla P - \partial_t v\big]$$

weakly in $L^2_{tx}(D\times[0,T])$. It then follows from (6.14) that

$$\Delta v - (v\cdot\nabla)v - \nabla P - \partial_t v = 0, \quad \text{in} \quad L^2\big(D\times(0,T]\big). \tag{6.18}$$

Similarly, by using the convergence (6.9)–(6.12) and (6.1), we can deduce from the identities (6.15)–(6.17) that

$$\begin{cases} \nabla\cdot v = 0, \quad \text{in} \quad L^2\big(D\times[0,T]\big), \\ v(\cdot,0) = v_0, \quad \text{in} \quad L^2(D), \\ v\cdot n = 0, \quad (\nabla\times v)\times n = 0, \quad \text{on} \quad L^2\big(\partial D\times(0,T]\big). \end{cases} \tag{6.19}$$

Hence, the combination of (6.18) and (6.19) shows that (v,P) is a strong solution of (1.3) (or equivalently (1.1)) on $D\times[0,T]$ with the initial data v_0 and the NHL boundary condition (1.5). Moreover, it follows from (6.8), (6.10) and (6.12) that

$$\int\limits_D |v(x,T)|^2\,dx + \frac{2}{3}\int\limits_0^T\int\limits_D |\nabla v(x,t)|^2\,dx\,dt \le \liminf_{m\to\infty}\int\limits_{D_m} |v_0^{(m)}(x)|^2\,dx = \int\limits_D |v_0(x)|^2\,dx,$$

where the last equality is due to (6.1). Thus, we indeed find a strong solution (v,P) which has the even-odd-odd symmetry and satisfies (1.18) and (1.19).

It remains to verify the uniqueness of the strong solution v. Suppose $(\tilde v,\tilde P)$ is another strong solution, with even-odd-odd symmetry, of (1.3) with the same initial data v_0 and the NHL boundary condition (1.5) on $D\times[0,T]$. We will prove that $\tilde v$ coincides with v. Let $f = v-\tilde v$ and $g = P-\tilde P$. Then f satisfies

$$\begin{cases} \Delta f - (f\cdot\nabla)v - (\tilde v\cdot\nabla)f - \nabla g - \partial_t f = 0 \quad \text{in} \quad D\times(0,T], \\ \nabla\cdot f = 0 \quad \text{in} \quad D\times(0,T], \\ f\cdot n = 0, \quad (\nabla\times f)\times n = 0 \quad \text{on} \quad \partial D\times(0,T], \\ f(\cdot,0) = 0 \quad \text{in} \quad D. \end{cases} \tag{6.20}$$

Since both v and $\tilde v$ are strong solutions, f belongs to the space $\mathscr{S}_T$ of test functions defined in (1.12). For any $0 < T_1 < T$, we test the first equation in (6.20) by f on $D\times[0,T_1]$ to find

$$\int_0^{T_1}\int_D (\Delta f)\cdot f\,dx\,dt - \int_0^{T_1}\int_D [(f\cdot\nabla)v]\cdot f\,dx\,dt$$
$$= \int_0^{T_1}\int_D [(\tilde{v}\cdot\nabla)f]\cdot f\,dx\,dt + \int_0^{T_1}\int_D (\nabla g)\cdot f\,dx\,dt + \int_0^{T_1}\int_D (\partial_t f)\cdot f\,dx\,dt.$$

Thanks to the boundary condition and the incompressibility condition of $\tilde{v}$ and f, we know $\int_0^{T_1}\int_D [(\tilde{v}\cdot\nabla)f]\cdot f\,dx\,dt = \int_0^{T_1}\int_D (\nabla g)\cdot f\,dx\,dt = 0$, so

$$\int_0^{T_1}\int_D (\Delta f)\cdot f\,dx\,dt - \int_0^{T_1}\int_D [(f\cdot\nabla)v]\cdot f\,dx\,dt = \frac{1}{2}\int_D f^2(x,T_1)\,dx. \tag{6.21}$$

Then by the similar computation as that in Section A.4, we know

$$\int_0^{T_1}\int_D (\Delta f)\cdot f\,dx\,dt = -\int_0^{T_1}\int_D |\nabla\times f|^2\,dx\,dt.$$

On the other hand, by definition,

$$\int_0^{T_1}\int_D [(f\cdot\nabla)v]\cdot f\,dx\,dt = \sum_{i,j=1}^{3}\int_0^{T_1}\int_D f_j(\partial_{x_j}v_i)f_i\,dx\,dt.$$

Then using integration by parts and taking advantage of the boundary condition and the incompressibility condition of f, we infer that

$$\int_0^{T_1}\int_D [(f\cdot\nabla)v]\cdot f\,dx\,dt = -\sum_{i,j=1}^{3}\int_0^{T_1}\int_D f_j v_i(\partial_{x_j}f_i)\,dx\,dt.$$

Plugging the above results into (6.21) yields

$$\frac{1}{2}\int_D |f(x,T_1)|^2\,dx + \int_0^{T_1}\int_D |\nabla\times f|^2\,dx\,dt = \sum_{i,j=1}^{3}\int_0^{T_1}\int_D f_j v_i(\partial_{x_j}f_i)\,dx\,dt. \tag{6.22}$$

Since $v\in L^\infty_{tx}$ and $f\in L^2_tH^1_x$ on $D\times[0,T]$, we deduce from (6.22) that

$$\begin{aligned}&\frac{1}{2}\int_D |f(x,T_1)|^2\,dx + \int_0^{T_1}\int_D |\nabla\times f|^2\,dx\,dt\\ &\le C\int_0^{T_1}\int_D |f(x,t)|^2\,dx\,dt + \frac{1}{6}\int_0^{T_1}\int_D |\nabla f|^2\,dx\,dt,\end{aligned} \tag{6.23}$$

where C only depends on $\|v\|_{L^{\infty}_{tx}(D\times[0,T])}$. Since both v and $\tilde{v}$ own the even-odd-odd symmetry, then so does f. Therefore, we are able to take advantage of the estimate in Remark 4.2 to find

$$\int_0^{T_1}\int_D |\nabla f|^2\,dx\,dt \le 3\int_0^{T_1}\int_D |\nabla\times f|^2\,dx\,dt.$$

Putting this estimate into (6.23) yields

$$\int_D |f(x,T_1)|^2\,dx \le 2C\int_0^{T_1}\int_D |f(x,t)|^2\,dx\,dt, \quad \forall\, 0<T_1\le T. \tag{6.24}$$

Finally, since both v and $\tilde{v}$ has the same initial data, $f(x,0)=0$ on D. As a result, it follows from (6.24) and Grönwall's inequality that $f=0$ on $D\times[0,T]$. So $\tilde{v}=v$ on $D\times[0,T]$, which justifies the uniqueness of the strong solution v. This completes the proof of Theorem 1.5. $\square$

7. Blowup solutions with finite energy on special cusp domains

As a byproduct of studying the NHL boundary condition (1.5), we will construct a class of blowup solutions to the ASNS (1.1) with finite energy on some cusp domains D_*. This type of domains was considered in [42] to establish the global existence of bounded solutions to (1.1) with finite energy for any smooth initial data under the Navier slip boundary condition as below:

$$v\cdot n=0, \qquad (\mathbb{S}(v)n)_{tan}=0, \quad \text{on} \quad \partial D_*. \tag{7.1}$$

Here, n is the unit outward normal of the smooth part of ∂D_*, $\mathbb{S}(v)=\frac{1}{2}\left[\nabla v+(\nabla v)^T\right]$ is the strain tensor and $(\mathbb{S}(v)n)_{tan}$ stands for the tangential component of the vector $\mathbb{S}(v)n$. Now we give a precise description of the domain D_*.

Definition 7.1. Let $\beta\in(1,\infty)$ be any number. Define the domain D_* as follows (also see Fig. 5).

$$\begin{aligned} &D_* := \bigcup_{m=1}^{\infty} D_m, \quad \text{with} \qquad D_m := \bigcup_{j=1}^{m} S_j, \\ &S_j := \{(r,x_3)\,|\,2^{-j}\le r<2^{-(j-1)},\ 0<x_3<2^{-\beta(j-1)}\}. \end{aligned} \tag{7.2}$$

In [42], one of us chose the parameter β in Definition 7.1 to lie in $(1,1.1)$ and proved that no finite-time blowup occurs under the Navier slip boundary condition (7.1). Now our observation is that when the domain D_* is sufficiently thin (say when $\beta>2$), then a

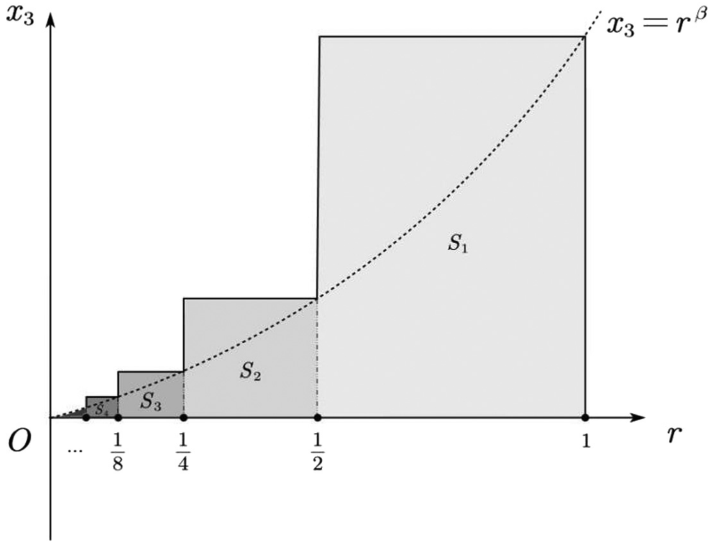

Fig. 5. Domain D_* in cylindrical coordinates.

mildly singular forcing term in the standard regularity class can generate infinite speed for the fluid under the NHL boundary condition (1.5).

Note that the boundary ∂D_* can be written as the union of horizontal and vertical parts, which are denoted by $\partial^H D_*$ and $\partial^V D_*$ respectively. Namely,

$$\partial D_* = \partial^H D_* \cup \partial^V D_*. \tag{7.3}$$

From (1.5), one sees that the NHL boundary condition can be expressed explicitly as

$$\begin{aligned} &v_3 = 0, \quad \omega_r = \omega_\theta = 0, \quad \text{on} \quad \partial^H D_*, \\ &v_r = 0, \quad \omega_3 = \omega_\theta = 0, \quad \text{on} \quad \partial^V D_*. \end{aligned} \tag{7.4}$$

Proposition 7.2. *Let D_* be the cusp domain in Definition 7.1 with $\beta > 2$. Let $\eta = \eta(t)$ be a smooth function of time $t \geq 0$ such that*

$$\eta(t) \begin{cases} = 0 & for \quad t \in [0, 1], \\ = 1 & for \quad t \geq 2. \end{cases}$$

Then $v := \frac{\eta(t)}{r} e_\theta$ is an unbounded solution of the forced axially symmetric Navier-Stokes equation:

$$\Delta v - v\nabla v - \nabla P - \partial_t v = -\frac{\eta'(t)}{r} e_\theta \quad on \quad D_* \times [0, \infty), \tag{7.5}$$

which satisfies the NHL boundary condition (1.5). *Moreover, for any $T > 0$, v is in the energy space with respect to the space time domain $D_* \times [0, T]$ and the forcing term is in the standard regularity class $L_t^\infty L_x^{1.5+}\big(D_* \times [0, T]\big)$.*

Proof. After setting $v_r = 0$ and $v_3 = 0$, (7.5) is reduced to:

$$\begin{cases} \frac{(v_\theta)^2}{r} - \partial_r P = 0, \\ \big(\Delta - \frac{1}{r^2}\big) v_\theta - \partial_t v_\theta = -\frac{\eta'(t)}{r}, \\ \partial_{x_3} P = 0. \end{cases} \tag{7.6}$$

Then by choosing $P = -\frac{\eta^2(t)}{2r^2}$, we see that $v_\theta = \frac{\eta(t)}{r}$ solves (7.6). As a result, $v := \frac{\eta(t)}{r} e_\theta$ is a solution of (7.5). Meanwhile, it is readily seen that $\nabla \times v = 0$, so the NHL boundary condition (7.4) is satisfied.

Thanks to the condition $\beta > 2$ in the definition of D_*, one can easily deduce

$$\int_{D_*} |v|^2(x,t)\, dx \le \eta^2(t) \int_{D_*} \frac{1}{r^2}\, dx \le \eta^2(t)\, 2\pi \int_0^1 \int_0^{2^\beta r^\beta} \frac{1}{r}\, dx_3\, dr < \infty,$$

$$\int_0^T \int_{D_*} |\nabla v|^2(x,t)\, dx \le \sup_{t \ge 0} \eta^2(t) \int_{D_*} \frac{1}{r^4}\, dx\, dt \le \sup_{t\ge 0} \eta^2(t)\, 2\pi \int_0^1 \int_0^{2^\beta r^\beta} \frac{1}{r^3}\, dx_3\, dr < \infty.$$

Therefore, v is in the energy space with respect to $D_* \times [0, T]$. It is also clear that the forcing term $-\frac{\eta'(t)}{r} e_\theta$ is in $L_t^\infty L_x^2 \subset L_t^\infty L_x^{1.5+}$ which is the standard regularity class. This proves the proposition. □

Finally, we recall the remarkable paper [2] in which nonuniqueness is established for the Navier-Stokes equations with a supercritical forcing term in $\mathbb{R}^3$. In contrast, in the aforementioned Proposition 7.2, the forcing term, with a scaling factor of -1, is subcritical, but the domains are special. It confirms the intuition that if the channel of a fluid is very thin, arbitrarily high speed in the classical sense can be attained under a mildly singular force which is physically reasonable in view that Newtonian gravity and Coulomb force have scaling factor -2. Actually, it is a standard physics fact that the force in (7.5) can be realized by the magnetic force generated by an electric current in a long and straight wire (i.e. Ampère force).

Data availability

No data was used for the research described in the article.

Acknowledgments

We wish to thank Professors Hongjie Dong and Zhen Lei for helpful discussions. Z. Li is supported by Natural Science Foundation of Jiangsu Province (No. BK20200803) and National Natural Science Foundation of China (No. 12001285). X. Pan is supported by National Natural Science Foundation of China (No. 11801268 and No. 12031006). X. Yang is supported by the startup fund from Southeast University (No. 4007012321). Q. S. Zhang is grateful to the support of the Simons Foundation through grant No. 710364. N. Zhao is supported by National Natural Science Foundation of China (No. 12301256), Shanghai Sailing Program (No. 22YF1412300) and the Shanghai University of Finance and Economics startup fund (No. 2021110478). Finally, we are grateful to the two anonymous referees for their careful review and constructive suggestions and comments.

Appendix A. Derivations of equations in spherical coordinates

The main purpose of this appendix is to give a short derivation for the equations of the key quantities K, F and Ω (see (2.14)) which are necessary in the proof of the boundedness of the velocity.

But at first, we will give an alternative derivation of the equations for the velocity and the vorticity of the Navier-Stokes system in the spherical coordinate system by using the tensor notation which seems succinct and accessible. Moreover, the equations of the velocity v and the vorticity ω may be slightly different from the classical ones since we will rewrite them using the divergence free condition to fit our purpose.

A.1. Velocity equation (2.7)

We will derive (2.7) from the results obtained for the cylindrical coordinates. First, it follows from (2.3) that

$$\begin{cases} e_r = \sin\phi\, e_\rho + \cos\phi\, e_\phi, \\ e_3 = \cos\phi\, e_\rho - \sin\phi\, e_\phi, \end{cases} \qquad \begin{cases} v_r = \sin\phi\, v_\rho + \cos\phi\, v_\phi, \\ v_3 = \cos\phi\, v_\rho - \sin\phi\, v_\phi, \end{cases} \tag{A.1}$$

where the basis (e_r, e_θ, e_3) and $(e_\rho, e_\phi, e_\theta)$ are defined as in (1.2) and (2.2) respectively. In addition, due to relation (2.1), we know

$$\begin{cases} \partial_r = \sin\phi\, \partial_\rho + \frac{\cos\phi}{\rho}\, \partial_\phi, \\ \partial_{x_3} = \cos\phi\, \partial_\rho - \frac{\sin\phi}{\rho}\, \partial_\phi. \end{cases} \tag{A.2}$$

Consequently,

$$\operatorname{div} v = \Big(\partial_r + \frac{1}{r}\Big) v_r + \partial_{x_3} v_3 = \Big(\partial_\rho + \frac{2}{\rho}\Big) v_\rho + \frac{1}{\rho}(\partial_\phi + \cot\phi) v_\phi \\ = \frac{1}{\rho^2}\partial_\rho(\rho^2 v_\rho) + \frac{1}{\rho\sin\phi}\partial_\phi(\sin\phi\, v_\phi), \tag{A.3}$$

and

$$\nabla = \partial_r \otimes e_r + \frac{1}{r}\partial_\theta \otimes e_\theta + \partial_{x_3} \otimes e_3 = \partial_\rho \otimes e_\rho + \frac{1}{\rho}\partial_\phi \otimes e_\phi + \frac{1}{\rho\sin\phi}\partial_\theta \otimes e_\theta. \tag{A.4}$$

Furthermore,

$$\begin{aligned} \Delta &= \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2 + \partial_{x_3}^2 \\ &= \frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho) + \frac{1}{\rho^2\sin\phi}\partial_\phi(\sin\phi\,\partial_\phi) + \frac{1}{\rho^2\sin^2\phi}\partial_\theta^2. \\ &= \partial_\rho^2 + \frac{2}{\rho}\,\partial_\rho + \frac{1}{\rho^2}\,\partial_\phi^2 + \frac{\cot\phi}{\rho^2}\,\partial_\phi + \frac{1}{\rho^2\sin^2\phi}\,\partial_\theta^2. \end{aligned} \tag{A.5}$$

Noticing

$$\begin{cases} \partial_\phi e_\rho = e_\phi \\ \partial_\phi e_\phi = -e_\rho \\ \partial_\phi e_\theta = 0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_\theta e_\rho = \sin\phi\, e_\theta \\ \partial_\theta e_\phi = \cos\phi\, e_\theta \\ \partial_\theta e_\theta = -\sin\phi\, e_\rho - \cos\phi\, e_\phi. \end{cases} \tag{A.6}$$

Under tensor notations and doing vector calculus under the spherical coordinates, one finds

$$\begin{aligned} \nabla v &= (\partial_\rho v) \otimes e_\rho + \frac{1}{\rho}\partial_\phi v \otimes e_\phi + \frac{1}{\rho\sin\phi}\partial_\theta v \otimes e_\theta \\ &= (\partial_\rho v_\rho\, e_\rho + \partial_\rho v_\phi\, e_\phi + \partial_\rho v_\theta\, e_\theta) \otimes e_\rho \\ &\quad + \frac{1}{\rho}\big[(\partial_\phi v_\rho - v_\phi)\, e_\rho + (v_\rho + \partial_\phi v_\phi)\, e_\phi + \partial_\phi v_\theta\, e_\theta\big] \otimes e_\phi \\ &\quad + \frac{1}{\rho}\big[- v_\theta\, e_\rho - v_\theta \cot\phi\, e_\phi + (v_\rho + v_\phi\cot\phi)\, e_\theta\big] \otimes e_\theta. \end{aligned}$$

It is convenient to denote $e_\rho \otimes e_\rho$, $e_\rho \otimes e_\phi$, $e_\rho \otimes e_\theta$, $\cdots$, $e_\theta \otimes e_\theta$ by the nine single-entry matrices in the standard basis for 3×3 matrices:

$$\begin{pmatrix} \vec{e}_\rho \otimes \vec{e}_\rho & \vec{e}_\rho \otimes \vec{e}_\phi & \vec{e}_\rho \otimes \vec{e}_\theta \\ \vec{e}_\phi \otimes \vec{e}_\rho & \vec{e}_\phi \otimes \vec{e}_\phi & \vec{e}_\phi \otimes \vec{e}_\theta \\ \vec{e}_\theta \otimes \vec{e}_\rho & \vec{e}_\theta \otimes \vec{e}_\phi & \vec{e}_\theta \otimes \vec{e}_\theta \end{pmatrix}. \tag{A.7}$$

Under this basis, ∇v is given by the following 3×3 matrix:

$$\nabla v = \begin{pmatrix} \partial_\rho v_\rho & \frac{1}{\rho}(\partial_\phi v_\rho - v_\phi) & -\frac{1}{\rho}\, v_\theta \\ \partial_\rho v_\phi & \frac{1}{\rho}(\partial_\phi v_\phi + v_\rho) & -\frac{\cot\phi}{\rho}\, v_\theta \\ \partial_\rho v_\theta & \frac{1}{\rho}\partial_\phi v_\theta & \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi) \end{pmatrix}. \tag{A.8}$$

As a result, the coordinate of $(v\cdot\nabla)v$ under the basis $\{e_\rho,\, e_\phi,\, e_\theta\}$ is given by

$$\begin{aligned}(v\cdot\nabla)v &= (\nabla v)v \\ &= \begin{pmatrix} \partial_\rho v_\rho & \frac{1}{\rho}(\partial_\phi v_\rho - v_\phi) & -\frac{1}{\rho}\, v_\theta \\ \partial_\rho v_\phi & \frac{1}{\rho}(\partial_\phi v_\phi + v_\rho) & -\frac{\cot\phi}{\rho}\, v_\theta \\ \partial_\rho v_\theta & \frac{1}{\rho}\partial_\phi v_\theta & \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi) \end{pmatrix} \begin{pmatrix} v_\rho \\ v_\phi \\ v_\theta \end{pmatrix}. \end{aligned} \tag{A.9}$$

In other words,

$$\begin{aligned}(v\cdot\nabla)v = &\Big[\Big(v_\rho\partial_\rho + \frac{1}{\rho}\, v_\phi\partial_\phi\Big)v_\rho - \frac{1}{\rho}\,(v_\phi^2 + v_\theta^2)\Big]e_\rho \\ &+ \Big(\Big[v_\rho\Big(\partial_\rho + \frac{1}{\rho}\Big) + \frac{1}{\rho}\, v_\phi\partial_\phi\Big]v_\phi - \frac{\cot\phi}{\rho}\, v_\theta^2\Big)e_\phi \\ &+ \Big[v_\rho\Big(\partial_\rho + \frac{1}{\rho}\Big)v_\theta + \frac{1}{\rho}\, v_\phi(\partial_\phi + \cot\phi)v_\theta\Big]e_\theta. \end{aligned} \tag{A.10}$$

Moreover,

$$\begin{aligned}\Delta v &= \Big(\partial_\rho^2 + \frac{2}{\rho}\,\partial_\rho + \frac{1}{\rho^2}\,\partial_\phi^2 + \frac{\cot\phi}{\rho^2}\,\partial_\phi + \frac{1}{\rho^2\sin^2\phi}\,\partial_\theta^2\Big)(v_\rho e_\rho + v_\phi e_\phi + v_\theta e_\theta) \\ &= \Big[\Big(\Delta - \frac{2}{\rho^2}\Big)v_\rho - \frac{2}{\rho^2}(\partial_\phi + \cot\phi)v_\phi\Big]e_\rho + \Big[\Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho\Big]\, e_\phi \\ &\quad + \Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)v_\theta\, e_\theta. \end{aligned} \tag{A.11}$$

If v is divergence free, that is $\operatorname{div} v = 0$, then it follows from (A.3) that

$$(\partial_\phi + \cot\phi)v_\phi = \frac{1}{\sin\phi}\,\partial_\phi(\sin\phi\, v_\phi) = -\frac{1}{\rho}\,\partial_\rho(\rho^2 v_\rho) = -2v_\rho - \rho\partial_\rho v_\rho.$$

Putting this relation into (A.11) yields

$$\Delta v = \Big(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\Big)v_\rho e_\rho + \Big[\Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)v_\phi + \frac{2}{\rho^2}\partial_\phi v_\rho\Big]e_\phi + \Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)v_\theta e_\theta. \tag{A.12}$$

Recalling that $b = v_\rho e_\rho + v_\phi e_\phi$, then the combination of (A.10), (A.12) and (A.3) yields (2.7) which is the equivalent expression of (1.1) or (1.3) under the spherical coordinates.

A.2. Vorticity field (2.12) and vorticity equation (2.13)

Recall the vorticity $\omega = \nabla \times v$. In cylindrical coordinates, $\omega = \omega_r e_r + \omega_\theta e_\theta + \omega_3 e_3$, where

$$\omega_r = -\partial_{x_3} v_\theta, \quad \omega_\theta = \partial_{x_3} v_r - \partial_r v_3, \quad \omega_3 = \partial_r v_\theta + \frac{1}{r} v_\theta. \tag{A.13}$$

In spherical coordinates, $\omega = \omega_\rho e_\rho + \omega_\phi e_\phi + \omega_\theta e_\theta$. According to the relation (2.3), $\omega_\rho = \sin\phi\, \omega_r + \cos\phi\, \omega_3$. Then the combination of (A.13) and the relation (A.2) yields

$$\begin{aligned}
\omega_\rho &= -\sin\phi\, \partial_{x_3} v_\theta + \cos\phi \left(\partial_r + \frac{1}{r}\right) v_\theta \\
&= -\sin\phi \left(\cos\phi\, \partial_\rho - \frac{\sin\phi}{\rho}\, \partial_\phi\right) v_\theta + \cos\phi \left(\sin\phi\, \partial_\rho + \frac{\cos\phi}{\rho}\, \partial_\phi + \frac{1}{\rho\sin\phi}\right) v_\theta \\
&= \frac{1}{\rho}(\partial_\phi + \cot\phi) v_\theta.
\end{aligned}$$

In a similar way, we can compute ω_ϕ and ω_θ in the spherical coordinates to verify (2.12).

Next, we will justify the vorticity equations (2.13) in spherical coordinates. First, we recall that the vorticity equation in the Cartesian coordinates is

$$\begin{cases} \Delta\omega - (v\cdot\nabla)\omega + (\omega\cdot\nabla)v - \partial_t \omega = 0, \\ \operatorname{div}\omega = 0. \end{cases} \tag{A.14}$$

Since $\operatorname{div}\omega = 0$, it follows from (A.12) that

$$\Delta\omega = \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)\omega_\rho e_\rho + \left[\left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)\omega_\phi + \frac{2}{\rho^2}\partial_\phi \omega_\rho\right] e_\phi + \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)\omega_\theta e_\theta. \tag{A.15}$$

Then analogous to (A.8), the coordinate of $(\omega\cdot\nabla)v$ under the basis $\{e_\rho, e_\phi, e_\theta\}$ is given by

$$\begin{aligned}
(\omega\cdot\nabla)v &= (\nabla v)\omega \\
&= \begin{pmatrix} \partial_\rho v_\rho & \frac{1}{\rho}(\partial_\phi v_\rho - v_\phi) & -\frac{1}{\rho}\, v_\theta \\ \partial_\rho v_\phi & \frac{1}{\rho}(\partial_\phi v_\phi + v_\rho) & -\frac{\cot\phi}{\rho}\, v_\theta \\ \partial_\rho v_\theta & \frac{1}{\rho}\partial_\phi v_\theta & \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi) \end{pmatrix} \begin{pmatrix} \omega_\rho \\ \omega_\phi \\ \omega_\theta \end{pmatrix}.
\end{aligned} \tag{A.16}$$

In other words,

$$\begin{aligned}
(\omega\cdot\nabla)v = &\left[(\partial_\rho v_\rho)\omega_\rho + \frac{1}{\rho}\,(\partial_\phi v_\rho - v_\phi)\omega_\phi - \frac{1}{\rho}\, v_\theta \omega_\theta\right] e_\rho \\
&+ \left[(\partial_\rho v_\phi)\omega_\rho + \frac{1}{\rho}\,(\partial_\phi v_\phi + v_\rho)\omega_\phi - \frac{\cot\phi}{\rho}\, v_\theta\omega_\theta\right] e_\phi
\end{aligned}$$

$$+\Big[(\partial_\rho v_\theta)\omega_\rho + \frac{1}{\rho}\,(\partial_\phi v_\theta)\omega_\phi + \frac{1}{\rho}\,(v_\rho + \cot\phi\, v_\phi)\omega_\theta\Big]\, e_\theta.$$

By switching ω and v,

$$\begin{aligned}(v\cdot\nabla)\omega = &\Big[(\partial_\rho \omega_\rho)v_\rho + \frac{1}{\rho}\,(\partial_\phi \omega_\rho - \omega_\phi)v_\phi - \frac{1}{\rho}\,\omega_\theta v_\theta\Big]\, e_\rho\\ &+\Big[(\partial_\rho \omega_\phi)v_\rho + \frac{1}{\rho}\,(\partial_\phi \omega_\phi + \omega_\rho)v_\phi - \frac{\cot\phi}{\rho}\,\omega_\theta v_\theta\Big]\, e_\phi\\ &+\Big[(\partial_\rho \omega_\theta)v_\rho + \frac{1}{\rho}\,(\partial_\phi \omega_\theta)v_\phi + \frac{1}{\rho}\,(\omega_\rho + \cot\phi\,\omega_\phi)v_\theta\Big]\, e_\theta.\end{aligned}$$

Consequently,

$$\begin{aligned}&-(v\cdot\nabla)\omega + (\omega\cdot\nabla)v\\ =&\Big[-\Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)\omega_\rho + \Big(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi\Big)v_\rho\Big]e_\rho\\ &+\Big[-\Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)\omega_\phi + \Big(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi\Big)v_\phi + \frac{1}{\rho}\Big(v_\rho\omega_\phi - \omega_\rho v_\phi\Big)\Big]e_\phi\\ &+\Big[-\Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)\omega_\theta + \Big(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi\Big)v_\theta + \frac{1}{\rho}(v_\rho\omega_\theta - \omega_\rho v_\theta)\\ &+\frac{\cot\phi}{\rho}(v_\phi\omega_\theta - \omega_\phi v_\theta)\Big]e_\theta.\end{aligned}$$

By taking advantage of the formulas for ω_ρ and ω_ϕ in (2.12), we are able to discover some cancellation and therefore simplify the above e_θ component. Actually,

$$\begin{aligned}&(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi)v_\theta + \frac{1}{\rho}(v_\rho\omega_\theta - \omega_\rho v_\theta) + \frac{\cot\phi}{\rho}(v_\phi\omega_\theta - \omega_\phi v_\theta)\\ =&\frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta + (\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi)v_\theta - \frac{1}{\rho}(\omega_\rho + \cot\phi\,\omega_\phi)v_\theta\\ =&\frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta - \frac{1}{\rho^2}\partial_\phi(v_\theta^2) + \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2).\end{aligned}$$

Meanwhile, recall $b = v_\rho e_\rho + v_\phi e_\phi$, so

$$\begin{aligned}&-(v\cdot\nabla)\omega + (\omega\cdot\nabla)v\\ =&\Big[-b\cdot\nabla\omega_\rho + \omega\cdot\nabla v_\rho\Big]e_\rho + \Big[-b\cdot\nabla\omega_\phi + \omega\cdot\nabla v_\phi + \frac{1}{\rho}\Big(v_\rho\omega_\phi - \omega_\rho v_\phi\Big)\Big]e_\phi\\ &+\Big[-b\cdot\nabla\omega_\theta + \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta - \frac{1}{\rho^2}\partial_\phi(v_\theta^2) + \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2)\Big]e_\theta.\end{aligned}\tag{A.17}$$

Putting the above formulas (A.15) and (A.17) into (A.14) leads to the following (A.18) which is exactly the same as (2.13).

$$\begin{cases} \left(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\right)\omega_\rho - b\cdot\nabla\omega_\rho + \omega\cdot\nabla v_\rho - \partial_t\omega_\rho = 0, \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)\omega_\phi - b\cdot\nabla\omega_\phi + \frac{2}{\rho^2}\partial_\phi\omega_\rho + \omega\cdot\nabla v_\phi + \frac{1}{\rho}(v_\rho\omega_\phi - \omega_\rho v_\phi) - \partial_t\omega_\phi = 0, \\ \left(\Delta - \frac{1}{\rho^2\sin^2\phi}\right)\omega_\theta - b\cdot\nabla\omega_\theta + \frac{1}{\rho}(v_\rho + \cot\phi\, v_\phi)\omega_\theta - \frac{1}{\rho^2}\partial_\phi(v_\theta^2) + \frac{\cot\phi}{\rho}\partial_\rho(v_\theta^2) - \partial_t\omega_\theta = 0, \\ \frac{1}{\rho^2}\partial_\rho(\rho^2\omega_\rho) + \frac{1}{\rho\sin\phi}\partial_\phi(\sin\phi\,\omega_\phi) = 0. \end{cases} \tag{A.18}$$

A.3. System (2.15) of K, F and Ω

Recall the definition (2.14) for K, F and Ω: $K = \frac{\omega_\rho}{\rho}$, $F = \frac{\omega_\phi}{\rho}$ and $\Omega = \frac{\omega_\theta}{\rho\sin\phi}$. In other words,

$$\omega_\rho = \rho K, \quad \omega_\phi = \rho F, \quad \omega_\theta = \rho\sin\phi\,\Omega. \tag{A.19}$$

Let's first deal with K. Based on the first equation in (A.18), we have

$$0 = \Big(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\Big)\omega_\rho - \Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)\omega_\rho + \Big(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi\Big)v_\rho - \partial_t\omega_\rho.$$

Putting the relation (A.19) into this equation yields

$$0 = \Big(\Delta + \frac{2}{\rho}\,\partial_\rho + \frac{2}{\rho^2}\Big)(\rho K) - \Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)(\rho K) + \Big(\rho K\partial_\rho + F\partial_\phi\Big)v_\rho - \partial_t(\rho K). \tag{A.20}$$

Noticing

$$\Delta(\rho K) = \rho\Delta K + 2\partial_\rho K + \frac{2}{\rho}\,K,$$

$$-\Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)(\rho K) + v_\rho K = -\rho v_\rho\partial_\rho K - v_\phi\partial_\phi K = -\rho b\cdot\nabla K,$$

and

$$\begin{aligned} \Big(\rho K\partial_\rho + F\partial_\phi\Big)v_\rho - Kv_\rho &= K(\rho\partial_\rho v_\rho - v_\rho) + F\partial_\phi v_\rho \\ &= \rho\omega_\rho\partial_\rho\Big(\frac{v_\rho}{\rho}\Big) + \rho\omega_\phi\frac{1}{\rho}\partial_\phi\Big(\frac{v_\rho}{\rho}\Big) = \rho\omega\cdot\nabla\Big(\frac{v_\rho}{\rho}\Big). \end{aligned}$$

Putting all these identities into (A.20) yields

$$0 = \Big(\rho\Delta + 4\partial_\rho + \frac{6}{\rho}\Big)K - \rho b\cdot\nabla K + \rho\omega\cdot\nabla\Big(\frac{v_\rho}{\rho}\Big) - \partial_t(\rho K).$$

Dividing by ρ leads to

$$0 = \Big(\Delta + \frac{4}{\rho}\partial_\rho + \frac{6}{\rho^2}\Big)K - b\cdot\nabla K + \omega\cdot\nabla\Big(\frac{v_\rho}{\rho}\Big) - \partial_t K.$$

This verifies the K equation in (2.15).

Now we will continue to discuss the case for F. Based on the second equation in (A.18), we know

$$\begin{aligned}0 = &\Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)\omega_\phi - \Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)\omega_\phi + \frac{2}{\rho^2}\partial_\phi\omega_\rho\\ &+ \Big(\omega_\rho\partial_\rho + \frac{1}{\rho}\omega_\phi\partial_\phi\Big)v_\phi + \frac{1}{\rho}(v_\rho\omega_\phi - \omega_\rho v_\phi) - \partial_t\omega_\phi.\end{aligned}$$

Putting the relation (A.19) into this equation yields

$$\begin{aligned}0 = &\Big(\Delta - \frac{1}{\rho^2\sin^2\phi}\Big)(\rho F) - \Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)(\rho F) + \frac{2}{\rho^2}\partial_\phi(\rho K)\\ &+ \Big(\rho K\partial_\rho + F\partial_\phi\Big)v_\phi + (v_\rho F - K v_\phi) - \partial_t(\rho F).\end{aligned} \tag{A.21}$$

Noticing

$$\Delta(\rho F) = \rho\Delta F + 2\partial_\rho F + \frac{2}{\rho}F,$$

$$-\Big(v_\rho\partial_\rho + \frac{1}{\rho}v_\phi\partial_\phi\Big)(\rho F) + v_\rho F = -\rho v_\rho\partial_\rho F - v_\phi\partial_\rho F = -\rho b\cdot\nabla F,$$

and

$$\begin{aligned}\Big(\rho K\partial_\rho + F\partial_\phi\Big)v_\phi - K v_\phi &= K(\rho\partial_\rho v_\phi - v_\phi) + F\partial_\phi v_\phi\\ &= \rho\omega_\rho\partial_\rho\Big(\frac{v_\phi}{\rho}\Big) + \rho\omega_\phi\frac{1}{\rho}\partial_\phi\Big(\frac{v_\phi}{\rho}\Big) = \rho\omega\cdot\nabla\Big(\frac{v_\phi}{\rho}\Big).\end{aligned}$$

Putting all these identities into (A.21) yields

$$0 = \Big(\rho\Delta + 2\partial_\rho + \frac{2}{\rho} - \frac{1}{\rho\sin^2\phi}\Big)F - \rho b\cdot\nabla F + \frac{2}{\rho^2}\partial_\phi(\rho K) + \rho\omega\cdot\nabla\Big(\frac{v_\phi}{\rho}\Big) - \partial_t(\rho F).$$

Dividing by ρ leads to

$$0 = \Big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{1-\cot^2\phi}{\rho^2}\Big)F - b\cdot\nabla F + \frac{2}{\rho^2}\partial_\phi K + \omega\cdot\nabla\Big(\frac{v_\phi}{\rho}\Big) - \partial_t F.$$

This verifies the F equation in (2.15).

Finally, the equation for Ω in spherical coordinates will be deduced. Rather than deriving its equation directly, it is helpful to take advantage of the result in the cylindrical coordinates case. In fact, it has already been known that Ω satisfies the following equation (see [39]).

$$\Delta\Omega - b\cdot\nabla\Omega + \frac{2}{r}\,\partial_r\Omega - \frac{2}{r^2}\,v_\theta\omega_r - \partial_t\Omega = 0. \tag{A.22}$$

Based on (A.2), we have

$$\frac{2}{r}\,\partial_r\Omega = \frac{2}{\rho\sin\phi}\Big(\sin\phi\,\partial_\rho + \frac{\cos\phi}{\rho}\,\partial_\phi\Big)\Omega = \frac{2}{\rho}\Big(\partial_\rho + \frac{\cot\phi}{\rho}\,\partial_\phi\Big)\Omega.$$

Applying the relation (A.1) to ω,

$$\frac{2}{r^2}\,v_\theta\omega_r = \frac{2v_\theta}{\rho^2\sin^2\phi}\,(\sin\phi\,\omega_\rho + \cos\phi\,\omega_\phi) = \frac{2v_\theta}{\rho\sin\phi}(K + \cot\phi\,F).$$

Putting the above two identities into (A.22) gives

$$\Big(\Delta + \frac{2}{\rho}\partial_\rho + \frac{2\cot\phi}{\rho^2}\partial_\phi\Big)\Omega - b\cdot\nabla\Omega - \frac{2v_\theta}{\rho\sin\phi}\,(K + \cot\phi\,F) - \partial_t\Omega = 0.$$

This verifies the Ω equation in (2.15).

The last equation $(2.15)_4$ can be derived immediately from the divergence free condition $(2.13)_4$ of ω.

A.4. Integration identity for strong solutions of (1.3) under the NHL boundary condition

The purpose of this subsection is to justify the integration identity (1.13). Assume $v \in \mathscr{S}_T$ (see (1.12)) and $P \in L^2_t H^1_x(D\times[0,T])$ such that (v,P) satisfies the following equations:

$$\begin{cases} \Delta v - (v\cdot\nabla)v - \nabla P - \partial_t v = 0 & \text{in} \quad D\times(0,T], \\ \nabla\cdot v = 0 & \text{in} \quad D\times(0,T], \\ v\cdot n = 0, \quad \omega\times n = 0 & \text{on} \quad \partial D\times(0,T], \\ v(\cdot,0) = v_0(\cdot) & \text{in} \quad D. \end{cases} \tag{A.23}$$

Then for any vector field $f \in \mathscr{S}_T$, we will prove the following integration identity:

$$\begin{aligned} &\int_D v(x,T)\cdot f(x,T)\,dx + \int_0^T\int_D (\nabla\times v)\cdot(\nabla\times f)\,dx\,dt \\ &= \int_D v_0(x)\cdot f(x,0)\,dx - \int_0^T\int_D [(v\cdot\nabla)v]\cdot f\,dx\,dt + \int_0^T\int_D v\cdot(\partial_t f)\,dx\,dt. \end{aligned} \tag{A.24}$$

Proof. Testing the first equation in (A.23) by f, we find

$$\begin{aligned}\underbrace{\int_0^T\int_D f\cdot(\Delta v)\,dx\,dt}_{I_1} &= \underbrace{\int_0^T\int_D [(v\cdot\nabla)v]\cdot f\,dx\,dt}_{I_2} + \underbrace{\int_0^T\int_D f\cdot\nabla P\,dx\,dt}_{I_3} \\ &\quad + \underbrace{\int_0^T\int_D f\cdot(\partial_t v)\,dx\,dt}_{I_4}.\end{aligned}\tag{A.25}$$

We first compute I_1. Since $\nabla\cdot v=0$, it holds that

$$\begin{aligned}\int_D f\cdot\Delta v\,dx &= \sum_{i,j=1}^3\int_D f_i\partial_j^2 v_i\,dx = \sum_{i,j=1}^3\int_D f_i\partial_j(\partial_j v_i-\partial_i v_j)\,dx + \sum_{i,j=1}^3\int_D f_i\partial_i(\partial_j v_j)\,dx \\ &= \sum_{i,j=1}^3\int_D f_i\partial_j(\partial_j v_i-\partial_i v_j)\,dx.\end{aligned}$$

Then using integration by parts,

$$\sum_{i,j=1}^3\int_D f_i\partial_j(\partial_j v_i-\partial_i v_j)\,dx = \sum_{i,j=1}^3\int_{\partial D} f_i(\partial_j v_i-\partial_i v_j)n_j\,dS - \sum_{i,j=1}^3\int_D(\partial_j f_i)(\partial_j v_i-\partial_i v_j)\,dx.$$

Since $\omega\times n=0$ on ∂D, then $\sum_{j=1}^3(\partial_j v_i-\partial_i v_j)n_j=0$ for any fixed i. Therefore, the above surface integral on the boundary ∂D vanishes and the equation reduces to

$$\sum_{i,j=1}^3\int_D f_i\partial_j(\partial_j v_i-\partial_i v_j)\,dx = -\sum_{i,j=1}^3\int_D(\partial_j f_i)(\partial_j v_i-\partial_i v_j)\,dx.$$

Denote $J_1=\sum_{i,j=1}^3\int_D(\partial_j f_i)(\partial_j v_i-\partial_i v_j)\,dx$. Then we can split J_1 to be $J_1=J_{11}+J_{12}$, where

$$J_{11}=\sum_{i,j=1}^3\int_D(\partial_j f_i-\partial_i f_j)(\partial_j v_i-\partial_i v_j)\,dx,\quad J_{12}=\sum_{i,j=1}^3\int_D(\partial_i f_j)(\partial_j v_i-\partial_i v_j)\,dx.$$

Noticing that $J_{11}=2\int_D(\nabla\times f)\cdot(\nabla\times v)\,dx$ and $J_{12}=-J_1$, we obtain $J_1=\int_D(\nabla\times f)\cdot(\nabla\times v)\,dx$. As a result,

$$I_1 = -\int_0^T \int_D (\nabla \times f) \cdot (\nabla \times v)\, dx.$$

Next, we compute the RHS of (A.25). For I_2, it is kept unchanged. For I_3, it follows from integration by parts and the property of the set $\mathscr{S}_T$ that

$$I_3 = \int_0^T \int_{\partial D} (f \cdot n) P\, dS - \int_0^T \int_D (\nabla \cdot f) P\, dx = 0.$$

For I_4, using integration by parts in the temporal variable yields

$$I_4 = \int_D v(x,T) \cdot f(x,T)\, dx - \int_D v_0(x) \cdot f(x,0)\, dx - \int_0^T \int_D v \cdot (\partial_t f)\, dx\, dt.$$

Plugging the above computations of I_1-I_4 into (A.25) leads to (A.24). □

References

[1] R.A. Adams, John J.F. Fournier, Sobolev Spaces, second edition, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] D. Albritton, E. Brué, M. Colombo, Non-uniqueness of Leray solutions of the forced Navier-Stokes equations, Ann. Math. (2) 196 (1) (2022) 415–455.

[3] H. Beirao da Veiga, J. Yang, Regularity criteria for Navier-Stokes equations with slip boundary conditions on non-flat boundaries via two velocity components, Adv. Nonlinear Anal. 9 (1) (2020) 633–643.

[4] J. Burke Loftus, Q.S. Zhang, A priori bounds for the vorticity of axially symmetric solutions to the Navier-Stokes equations, Adv. Differ. Equ. 15 (5–6) (2010) 531–560.

[5] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Commun. Pure Appl. Math. 35 (6) (1982) 771–831.

[6] J.-Y. Chemin, P. Zhang, Z. Zhang, On the critical one component regularity for 3-D Navier-Stokes system: general case, Arch. Ration. Mech. Anal. 224 (3) (2017) 871–905.

[7] C.-C. Chen, R.M. Strain, T.-P. Tsai, H.-T. Yau, Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations, Int. Math. Res. Not. (9) (2008) rnn016.

[8] C.-C. Chen, R.M. Strain, T.-P. Tsai, H.-T. Yau, Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II, Commun. Partial Differ. Equ. 34 (1–3) (2009) 203–232.

[9] G.-Q. Chen, Z. Qian, A study of the Navier-Stokes equations with the kinematic and Navier boundary conditions, Indiana Univ. Math. J. 59 (2) (2010) 721–760.

[10] H. Chen, D. Fang, T. Zhang, Regularity of 3D axisymmetric Navier-Stokes equations, Discrete Contin. Dyn. Syst. 37 (4) (2017) 1923–1939.

[11] H. Chen, T.-P. Tsai, T. Zhang, Remarks on local regularity of axisymmetric solutions to the 3D Navier–Stokes equations, arXiv:2201.01766.

[12] P. Deuring, W. von Wahl, Strong solutions of the Navier-Stokes system in Lipschitz bounded domains, Math. Nachr. 171 (1995) 111–148.

[13] T.M. Elgindi, I.-J. Jeong, Finite-time singularity formation for strong solutions to the axi-symmetric 3D Euler equations, Ann. PDE 5 (2) (2019) 16.

[14] T.Y. Hou, Z. Lei, C. Li, Global regularity of the 3D axi-symmetric Navier-Stokes equations with anisotropic data, Commun. Partial Differ. Equ. 33 (7–9) (2008) 1622–1637.

[15] J.P. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane, SIAM J. Math. Anal. 38 (1) (2006) 210–232.

[16] D. Kim, S. Ryu, The weak maximum principle for second-order elliptic and parabolic conormal derivative problems, Commun. Pure Appl. Anal. 19 (1) (2020) 493–510.
[17] G. Koch, N. Nadirashvili, G.A. Seregin, V. Sverak, Liouville theorems for the Navier-Stokes equations and applications, Acta Math. 203 (1) (2009) 83–105.
[18] O.A. Ladyzhenskaya, Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry, Zap. Naučn. Semin. LOMI 7 (1968) 155–177.
[19] Z. Lei, Q.S. Zhang, Criticality of the axially symmetric Navier-Stokes equations, Pac. J. Math. 289 (1) (2017) 169–187.
[20] S. Leonardi, J. Malek, J. Necas, M. Pokorny, On axially symmetric flows in $\mathbf{R}^3$, Z. Anal. Anwend. 18 (3) (1999) 639–649.
[21] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1) (1934) 193–248.
[22] F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, Commun. Pure Appl. Math. 51 (3) (1998) 241–257.
[23] A. Mahalov, E.S. Titi, S. Leibovich, Invariant helical subspaces for the Navier-Stokes equations, Arch. Ration. Mech. Anal. 112 (3) (1990) 193–222.
[24] N. Masmoudi, F. Rousset, Uniform regularity for the Navier-Stokes equation with Navier boundary condition, Arch. Ration. Mech. Anal. 203 (2) (2012) 529–575.
[25] M. Mitrea, S. Monniaux, The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains, Differ. Integral Equ. 22 (3–4) (2009) 339–356.
[26] C.L.M.H. Navier, Sur les lois du mouvement des fuides, Mem. Acad. R. Sci. Inst. Fr. 6 (1823) 389–440.
[27] J. Neustupa, M. Pokorny, An interior regularity criterion for an axially symmetric suitable weak solution to the Navier-Stokes equations, J. Math. Fluid Mech. 2 (4) (2000) 381–399.
[28] X. Pan, Regularity of solutions to axisymmetric Navier-Stokes equations with a slightly supercritical condition, J. Differ. Equ. 260 (12) (2016) 8485–8529.
[29] L.E. Payne, H.F. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Ration. Mech. Anal. 5 (1960) 286–292.
[30] D. Phan, S.S. Rodrigues, Gevrey regularity for Navier-Stokes equations under Lions boundary conditions, J. Funct. Anal. 272 (7) (2017) 2865–2898.
[31] D. Phan, S.S. Rodrigues, Approximate controllability for Navier-Stokes equations in 3D rectangles under Lions boundary conditions, J. Dyn. Control Syst. 25 (3) (2019) 351–376.
[32] J.C. Robinson, J.L. Rodrigo, W. Sadowski, The Three-Dimensional Navier-Stokes Equations: Classical Theory, Cambridge Studies in Advanced Mathematics, vol. 157, Cambridge University Press, Cambridge, 2016.
[33] G. Seregin, A note on local regularity of axisymmetric solutions to the Navier–Stokes equations, J. Math. Fluid Mech. 24 (1) (2022) 27.
[34] G. Seregin, A slightly supercritical condition of regularity of axisymmetric solutions to the Navier-Stokes equations, J. Math. Fluid Mech. 24 (1) (2022) 18.
[35] G. Seregin, V. Sverak, On type I singularities of the local axi-symmetric solutions of the Navier-Stokes equations, Commun. Partial Differ. Equ. 34 (1–3) (2009) 171–201.
[36] T. Tao, Why global regularity for Navier–Stokes is hard, Online blog, https://tinyurl.com/tao-navier-stokes, 2007.
[37] T. Tao, Quantitative bounds for critically bounded solutions to the Navier-Stokes equations, in: Nine Mathematical Challenges—an Elucidation, in: Proc. Sympos. Pure Math., vol. 104, Amer. Math. Soc., Providence, RI, 2021, pp. 149–193.
[38] R. Temam, M. Ziane, Navier-Stokes equations in three-dimensional thin domains with various boundary conditions, Adv. Differ. Equ. 1 (4) (1996) 499–546.
[39] M.R. Ukhovskii, V.I. Yudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, J. Appl. Math. Mech. 32 (1968) 52–61.
[40] D. Wei, Regularity criterion to the axially symmetric Navier-Stokes equations, J. Math. Anal. Appl. 435 (1) (2016) 402–413.
[41] Y. Xiao, Z. Xin, On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, Commun. Pure Appl. Math. 60 (7) (2007) 1027–1055.
[42] Q.S. Zhang, Bounded solutions to the axially symmetric Navier Stokes equation in a cusp region, J. Differ. Equ. 312 (2022) 407–473.