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# On global smooth solutions of the 3D spherically symmetric Euler equations with time-dependent damping and physical vacuum

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**Abstract**

In this paper, we consider the global existence and convergence of smooth solutions for the three dimensional spherically symmetric compressible Euler equations with time-dependent damping and physical vacuum. The damping coefficient decays with time and the sound speed is $C^{1/2}$-Hölder continuous across the physical vacuum boundary. Both the degeneration of the damping coefficient at time infinity and the non $C^1$ continuity of the sound speed across the vacuum boundary will cause difficulty in proving the global existence of smooth solutions. Under suitable assumptions on the decayed damping coefficients, the globally in-time smooth solutions and convergence to the modified Barenblatt solution will be given. Also obtained are the pointwise convergence rate of the density, velocity and the expanding rate of the physical vacuum boundary. Our result extends that in Zeng (2017 *Arch. Ration. Mech. Anal.* **226** 33–82) by considering the degenerate damping coefficient instead of the constant damping coefficient and that in Pan (2021 *Calc. Var. Partial Differ. Equ.* **60** 5) from the one dimensional case to the three dimensional case with spherically symmetric data.

Keywords: global smooth solutions, compressible Euler equations, time-dependent damping, physical vacuum

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(Some figures may appear in colour only in the online journal)

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## 1. Introduction

In this paper, the following 3D compressible Euler equations with time-dependent damping and physical vacuum will be considered.

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0 & \text{in } \Omega(t), \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_{\boldsymbol{x}} p(\rho) = -\dfrac{\mu}{(1+t)^\lambda} \rho \boldsymbol{u} & \text{in } \Omega(t), \\ \rho > 0 & \text{in } \Omega(t), \\ \rho = 0 & \text{on } \Gamma(t) := \partial\Omega(t), \\ \mathcal{V}(\Gamma(t)) = \boldsymbol{u} \cdot \boldsymbol{n}, & \\ (\rho, \boldsymbol{u})\big|_{t=0} = (\rho_0, \boldsymbol{u}_0) & \text{on } \Omega := \Omega(0). \end{cases} \tag{1.1}$$

Here $\rho$, $\boldsymbol{u}$, and $p$ represent the density, velocity, and pressure, respectively. $\Omega(t)$, $\Gamma(t)$, $\mathcal{V}(\Gamma(t))$, and $\boldsymbol{n}$ denote the domain where the gas exists, the physical vacuum boundary, the normal velocity on $\Gamma(t)$ and the unit outward normal vector on $\Gamma(t)$, respectively. The term $-\frac{\mu}{(1+t)^\lambda}\rho\boldsymbol{u}$, which decays with power $-\lambda$ in time, is the frictional damping. The gas is assumed to be isentropic, which means the pressure satisfies the $\gamma$ law:

$$p(\rho) = \rho^\gamma \quad \text{for } \gamma > 1.$$

Then the sound speed $c := \sqrt{p'(\rho)}$. If

$$-\infty < \frac{\partial c^2}{\partial \boldsymbol{n}} < 0 \tag{1.2}$$

in a small neighborhood of the boundary, we call that a vacuum boundary is physical.

Damping can affect the asymptotic behavior of solutions of the Euler equations. When the damping vanishes (the damping coefficient $\mu \equiv 0$), shock will form. See [1, 2, 42, 44] and references therein for more details. While for the Euler equations with constant-coefficient damping ($\lambda = 0, 0 < \mu$), global existence and asymptotic behavior of smooth solutions away from vacuum can be founded in [15, 34, 47] and references therein. It is natural to ask whether there are some global or blow-up results of solutions of the Euler equations with variant-coefficient damping, which decays in time. A typical type of time decayed damping coefficient is $\frac{\mu}{(1+t)^\lambda}$. Actually now there are already numerous works concerning on the global existence, finite-time blow up, and asymptotic behaviors of smooth solutions for system (1.1) with initial data away from vacuum. A critical couple of numbers $(\lambda,\ \mu)$, depending on the space dimension, is given to separate the global existence and finite-time blow up of smooth solutions in Hou–Witt–Yin [18, 19] and Pan [35–37]. Later, various results are shown in this aspect. Reader can refer to [3, 7, 12, 23, 24, 30, 41] and references therein.

If the initial data contains physical vacuum, Luo–Zeng [33] and Zeng [48] proved the global existence of smooth solutions and convergence to the Barenblatt solutions of the porous media equation for the one dimensional and three dimensional spherically symmetric Euler equations with constant-coefficient damping. Recently the result in [33] was extended in [38] to the time decayed damping system. Also a stability results of smooth solutions for system (1.1) with one side physical vacuum was established in [39]. Our main purpose of this paper is to extend the result in [48] to system (1.1) with spherically initial data and $0 < \lambda$.

As shown in Pan [38] in one dimensional case, system (1.1) is closedly related to the related porous media equations with time-dependent dissipation, read as follows

$$\begin{cases} \rho_t + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u}) = 0, \\ \nabla_{\boldsymbol{x}} p(\rho) = -\dfrac{\mu}{(1+t)^{\lambda}} \rho \boldsymbol{u}. \end{cases} \tag{1.3}$$

Actually, the above equations enjoys the following space and time variables scaling and translation.

For any constant $\mathfrak{c} > 0$, set

$$\tilde{\rho}(\tilde{\boldsymbol{x}}, \tilde{t}) = \mathfrak{c}^{\frac{2\lambda}{\gamma-1}} \rho(\mathfrak{c}\tilde{\boldsymbol{x}}, \mathfrak{c}^2 \tilde{t}) = \mathfrak{c}^{\frac{2\lambda}{\gamma-1}} \rho(\boldsymbol{x}, t-1), \tag{1.4}$$

and

$$\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}, \tilde{t}) = \mathfrak{c}\boldsymbol{u}(\mathfrak{c}\tilde{x}, \mathfrak{c}^2 \tilde{t}) = \mathfrak{c}\boldsymbol{u}(\boldsymbol{x}, t-1), \tag{1.5}$$

then we can see that $\tilde{\rho}$ and $\tilde{\boldsymbol{u}}$ satisfy

$$\begin{cases} \tilde{\rho}_{\tilde{t}} + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\boldsymbol{u}}) = 0, \\ \nabla_{\tilde{x}} p(\tilde{\rho}) = -\dfrac{\mu}{\tilde{t}^{\lambda}} \tilde{\rho}\tilde{\boldsymbol{u}}. \end{cases} \tag{1.6}$$

If we make the same scaling and translation (1.4) and (1.5) to system (1.1), we can see that the first two equations of system (1.1) are transformed to

$$\begin{cases} \tilde{\rho}_{\tilde{t}} + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\boldsymbol{u}}) = 0 \\ \mathfrak{c}^{2(\lambda-1)} [(\tilde{\rho}\tilde{\boldsymbol{u}})_{\tilde{t}} + \nabla_{\tilde{x}} \cdot (\tilde{\rho}\tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}})] + \nabla_{\tilde{x}} p(\tilde{\rho}) = -\dfrac{\mu}{\tilde{t}^{\lambda}} \tilde{\rho}. \end{cases} \tag{1.7}$$

Since we are considering the global existence and large time behavior of solutions, when $\lambda < 1$, letting $\mathfrak{c} \to +\infty$ indicates that solutions of (1.7) approach to solutions of (1.6) formally. So we can conjecture that solutions of system (1.1) will asymptotically converge to that of (1.3) if their initial data is close. Actually we will see the convergence is still valid for $\lambda = 1$ and suitably large $\mu$.

Let $M \in (0, \infty)$ be the initial total mass of system (1.1). Taking 'div' to $(1.3)_2$ and then inserting it to $(1.3)_1$, we have

$$\begin{cases} \rho_t - \dfrac{(1+t)^{\lambda}}{\mu} \Delta p(\rho) = 0, \\ \nabla_{\boldsymbol{x}} p(\rho) = -\dfrac{\mu}{(1+t)^{\lambda}} \rho \boldsymbol{u}. \end{cases} \tag{1.8}$$

The self-similar solution of $(1.8)_1$ with finite mass $M$ is given by

$$\bar{\rho}(\boldsymbol{x}, t) = \bar{\rho}(r, t) = (1+t)^{-\frac{3(\lambda+1)}{3\gamma-1}} \left[ A - B(1+t)^{-\frac{2(\lambda+1)}{3\gamma-1}} r^2 \right]^{\frac{1}{\gamma-1}} \quad \text{with } r = |\boldsymbol{x}|, \tag{1.9}$$

where

$$B = \frac{\mu(\lambda+1)(\gamma-1)}{2\gamma(3\gamma-1)} \quad \text{and} \quad (\gamma A)^{\frac{3\gamma-1}{2(\gamma-1)}}$$
$$= M\gamma^{\frac{1}{\gamma-1}}(\gamma B)^{\frac{3}{2}}\left(\int_0^1 y^2\left(1-y^2\right)^{\frac{1}{\gamma-1}}\mathrm{d}y\right)^{-1}. \tag{1.10}$$

We call this solution (1.9) to be the modified Barenblatt solution since it comes from the Barenblatt solution of the porous media equation $\rho_t - \Delta p(\rho) = 0$. Here the constant $A$ is chosen such that it has the same total mass as that for the solution of (1.1):

$$\int_0^{\bar{R}(t)} r^2\bar{\rho}(r,t)\mathrm{d}r = M \quad \text{for } t \geqslant 0 \text{ and } \bar{R}(t) = \sqrt{A/B}(1+t)^{\frac{\lambda+1}{3\gamma-1}}. \tag{1.11}$$

Here we give a calculation of the equality (1.11) by using the relation between $A$ and $B$ in (1.10). By using a variable change, we have

$$\begin{aligned}
&\int_0^{\bar{R}(t)} r^2\bar{\rho}(r,t)\mathrm{d}r \\
&= \int_0^{\sqrt{A/B}(1+t)^{\frac{\lambda+1}{3\gamma-1}}} (1+t)^{-\frac{3(\lambda+1)}{3\gamma-1}} r^2\left[A - B(1+t)^{-\frac{2(\lambda+1)}{3\gamma-1}}r^2\right]^{\frac{1}{\gamma-1}} \mathrm{d}r \\
&\overset{\sqrt{B/A}(1+t)^{-\frac{\lambda+1}{3\gamma-1}}r=y}{=\!=\!=\!=\!=} \left(\frac{A}{B}\right)^{3/2} A^{\frac{1}{\gamma-1}} \int_0^1 y^2(1-y^2)^{\frac{1}{\gamma-1}}\mathrm{d}y \\
&\overset{\text{by using (1.10)}}{=\!=\!=\!=} \left(\frac{A}{B}\right)^{3/2} A^{\frac{1}{\gamma-1}} M\gamma^{\frac{1}{\gamma-1}}(\gamma B)^{\frac{3}{2}}(\gamma A)^{-\frac{3\gamma-1}{2(\gamma-1)}} \\
&= M(\gamma A)^{\frac{3}{2}+\frac{1}{\gamma-1}-\frac{3\gamma-1}{2(\gamma-1)}} = M.
\end{aligned}$$

Here at the last line of the above equality, the exponent on $\gamma A$ is zero.

From $(1.8)_2$, the corresponding velocity is calculated by

$$\bar{\boldsymbol{u}}(\boldsymbol{x},t) = -\frac{(1+t)^\lambda}{\mu}\frac{\nabla_{\boldsymbol{x}}p(\bar{\rho})}{\bar{\rho}} = \bar{u}(r,t)\frac{\boldsymbol{x}}{r}, \tag{1.12}$$

where $\bar{u}(r,t) = \frac{(\lambda+1)r}{(3\gamma-1)(1+t)}$ and $\bar{u}(0,t) = 0$.

$(\bar{\rho}, \bar{\boldsymbol{u}})$, defined in (1.9) and (1.12), have a physical vacuum boundary $r = \bar{R}(t)$. Our main purpose of this paper is to prove convergence of spherically symmetric smooth solutions for system (1.1) to $(\bar{\rho}, \bar{\boldsymbol{u}})$ if their initial data are close and have the same total mass.

Since the modified Barenblatt solution (1.9) and (1.12) are spherically symmetric solutions, it is reasonable first to study spherically symmetric solutions of system (1.1) and to pursue the more generalized case in the future.

For this purpose, we seek solutions with symmetry to problem (1.1) of the form:

$$\Omega(t) = B_{R(t)}(\boldsymbol{0}), \qquad \rho(\boldsymbol{x},t) = \rho(r,t), \qquad \boldsymbol{u}(\boldsymbol{x},t) = u(r,t)\frac{\boldsymbol{x}}{r} \quad \text{with } r = |\boldsymbol{x}|.$$

Then problem (1.1) reduces to

$$\begin{cases} \left(r^2\rho\right)_t + \left(r^2\rho u\right)_r = 0 & \text{in } (0, R(t)), \\ \rho\left(u_t + uu_r\right) + p_r = -\dfrac{\mu}{(1+t)^\lambda}\rho u & \text{in } (0, R(t)), \\ \rho > 0 & \text{in } [0, R(t)), \\ \rho(R(t), t) = 0, \quad u(0,t) = 0, \\ \dot{R}(t) = u(R(t), t) \quad \text{with } R(0) = R_0, \\ (\rho, u)(r,t)|_{t=0} = (\rho_0, u_0)(r) & \text{on } (0, R_0), \end{cases} \tag{1.13}$$

so that $R(t)$ is the radius of the domain occupied by the gas at time $t$ and $r = R(t)$ represents the vacuum boundary which issues from $r = R_0$ and moves with the fluid velocity.

In the spherically symmetric setting, the physical vacuum boundary condition (1.2) reduces to $-\infty < \left(c^2\right)_r < 0$ in a small neighborhood of the boundary. To capture this singularity, the initial domain is taken to be a ball $\{0 \leqq r \leqq R_0\}$ and the initial density is assumed to satisfy

$$\begin{cases} \rho_0(r) > 0 \quad \text{for } r \in [0, R_0), \quad \rho_0(R_0) = 0, \\ -\infty < \left(\rho_0^{\gamma-1}\right)_r(R_0) < 0. \end{cases}$$

Since we have assumed the initial total mass of the Euler equation is $M$, according to the mass conversation equation $(1.13)_1$, we have for $t \geqslant 0$,

$$\int_0^{R(t)} r^2\rho(r,t)(r)\mathrm{d}r = \int_0^{R_0} r^2\rho_0(r)\mathrm{d}r = \int_0^{\bar{R}(0)} r^2\bar{\rho}_0(r)\mathrm{d}r = M.$$

The physical vacuum problem of the compressible Euler equations is a challenging and interesting problem in the study of free boundary problems for compressible fluids since standard methods of symmetric hyperbolic systems developed in [8, 25, 28] do not apply. Since system (1.1) is a degenerate and characteristic hyperbolic system, near the vacuum boundary of which the uniform Kreiss–Lopatinskii condition (see [26]) is violated, even the local-in-time existence theory is hard to prove. Only recently, the local well-posedness theory has been established for the compressible Euler equations with physical vacuum in one and three dimensions by the space weighted energy estimate. See Coutand *et al* [4–6] and Jang–Masmoudi [21, 22]. In order to understand the behavior and long-time dynamics of physical vacuum boundaries, study on the global-in-time regularity of solutions is essential.

For the compressible Euler equations with damping and physical vacuum, there already exist some results concerning on the global existence of solutions. See [9, 16, 17, 33, 38, 48]. In one space dimension, the authors in [9, 16, 17] gave the $L^\infty$ weak solutions and $L^p$ convergence to the Barenblatt solutions by using the method based on entropy-type estimates. The global existence of smooth solutions and pointwise convergence was proved in [33]. This result was extended to the time-dependent damping system in [38]. In three space dimensions, [48] proved convergence of smooth solutions to the Barenblatt solutions with spherically symmetric data for system (1.1) with constant-coefficient damping $\lambda = 0$. In [33, 48], the authors introduced the space and time weighted energy to characterize the large-time behavior of solutions. The a prior estimates for the weighted energy there can be closed globally in time relies heavily on the constant-coefficient damping term $-\rho u$. Based on some refined analysis, in this paper, we will consider the generalized case of system (1.1) with spherically symmetric data and $0 < \lambda$, which corresponds to a degenerate damping coefficient as $t \to +\infty$. This issue is more challenging since both the vacuum boundary and the damping coefficient are degenerate.

At last, we mention some other results on vacuum free boundary problems in the author's interest. When the degeneration near the boundary is mild, which means the sound is $c^{\alpha}$ $(0 < \alpha \leqslant 1)$ smooth across the boundary, the local existence theory was proved in Liu–Yang [32] for the one-dimensional Euler equations with constant coefficient damping. The local existence and uniqueness for the three dimensional compressible Euler equations modeling the liquid rather than the gas with vacuum boundary were established in [29] by using Lagrangian variables and Nash–Moser iteration. An alternative proof under Eulerian coordinates and extension to that of non-isentropic case can be found in [45]. Gu–Lei [10, 11] investigated the local-in-time well-posedness of the physical vacuum free boundary problem for the one-dimensional and three dimensional Euler–Poisson equations, respectively. The stabilizing and unstabilizing mechanism for the Euler and Euler–Poisson equations with physical vacuum have been shown in [13, 14, 20, 40] and references therein. See also [49, 50] for recent progress on the compressible Euler equations with constant-coefficient damping.

Throughout the rest of paper, $C_{\alpha,\beta,\gamma,\ldots}$ denotes a positive constant depending on $\alpha, \beta, \gamma, \ldots$ which may be different from line to line. We will employ the notation $a \lesssim_{\alpha,\beta,\gamma,\ldots} b$ to denote $a \leqslant C_{\alpha,\beta,\gamma,\ldots} b$ and $a \approx_{\alpha,\beta,\gamma,\ldots} b$ to denote $C^{-1}_{\alpha,\beta,\gamma,\ldots} b \leqslant a \leqslant C_{\alpha,\beta,\gamma,\ldots} b$. Usually $\alpha, \beta, \gamma, \ldots$ in the constant $C_{\alpha,\beta,\gamma,\ldots}$ will be ignored if no confusion is caused.

## 2. Reformulation of the problem and main results

### *2.1. Reformulation to Lagrangian variables*

Define a diffeomorphism

$$\eta_0 : \left(0, \bar{R}(0)\right) \to (0, R_0)$$

by

$$\int_0^{\eta_0(r)} s^2 \rho_0(s) \mathrm{d}s = \int_0^r s^2 \bar{\rho}_0(s) \mathrm{d}s \quad \text{for } r \in \left(0, \bar{R}(0)\right)$$

where $\left(0, \bar{R}(0)\right)$ is the initial interval of the modified Barenblatt solution (1.9), taken as reference interval, and $\bar{\rho}_0(r) := \bar{\rho}(r, 0)$ is the initial density of the solution (1.9).

Taking derivative of the above equality on $r$ indicates

$$\eta_0^2(r) \rho_0\left(\eta_0(r)\right) \eta_{0,r}(r) = r^2 \bar{\rho}_0(r) \quad \text{for } r \in \left(0, \bar{R}(0)\right). \tag{2.1}$$

Here $_{,r}$ means derivative on $r$. Set the reference interval

$$\mathcal{I} := \left(0, \bar{R}(0)\right) = \left(0, \sqrt{AB^{-1}}\right).$$

For $r \in \mathcal{I}$, we define the Lagrangian variable $\eta(r,t)$ by

$$\begin{cases} \eta_t(r,t) = u(\eta(r,t),t) \quad \text{for } t > 0, \\ \eta(r,0) = \eta_0(r), \end{cases}$$

and set the Lagrangian density and velocity by

$$f(r,t) = \rho(\eta(r,t),t) \quad \text{and} \quad v(r,t) = u(\eta(r,t),t).$$

Then in Lagrangian variables, system (1.13) has fixed boundary and can be written on the reference domain $\mathcal{I}$ as

$$\begin{cases} (\eta^2 f)_t + \eta^2 f v_r/\eta_r = 0 & \text{in } \mathcal{I} \times (0,\infty), \\ f v_t + (f^\gamma)_r/\eta_r = -\dfrac{\mu}{(1+t)^\lambda} f v & \text{in } \mathcal{I} \times (0,\infty), \\ f > 0 \quad \text{in } \mathcal{I} \times (0,\infty), \quad f\left(\sqrt{AB^{-1}}, t\right) = 0 \quad \text{on } (0,\infty), & \\ v(0,t) = 0 & \text{on } (0,\infty), \\ (f, v) = (\rho_0(\eta_0), u_0(\eta_0)) & \text{on } \mathcal{I} \times \{t = 0\}. \end{cases} \tag{2.2}$$

The map $\eta(\cdot,t)$ defined above can be extended to $\overline{\mathcal{I}} = \left[0, \sqrt{AB^{-1}}\right]$. In the setting, the vacuum free boundary for problem (1.13) is given by

$$R(t) = \eta\left(\bar{R}(0), t\right) = \eta(\sqrt{AB^{-1}}, t) \quad \text{for } t \geqslant 0.$$

By solving $(2.2)_1$ and using (2.1), we see that

$$f(r,t)\eta^2(r,t)\eta_r(r,t) = \rho_0(\eta_0(r))\,\eta_0^2(r)\eta_{0,r}(r) = r^2\bar{\rho}_0(r), \quad r \in \mathcal{I}. \tag{2.3}$$

It should be noticed that we need $\eta_r(r,t) > 0$ for $r \in \mathcal{I}$ and $t \geqslant 0$ to make the Lagrangian transformation reasonable, which will be verified later. Inserting (2.3) into $(2.2)_2$, we can get the following $(2.4)_1$. Then the initial density, $\bar{\rho}_0$, can be regarded as a parameter, and system (2.2) can be rewritten as

$$\begin{cases} \bar{\rho}_0\eta_{tt} + \dfrac{\mu}{(1+t)^\lambda}\bar{\rho}_0\eta_t + \left(\dfrac{\eta}{r}\right)^2\left[\left(\dfrac{r^2}{\eta^2}\dfrac{\bar{\rho}_0}{\eta_r}\right)^\gamma\right]_r = 0 & \text{in } \mathcal{I} \times (0,\infty), \\ \eta(0,t) = 0 & \text{on } (0,\infty), \\ (\eta, \eta_t) = (\eta_0, u_0(\eta_0)) & \text{on } \mathcal{I} \times \{t = 0\}. \end{cases} \tag{2.4}$$

### 2.2. *Correction of the modified Barenblatt solutions*

The Lagrangian variable $\bar{\eta}(r,t)$ for the modified Barenblatt flow in $\overline{\mathcal{I}}$ is defined by

$$\bar{\eta}_t(r,t) = \bar{u}(\bar{\eta}(r,t),t) = \frac{(\lambda+1)\bar{\eta}(r,t)}{(3\gamma-1)(1+t)} \quad \text{for } t > 0 \text{ and } \bar{\eta}(r,0) = r.$$

Solving the above ODE gives that

$$\bar{\eta}(r,t) = r(1+t)^{\frac{\lambda+1}{3\gamma-1}} \quad \text{for } (r,t) \in \overline{\mathcal{I}} \times [0,\infty), \tag{2.5}$$

and by direct calculation, it satisfies

$$\frac{\mu}{(1+t)^\lambda}\bar{\rho}_0\bar{\eta}_t + \left(\frac{\bar{\eta}}{r}\right)^2 \left[\left(\frac{r^2}{\bar{\eta}^2}\frac{\bar{\rho}_0}{\bar{\eta}_r}\right)^\gamma\right]_r = 0 \quad \text{in } \mathcal{I} \times (0,\infty).$$

Since $\bar{\eta}$ does not solve (2.4)$_1$ exactly, we introduce a correction $h(t)$, and set

$$\tilde{\eta}(r,t) := \bar{\eta}(r,t) + rh(t), \tag{2.6}$$

so that

$$\begin{cases} \bar{\rho}_0\tilde{\eta}_{tt} + \dfrac{\mu}{(1+t)^\lambda}\bar{\rho}_0\tilde{\eta}_t + \left(\dfrac{\tilde{\eta}}{r}\right)^2 \left[\left(\dfrac{r^2}{\tilde{\eta}^2}\dfrac{\bar{\rho}_0}{\tilde{\eta}_r}\right)^\gamma\right]_r = 0 \quad \text{in } \mathcal{I} \times (0,\infty), \\ \tilde{\eta}(r,0) = \bar{\eta}(r,0), \quad \tilde{\eta}_t(r,0) = \bar{\eta}_t(r,0). \end{cases} \tag{2.7}$$

Then $h(t)$ is the solution of the following initial value problem of ordinary differential equations:

$$\begin{cases} h_{tt} + \dfrac{\mu}{(1+t)^\lambda}h_t - \dfrac{\mu(\lambda+1)}{3\gamma-1}(\bar{\eta}_r + h)^{2-3\gamma} + \bar{\eta}_{rtt} + \dfrac{\mu}{(1+t)^\lambda}\bar{\eta}_{rt} = 0, \\ h|_{t=0} = h_t|_{t=0} = 0. \end{cases}$$

Notice that from (2.5), $\bar{\eta}_r, \bar{\eta}_{rt}$, and $\bar{\eta}_{rtt}$ are independent of $r$.

Actually $h$ is a positive bounded function and $\tilde{\eta}$ behaves similarly to $\bar{\eta}$. That is, there exist positive constants $L$ and $c_k$ independent of time $t$ such that for all $t \geqslant 0$,

If $\mathbf{0 < \boldsymbol{\lambda} < 1}$:

$$\begin{aligned} &(1+t)^{\frac{\lambda+1}{3\gamma-1}} \leqslant \tilde{\eta}_r(t) \leqslant L(1+t)^{\frac{\lambda+1}{3\gamma-1}}, \quad \tilde{\eta}_{rt}(t) \geqslant 0, \\ &\left|\frac{\mathrm{d}^k\tilde{\eta}_r(t)}{\mathrm{d}t^k}\right| \leqslant c_k(1+t)^{\frac{\lambda+1}{3\gamma-1}-k} \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{2.8}$$

**If $\boldsymbol{\lambda} = \mathbf{1}$:**

$$\begin{aligned} &(1+t)^{\frac{2}{3\gamma-1}} \leqslant \tilde{\eta}_r(t) \leqslant L(1+t)^{\frac{2}{3\gamma-1}}, \quad \tilde{\eta}_{rt}(t) \geqslant 0, \\ &\left|\frac{\mathrm{d}^k\tilde{\eta}_r(t)}{\mathrm{d}t^k}\right| \leqslant \begin{cases} c_k(1+t)^{\frac{2}{3\gamma-1}-k} & \text{for } k < \mu + \dfrac{2}{3\gamma-1} \text{ and } k \in \mathbb{N}, \\ c_k(1+t)^{-\mu}\ln(1+t) & \text{for } k \geqslant \mu + \dfrac{2}{3\gamma-1} \text{ and } k \in \mathbb{N}. \end{cases} \end{aligned} \tag{2.9}$$

The proofs of (2.8) and (2.9) are almost the same as that in [38, appendix] with $\gamma + 1$ there replaced by $3\gamma - 1$ here. Here we omit the details.

### *2.3. Main results*

Let

$$w(r,t) = \frac{\eta(r,t)}{r} - \frac{\tilde{\eta}(r,t)}{r}. \tag{2.10}$$

Then subtract (2.7) from $(2.4)_1$, we see that $w$ satisfies

$$\begin{cases} r\bar{\rho}_0 w_{tt} + \dfrac{\mu}{(1+t)^{\lambda}} r\bar{\rho}_0 w_t - \tilde{\eta}_r^{2-3\gamma}(\bar{\rho}_0^{\gamma})_r & \\ \quad + (\tilde{\eta}_r + w)^2 \left[\bar{\rho}_0^{\gamma}(\tilde{\eta}_r + w)^{-2\gamma}(\tilde{\eta}_r + w + rw_r)^{-\gamma}\right]_r = 0 & \text{in } \mathcal{I} \times (0,\infty), \\ (w, w_t) = \left(\dfrac{\eta_0}{r} - 1, \dfrac{u_0(\eta_0)}{r} - \dfrac{\lambda+1}{3\gamma-1}\right) & \text{on } \mathcal{I} \times \{t=0\}. \end{cases} \tag{2.11}$$

Denote $\alpha = \frac{1}{\gamma-1}$ and set $m := 4 + [\alpha]$. Let $\delta \in (0, \frac{2(\lambda+1)}{3\gamma-1})$. For $j = 0, \ldots, m$ and $i = 0, \ldots, m-j$, we set

$$\begin{aligned} \mathcal{E}_j(t) &:= (1+t)^{2j-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[ r^4 \bar{\rho}_0 (\partial_t^j w)^2 + r^2 \bar{\rho}_0^{\gamma} \left( (\partial_t^j w)^2 + (r\partial_t^j w_r)^2 \right) \\ &\quad + (1+t)^{\lambda+1} r^4 \bar{\rho}_0 (\partial_t^{j+1} w)^2 \Big] (r,t)\mathrm{d}r, \\ \mathcal{E}_{j,i}(t) &:= (1+t)^{2j-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[ r^2 \bar{\rho}_0^{1+(i-1)(\gamma-1)} (\partial_t^j \partial_r^i w)^2 \\ &\quad + r^4 \bar{\rho}_0^{1+(i+1)(\gamma-1)} (\partial_t^j \partial_r^{i+1} w)^2 \Big] (r,t)\mathrm{d}r, \end{aligned} \tag{2.12}$$

where $\mathbf{1}_{\lambda<1}$ is the characteristic function on the set $\{\lambda < 1\}$, which means

$$\boldsymbol{1}_{\lambda<1} = \begin{cases} 1, & \text{if } \lambda < 1, \\ 0, & \text{if } \lambda = 1. \end{cases}$$

If we set

$$\sigma(x) := \bar{\rho}_0^{\gamma-1}(x) = A - Br^2, \quad r \in \mathcal{I},$$

then $\mathcal{E}_j$ and $\mathcal{E}_{j,i}$ can be rewritten as

$$\begin{aligned} \mathcal{E}_j(t) &= (1+t)^{2j-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[ r^4 \sigma^{\alpha} (\partial_t^j w)^2 + r^2 \sigma^{\alpha+1} \left( (\partial_t^j w)^2 + (r\partial_t^j w_r)^2 \right) \\ &\quad + (1+t)^{\lambda+1} r^4 \sigma^{\alpha} (\partial_t^{j+1} w)^2 \Big] (r,t)\mathrm{d}r, \\ \mathcal{E}_{j,i}(t) &= (1+t)^{2j-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[ r^2 \sigma^{\alpha+i-1} (\partial_t^j \partial_r^i w)^2 + r^4 \sigma^{\alpha+i+1} (\partial_t^j \partial_r^{i+1} w)^2 \Big] (r,t)\mathrm{d}r. \end{aligned} \tag{2.13}$$

The total energy is defined by

$$\mathcal{E}(t) := \sum_{j=0}^{m} \left( \mathcal{E}_j(t) + \sum_{i=1}^{m-j} \mathcal{E}_{j,i}(t) \right). \tag{2.14}$$

Now, we give the following main result.

**Theorem 2.1.** *Let* $\lambda = 1, \mu + \frac{2}{3\gamma-1} > m$ *or* $0 < \lambda < 1,\ \mu > 0$. *Then there exists a constant* $\epsilon_0$ *such that if* $\mathcal{E}(0) \leqslant \epsilon_0$, *the problem* (2.11) *admits a globally unique smooth solution in* $\mathcal{I} \times [0, \infty)$, *satisfying for all* $t \geqslant 0$

$$\mathcal{E}(t) \leqslant C\mathcal{E}(0),$$

*where* $C$ *is a positive constant independent of* $t$.

**Remark 2.2.** When $\lambda = 1$, the assumption $\mu + \frac{2}{3\gamma-1} > m$ can be relaxed to $\mu > 2$. See the basic nonlinear energy estimates in lemma 3.7. However due to the decay estimates in (2.9) for $\tilde{\eta}$, the time weight $(1+t)^{2j}$ in the energy functionals $\mathcal{E}_j(t)$ and $\mathcal{E}_{j,i}(t)$ in (2.12) and (2.13) need to be adjusted to $(1+t)^{2\tilde{j}}$, where

$$\tilde{j} = \begin{cases} j, & j < \mu + \dfrac{2}{3\gamma - 2}, \\ \left(\mu + \dfrac{2}{3\gamma-2}\right)^{-}, & j \geqslant \mu + \dfrac{2}{3\gamma-2}. \end{cases}$$

Here for a constant $a$, $a^{-}$ denote a constant which is smaller than but can be arbitrarily close to $a$.

**Remark 2.3.** When $\lambda < 1$, the damping coefficient in this case is stronger than that for $\lambda = 1$. However the time weight in our energy functionals in (2.13) seems to be weaker by an order $\delta$. Actually, by an elaborated analysis and refined calculation, the time weight can be replaced by $(1+t)^{2j+\tilde{\delta}}$ with some positive $\tilde{\delta}$. The strategy is to perform weighted energy estimates for equation (3.9) in lemma 3.7 by multiplying an additional weight $\mathrm{e}^{a\frac{r^2}{(1+t)^{1+\lambda}}}$ for a suitable constant $a$. This exponent weight is equivalent to 1 for $r \in [0, \sqrt{AB^{-1}}]$. The idea can date back to [43, 46]. See also [37, appendix]. Since the computation is more complicated and involved, but the optimal time-decay estimate is not our main target, we do not pursue this energy estimates any further.

If we go back to the Eulerian coordinates from the Lagrangian coordinates, from theorem 2.1, we have the following theorem for solutions to the original vacuum free boundary problem (1.1).

**Theorem 2.4.** *Let* $\lambda = 1, \mu + \frac{2}{3\gamma-1} > m$ *or* $0 < \lambda < 1,\ \mu > 0$. *Then there exists a constant* $\epsilon_0 > 0$ *such that if* $\mathcal{E}(0) \leqslant \epsilon_0$, *the problem* (1.1) *admits a global unique smooth solution* $(\rho, u, R(t))$ *for* $t \in [0, \infty)$ *satisfying*

$$\begin{aligned} &|\rho(\eta(r,t),t) - \bar{\rho}(\bar{\eta}(r,t),t)| \\ &\leqslant \left(A - Br^2\right)^{\frac{1}{\gamma-1}} (1+t)^{-\frac{4(\lambda+1)}{3\gamma-1}} \left((1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}} \sqrt{\mathcal{E}(0)} + 1\right), \end{aligned} \tag{2.15}$$

$$|u(\eta(x,t),t) - \bar{u}(\bar{\eta}(x,t),t)| \lesssim r(1+t)^{-1} \left((1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}} \sqrt{\mathcal{E}(0)} + 1\right), \tag{2.16}$$

$$R(t) \approx (1+t)^{\frac{\lambda+1}{3\gamma-1}}, \tag{2.17}$$

$$\left|\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k}\right| \leqslant C(1+t)^{\frac{\lambda+1}{3\gamma-1}-k}, \quad k = 1, 2, 3, \tag{2.18}$$

*for all* $r \in \mathcal{I}$ *and* $t \geqslant 0$. *Here C is a positive constant, depending on* $\epsilon_0$ *and the upper bound of h but independent of t.*

Here (2.15) and (2.16) give the pointwise convergence of the density and velocity for the vacuum free boundary problem (1.1) to that of the modified Barenblatt solution, respectively. The precise expanding rate of the vacuum boundaries, which is the same as that for the modified Barenblatt solution is given in (2.17).

### 2.4. Notations and Hardy inequality

In this subsection, we present some embedding estimates for weighted Sobolev spaces that will be used later.

Set

$$\begin{aligned} &d(r) := \operatorname{dist}(r, \partial\mathcal{I}) = \min\{r, \sqrt{AB^{-1}} - r\}, \\ &r \in \mathcal{I} = (0, \sqrt{AB^{-1}}). \end{aligned}$$

For any $a > 0$ and nonnegative integer $b$, the weighted Sobolev space $H^{a,b}(\mathcal{I})$ is defined by

$$H^{a,b}(\mathcal{I}) := \left\{ d^{a/2}F \in L^2(\mathcal{I}) : \int_{\mathcal{I}} d^a \left|\partial_r^k F\right|^2 \mathrm{d}r < \infty, \quad 0 \leqslant k \leqslant b \right\}$$

with the norm

$$\|F\|^2_{H^{a,b}(\mathcal{I})} := \sum_{k=0}^{b} \int_{\mathcal{I}} d^a \left|\partial_r^k F\right|^2 \mathrm{d}r.$$

Then for $b \geqslant a/2$, the following embedding of weighted Sobolev spaces holds (cf [27]):

$$H^{a,b}(\mathcal{I}) \hookrightarrow H^{b-a/2}(\mathcal{I})$$

with the estimate

$$\|F\|_{H^{b-a/2}(\mathcal{I})} \leqslant C_{a,b} \|F\|_{H^{a,b}(\mathcal{I})} \tag{2.19}$$

for some positive constant $C_{a,b}$.

The following general version of the Hardy inequality, whose proof can be found in [27], will also be used frequently in this paper. Let $\theta > 1$ be a given real number and $F$ be a function satisfying

$$\int_0^L r^\theta \left(F^2 + F_r^2\right) \mathrm{d}r < \infty,$$

where $L$ is a positive constant; then it holds that

$$\int_0^L r^{\theta-2} F^2 \,\mathrm{d}r \leqslant C_{\theta,L} \int_0^L r^\theta \left(F^2 + F_r^2\right) \mathrm{d}r. \tag{2.20}$$

It is easy to note that $\sigma \approx_{A,B} \sqrt{A/B} - r$. If we divide $\mathcal{I}$ into $\mathcal{I}_1 := (0, \sqrt{A/4B})$ and $\mathcal{I}_2 := (\sqrt{A/4B}, \sqrt{A/B})$, as a consequence of (2.20) and by making a simple variable change, we have

$$\begin{aligned}\int_{\mathcal{I}_2} \sigma^{\theta-2} F^2 \, \mathrm{d}r &\approx_{A,B} \int_{\mathcal{I}_2} (\sqrt{A/B} - r)^{\theta-2} F^2 \, \mathrm{d}r \\ &\lesssim_{A,B} \int_{\mathcal{I}_2} (\sqrt{A/B} - r)^{\theta} (F^2 + F_r^2) \mathrm{d}r \\ &\approx_{A,B} \int_{\mathcal{I}_2} \sigma^{\theta} (F^2 + F_r^2) \mathrm{d}r.\end{aligned} \tag{2.21}$$

**Remark 2.5.** In later calculation for the weighted energy estimates, (2.20) and (2.21) will be frequently used. The choice of $m := [\alpha] + 4$ is due to the restriction of $\theta > 1$.

In the rest of the paper, we will use the notation

$$\int = : \int_{\mathcal{I}}, \quad \| \cdot \| = : \| \cdot \|_{L^2(\mathcal{I})}, \quad \text{and} \quad \| \cdot \|_{L^\infty} = : \| \cdot \|_{L^\infty(\mathcal{I})}.$$

For a function $f(t, r)$, sometimes for simplicity, we use $\int f(t, r)$ and $\int_0^t \int f(\tau, r)$ to denote $\int f(t, r) \mathrm{d}r$ and $\int_0^t \int f(\tau, r) \mathrm{d}r \, \mathrm{d}\tau$, respectively if no confusion is caused.

## 3. Proof of theorem 2.1

At the beginning, we give a weighted Sobolev $L^\infty$ embedding lemma for later use.

**Lemma 3.1.** *Let $\mathcal{E}_\infty(t)$ be the following weighted $L^\infty$ norm*

$$\begin{aligned}\mathcal{E}_\infty(t) := &\sum_{2i+j \leqslant 2} (1+t)^{2j - \delta \mathbf{1}_{\lambda<1}} \left\| \partial_t^j \partial_r^i w(\cdot, t) \right\|_\infty^2 \\ &+ \sum_{\substack{i+j \leqslant m-2 \\ 2i+j \geqslant 3}} (1+t)^{2j - \delta \mathbf{1}_{\lambda<1}} \left\| \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i w(\cdot, t) \right\|_\infty^2 \\ &+ \sum_{i+j = m-1} (1+t)^{2j - \delta \mathbf{1}_{\lambda<1}} \left\| r \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i w(\cdot, t) \right\|_\infty^2 \\ &+ \sum_{i+j = m} (1+t)^{2j - \delta \mathbf{1}_{\lambda<1}} \left\| r^2 \sigma^{\frac{2i+j-3}{2}} \partial_t^j \partial_r^i w(\cdot, t) \right\|_\infty^2.\end{aligned} \tag{3.1}$$

*Assume that $\mathcal{E}(t)$ is finite, then it holds that*

$$\mathcal{E}_\infty(t) \leqslant C \mathcal{E}(t).$$

The proof of lemma 3.1 can follow the same line with that in [48, p 69, lemma 3.7] by using (2.19) repeatedly. Here we omit the details.

The proof of theorem 2.1 is based on the local existence and uniqueness of smooth solutions (cf [5, 21, 31]) and continuation arguments. In order to prove the global existence of smooth solutions, we need to obtain the uniform-in-time *a priori* estimates on any given time interval $[0, T]$ satisfying $\sup_{t\in[0,T]} \mathcal{E}(t) < \infty$. To this end, we use a bootstrap argument by making the following *a priori* assumption: there exists a suitably small fixed positive number $\epsilon_0 \in (0, 1)$ independent of $t$ such that

$$\sup_{0\leqslant t\leqslant T} \mathcal{E}(t) \leqslant M\epsilon_0, \tag{3.2}$$

for some constant $M$, independent of $\epsilon_0$, to be determined later. Under this *a priori* assumption, and by using lemma 3.1, we see that for $0 \leqslant t \leqslant T$,

$$\mathcal{E}_\infty(t) \leqslant CM\epsilon_0. \tag{3.3}$$

By assuming that $M\epsilon_0$ is sufficiently small such that $M\epsilon_0 \ll 1$, then the following elliptic estimates:

$$\mathcal{E}_{j,i}(t) \leqslant C\sum_{\ell=0}^{i+j} \mathcal{E}_\ell(t) \quad \text{when } i,\ \ j \geqslant 0,\ \ i + j \leqslant m, \tag{3.4}$$

will be shown in subsection 3.1, where $C$ is a positive constant independent of $t$.

With (3.3) and elliptic estimates (3.4), we show in subsection 3.2 the following nonlinear weighted energy estimate: for some positive constant $C$ independent of $t$

$$\mathcal{E}_j(t) \leqslant C\sum_{\ell=0}^{j} \mathcal{E}_\ell(0), \qquad j = 0, 1, \ldots, m. \tag{3.5}$$

Remembering the definition of the total energy $\mathcal{E}(t)$ in (2.14) and combining (3.4) and (3.5), we see that

$$\mathcal{E}(t) \leqslant C_*\mathcal{E}(0), \tag{3.6}$$

for some constant $C_*$ independent of $t$ and $M$. By choosing $M = 2C_*$, we see that

$$\mathcal{E}(t) \leqslant \frac{1}{2}M\epsilon_0,$$

which closes energy estimates.

In order to simplify the presentation, we will extract the main term in the equation $(2.11)_1$ and perform our elliptic and nonlinear weighted energy estimates in the next two subsections on the simplified equation. See (3.9) below.

Under the assumption of (3.2) and using lemma 3.1, we see $\tilde{\eta}_r^{-1}w$ and $\tilde{\eta}_r^{-1}rw_r$ is small since $\tilde{\eta}_r^{-1} \approx (1+t)^{-\frac{\lambda+1}{3\gamma-1}}$ and $\delta/2 \in (0, \frac{\lambda+1}{3\gamma-1})$. First we rewrite $(2.11)_1$ as follows by remembering that $\bar{\rho}_0 = \sigma^\alpha$ and $\bar{\rho}_0^\gamma = \sigma^{\alpha+1}$,

$$\begin{aligned} &r\sigma^\alpha w_{tt} + \frac{\mu}{(1+t)^\lambda} r\sigma^\alpha w_t - \tilde{\eta}_r^{2-3\gamma}(\sigma^{\alpha+1})_r \\ &\quad + (\tilde{\eta}_r + w)^2\left[\sigma^{\alpha+1}(\tilde{\eta}_r + w)^{-2\gamma}(\tilde{\eta}_r + w + rw_r)^{-\gamma}\right]_r = 0. \end{aligned} \tag{3.7}$$

Then for the space derivative term, by using Taylor expansion and smallness of $\tilde{\eta}_r^{-1}w$ and $\tilde{\eta}_r^{-1}rw_r$, we have

$$
\begin{aligned}
&(1+\tilde{\eta}_r^{-1}w)^2 = \left(1+2\tilde{\eta}_r^{-1}w + o(\tilde{\eta}_r^{-1}|w|)\right),\\
&(1+\tilde{\eta}_r^{-1}w)^{-2\gamma} = 1-2\gamma\tilde{\eta}_r^{-1}w + o(\tilde{\eta}_r^{-1}|w|),\\
&\left(1+\tilde{\eta}_r^{-1}(w+rw_r)\right)^{-\gamma} = 1-\gamma\tilde{\eta}_r^{-1}(w+rw_r) + o(\tilde{\eta}_r^{-1}|w,rw_r|).
\end{aligned}
$$

Inserting the above Taylor expansion into (3.7), we have

$$
\begin{aligned}
&(\tilde{\eta}_r+w)^2\left[\sigma^{\alpha+1}(\tilde{\eta}_r+w)^{-2\gamma}(\tilde{\eta}_r+w+rw_r)^{-\gamma}\right]_r - \tilde{\eta}_r^{2-3\gamma}(\sigma^{\alpha+1})_r\\
&\quad= \tilde{\eta}_r^{2-3\gamma}(1+\tilde{\eta}_r^{-1}w)^2\left[\sigma^{\alpha+1}(1+\tilde{\eta}_r^{-1}w)^{-2\gamma}\left(1+\tilde{\eta}_r^{-1}(w+rw_r)\right)^{-\gamma}\right]_r\\
&\qquad - \tilde{\eta}_r^{2-3\gamma}(\sigma^{\alpha+1})_r\\
&\quad= \tilde{\eta}_r^{2-3\gamma}\left(1+2\tilde{\eta}_r^{-1}w+o(\tilde{\eta}_r^{-1}|w|)\right)\left[\sigma^{\alpha+1}\left(1-2\gamma\tilde{\eta}_r^{-1}w-\gamma\tilde{\eta}_r^{-1}(w+rw_r)\right.\right.\\
&\qquad \left.\left.+\,o(\tilde{\eta}_r^{-1}|w,rw_r|)\right)\right]_r - \tilde{\eta}_r^{2-3\gamma}(\sigma^{\alpha+1})_r\\
&\quad= -\tilde{\eta}_r^{1-3\gamma}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma rw_r)\right]_r - 2w[\sigma^{\alpha+1}]_r\right\} + \tilde{\eta}_r^{2-3\gamma}o(\tilde{\eta}_r^{-1}|w,rw_r|).
\end{aligned}
\tag{3.8}
$$

Then (3.7) is simplified to

$$
r\sigma^\alpha w_{tt} + \frac{\mu}{(1+t)^\lambda}r\sigma^\alpha w_t - (1+o(1))\tilde{\eta}_r^{1-3\gamma}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma rw_r)\right]_r - 2w[\sigma^{\alpha+1}]_r\right\} = 0.
\tag{3.9}
$$

**Remark 3.2.** The exact formulation of the quadratic term $\tilde{\eta}_r^{2-3\gamma}o(\tilde{\eta}_r^{-1}|w,rw_r|)$ in (3.8) can be traced to the terms $\mathfrak{J}_1$ and $\mathfrak{J}_1$ in reference [48, p 47]. When we perform higher order elliptic estimates and nonlinear energy estimates in subsections 3.1 and 3.2, we need to first take derivatives $\partial_t^j\partial_r^{i-1}$ and $\partial_t^j$ to the equation (3.7), and then apply the Taylor expansion rather than take derivatives on equations (3.8) or (3.9) directly since small '$o$' term may not be small after taking derivatives. Actually, when we take derivatives on the quadratic term $\tilde{\eta}_r^{2-3\gamma}o(\tilde{\eta}_r^{-1}|w,rw_r|)$, it corresponds to the term $\mathfrak{B}_2$ in [p 49, equation (3.20)] and the terms $K_1, K_2$ and $K_3$ in [p 58, equation (3.38)] of reference [48]. Handing of these terms are almost the same as that in reference [48] with the only difference being that here we have a little different time weight, which cause no problem for the corresponding estimates. So in the following, when we perform higher order estimates, we will formally take derivatives on equation (3.9) directly and only take care of the linear highest order derivative term. The linear highest order derivative term reveals the essential structure of equation (3.7). The nonlinear quadratic and multi-power's lower order derivative terms coming from Taylor's expansion of (3.7) can be handled by the weighted $L^2$ norm multiplied by the weighted $L^\infty$ norm in lemma 3.1. Then it can be absorbed by the linear higher order weighted $L^2$ norm due to the smallness of the energy assumption (3.2). Readers can refer to Zeng [48, lemmas 3.3 and 3.6] for more details on how to performing the low order derivative estimates.

### 3.1. *Elliptic Estimates*

We prove the following elliptic estimates in this subsection.

**Proposition 3.3.** *Under the assumption of (3.2) for suitably small positive number $\epsilon_0 \in (0, 1)$, then for $0 \leqslant t \leqslant T$, we have*

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\ell=0}^{i+j} \mathcal{E}_\ell(t) \quad \text{when } i,\ j \geqslant 0,\ i+j \leqslant m.$$

The proof of this proposition consists of lemmas 3.4 and 3.5.

**First-order elliptic estimates**

Divide (3.9) by $\sigma^\alpha$ to obtain

$$\begin{aligned}&\gamma \tilde{\eta}_r^{1-3\gamma} \left[ r\sigma w_{rr} + 4\sigma w_r + (\alpha+1) r w_r \sigma_r \right] \\ =&(1+o(1)) \left[ r w_{tt} + \frac{\mu}{(1+t)^\lambda} r w_t + (\alpha+1)(2-3\gamma)\sigma_r \tilde{\eta}_r^{1-3\gamma} w \right].\end{aligned} \tag{3.10}$$

**Lemma 3.4.** *Under the assumption of (3.2) for suitably small positive number $\epsilon_0 \in (0, 1)$, we have*

$$\mathcal{E}_{0,0}(t) + \mathcal{E}_{1,0}(t) + \mathcal{E}_{0,1}(t) \lesssim \mathcal{E}_0(t) + \mathcal{E}_1(t), \quad 0 \leqslant t \leqslant T.$$

**Proof.** When $i = 0$, first we see that

$$\begin{aligned}\mathcal{E}_{j,0}(t) :=& (1+t)^{2j-\delta \mathbf{1}_{\lambda<1}} \int \left[ r^4 \sigma^{\alpha+1} (\partial_t^j w_r)^2 + r^2 \sigma^{\alpha-1} (\partial_t^j w)^2 \right] \\ \leqslant& \mathcal{E}_j(t) + (1+t)^{2j-\delta \mathbf{1}_{\lambda<1}} \int r^2 \sigma^{\alpha-1} (\partial_t^j w)^2 \\ =& \mathcal{E}_j(t) + (1+t)^{2j-\delta \mathbf{1}_{\lambda<1}} \left\{ \int_{\mathcal{I}_1} + \int_{\mathcal{I}_2} \right\} r^2 \sigma^{\alpha-1} (\partial_t^j w)^2.\end{aligned} \tag{3.11}$$

In $\mathcal{I}_1$, $\sigma \approx_{A,B} 1$, then

$$\int_{\mathcal{I}_1} r^2 \sigma^{\alpha-1} (\partial_t^j w)^2 \lesssim \int_{\mathcal{I}_1} r^2 \sigma^{\alpha+1} (\partial_t^j w)^2.$$

And in $\mathcal{I}_2$, $r \approx_{A,B} 1$, by using (2.21), we have

$$\begin{aligned}\int_{\mathcal{I}_2} r^2 \sigma^{\alpha-1} (\partial_t^j w)^2 \lesssim& \int_{\mathcal{I}_2} \sigma^{\alpha+1} \left( (r\partial_t^j w)^2 + \left( (r\partial_t^j w)_r \right)^2 \right) \\ \lesssim& \int_{\mathcal{I}_2} r^2 \sigma^{\alpha+1} \left( (\partial_t^j w)^2 + (r\partial_t^j w_r)^2 \right).\end{aligned}$$

Inserting the above two inequalities into (3.11) implies that $\mathcal{E}_{0,0}(t) + \mathcal{E}_{1,0}(t) \lesssim \mathcal{E}_0(t) + \mathcal{E}_1(t)$. We mainly focus on the proof of $\mathcal{E}_{0,1}(t) \lesssim \mathcal{E}_0(t) + \mathcal{E}_1(t)$.

Remembering that $\tilde{\eta}_r^{3\gamma-1} \approx (1+t)^{\lambda+1}$, multiply equation (3.10) by $\tilde{\eta}_r^{3\gamma-1} r\sigma^{\alpha/2}$ and perform the spatial $L^2$-norm to obtain

$$\begin{aligned}
&\left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} + 4r\sigma^{1+\frac{\alpha}{2}} w_r + (\alpha+1) r^2\sigma^{\frac{\alpha}{2}}\sigma_r w_r \right\|^2 \\
&\quad \lesssim (1+t)^{2(\lambda+1)} \left\| r^2\sigma^{\frac{\alpha}{2}} w_{tt} \right\|^2 + (1+t)^2 \left\| r^2\sigma^{\frac{\alpha}{2}} w_t \right\|^2 + \left\| r^2\sigma^{\frac{\alpha}{2}} w \right\|^2 \\
&\quad \leqslant (1+t)^{\delta \mathbf{1}_{\lambda<1}} \left(\mathcal{E}_0 + \mathcal{E}_1\right).
\end{aligned} \tag{3.12}$$

In what follows, we analyze the left-hand side of (3.12), which can be expanded as

$$\begin{aligned}
&\left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} + 4r\sigma^{1+\frac{\alpha}{2}} w_r + (\alpha+1) r^2\sigma^{\frac{\alpha}{2}}\sigma_r w_r \right\|^2 \\
=& \left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} \right\|^2 + 16 \left\| r\sigma^{1+\frac{\alpha}{2}} w_r \right\|^2 + (\alpha+1)^2 \left\| r^2\sigma^{\frac{\alpha}{2}}\sigma_r w_r \right\|^2 \\
&+ \int \left[4r^3\sigma^{2+\alpha} + (\alpha+1) r^4\sigma^{1+\alpha}\sigma_r\right] \left(w_r^2\right)_r + 8(\alpha+1) \int r^3\sigma^{1+\alpha}\sigma_r w_r^2.
\end{aligned} \tag{3.13}$$

With the help of the integration by parts and the fact $\sigma_r = -2Br$, one has

$$\begin{aligned}
&\int \left[4r^3\sigma^{2+\alpha} + (\alpha+1) r^4\sigma^{1+\alpha}\sigma_r\right] \left(w_r^2\right)_r \\
=& -12 \int r^2\sigma^{2+\alpha} w_r^2 - (\alpha+1)^2 \int r^4\sigma^{\alpha}\sigma_r^2 w_r^2 + 8B(2\alpha+3) \int r^4\sigma^{1+\alpha} w_r^2.
\end{aligned}$$

Substitute this into (3.13) and use $\sigma_r = -2Br$ to give

$$\begin{aligned}
&\left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} + 4r\sigma^{1+\frac{\alpha}{2}} w_r + (\alpha+1) r^2\sigma^{\frac{\alpha}{2}}\sigma_r w_r \right\|^2 \\
=& \left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} \right\|^2 + 4 \left\| r\sigma^{1+\frac{\alpha}{2}} w_r \right\|^2 + 8B \int r^4\sigma^{1+\alpha} w_r^2.
\end{aligned}$$

In view of (3.12), we then see that

$$\left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} \right\|^2 + 4 \left\| r\sigma^{1+\frac{\alpha}{2}} w_r \right\|^2 \lesssim (1+t)^{\delta \mathbf{1}_{\lambda<1}} \left(\mathcal{E}_0 + \mathcal{E}_1\right). \tag{3.14}$$

Besides, by using (2.21), we have

$$\begin{aligned}
\int_{\mathcal{I}} r^2\sigma^{\alpha} w_r^2 =& \int_{\mathcal{I}_1} r^2\sigma^{\alpha} w_r^2 + \int_{\mathcal{I}_2} r^2\sigma^{\alpha} w_r^2 \\
\lesssim& \int_{\mathcal{I}_1} r^2\sigma^{\alpha+2} w_r^2 + \int_{\mathcal{I}_2} \sigma^{\alpha+2} \left(r^2 w_r^2 + w_r^2 + r^2 w_{rr}^2\right) \\
\lesssim& \left\| r\sigma^{1+\frac{\alpha}{2}} w_r \right\|^2 + \left\| r^2\sigma^{1+\frac{\alpha}{2}} w_{rr} \right\|^2.
\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we get

$$\mathcal{E}_{0,1}(t) \lesssim \mathcal{E}_0(t) + \mathcal{E}_1(t).$$

□

**Higher-order elliptic estimates**

For $i \geqslant 1$ and $j \geqslant 0$, applying $\partial_t^j \partial_r^{i-1}$ to (3.10) yields that

$$\begin{aligned}
&\gamma \tilde{\eta}_r^{1-3\gamma} \left[ r\sigma \partial_t^j \partial_r^{i+1} w + (i+3)\sigma \partial_r^i w + (\alpha+i)\, r\sigma_r \partial_t^j \partial_r^i w \right] \\
&= (1+o(1)) \left[ r\partial_t^{j+2} \partial_r^{i-1} w + \frac{\mu}{(1+t)^\lambda} r\partial_t^{j+1} \partial_r^{i-1} w \right] \\
&\quad + \mathcal{A}_1 + (1+o(1))\, \mathcal{A}_2,
\end{aligned} \tag{3.16}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined as follows

$$\begin{aligned}
\mathcal{A}_1 :=& -\gamma \sum_{\ell=1}^{j} \left[ \partial_t^\ell \left( \tilde{\eta}_r^{1-3\gamma} \right) \right] \partial_t^{j-\ell} \left[ r\sigma \partial_r^{i+1} w + (i+3)\sigma \partial_r^i w + (\alpha+i)\, r\sigma_r \partial_r^i w \right] \\
& -\gamma \partial_t^j \left\{ \tilde{\eta}_r^{1-3\gamma} \left[ \sum_{\ell=1}^{i-1} C_{i-1}^\ell \partial_r^\ell (r\sigma) \partial_r^{i+1-\ell} w + 4 \sum_{\ell=1}^{i-1} C_{i-1}^\ell \partial_r^\ell \sigma \partial_r^{i-\ell} w \right.\right. \\
& \left.\left. + (\alpha+1) \sum_{\ell=1}^{i-1} C_{i-1}^\ell \partial_r^\ell (r\sigma_r) \partial_r^{i-\ell} w \right] \right\}, \\
\mathcal{A}_2 :=& (i-1) \left( \partial_t^{j+2} \partial_r^{i-2} w + \frac{\mu}{(1+t)^\lambda} \partial_t^{j+1} \partial_r^{i-2} w \right) \\
& -2B(\alpha+1)(2-3\gamma) \partial_t^j \left[ \tilde{\eta}_r^{1-3\gamma} \left( r\partial_r^{i-1} w + (i-1)\partial_r^{i-2} w \right) \right] \\
& +\mu \sum_{\ell=1}^{j} C_j^\ell \partial_t^\ell (1+t)^{-\lambda} \left[ r\partial_t^{j+1-\ell} \partial_r^{i-1} w + (i-1) \partial_t^{j+1-\ell} \partial_r^{i-2} w \right].
\end{aligned} \tag{3.17}$$

Summations $\sum_{\ell=1}^{i-1}$ is set to be 0 when $i = 1$.

Multiply equation (3.16) by $\tilde{\eta}_r^{3\gamma-1} r\sigma^{(\alpha+i-1)/2}$ and perform the spatial $L^2$-norm of the product to give

$$\begin{aligned}
&\left\| r^2 \sigma^{\frac{\alpha+i+1}{2}} \partial_t^j \partial_r^{i+1} w + (i+3) r\sigma^{\frac{\alpha+i+1}{2}} \partial_t^j \partial_r^i w \right. \\
&\left. + (\alpha+i) r^2 \sigma^{\frac{\alpha+i-1}{2}} \sigma_r \partial_t^j \partial_r^i w \right\|^2 \\
&\lesssim (1+t)^{2(\lambda+1)} \left\| r^2 \sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j+2} \partial_r^{i-1} w \right\|^2 + (1+t)^2 \left\| r^2 \sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j+1} \partial_r^{i-1} w \right\|^2 \\
&+ (1+t)^{2(\lambda+1)} \left\| r\sigma^{\frac{\alpha+i-1}{2}} (\mathcal{A}_1, \mathcal{A}_2) \right\|^2.
\end{aligned}$$

Similar to the derivation of (3.12) to (3.15), we can then obtain

$$
\begin{aligned}
&(1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j,i}\\
&\quad= \left\| r^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_r^{i+1}w\right\|^2 + \left\| r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^j\partial_r^i w\right\|^2\\
&\quad\lesssim (1+t)^{2(\lambda+1)}\left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_r^{i-1}w\right\|^2 + (1+t)^2\left\| r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_r^{i-1}w\right\|^2\\
&\qquad + (1+t)^{2(\lambda+1)}\left\| r\sigma^{\frac{\alpha+i-1}{2}}(\mathcal{A}_1,\mathcal{A}_2)\right\|^2\\
&\quad\lesssim (1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\left(\mathcal{E}_{j+2,i-2}\mathbf{1}_{i\geqslant 2} + \mathcal{E}_{j+1}\mathbf{1}_{i=1} + \mathcal{E}_{j+1,i-2}\mathbf{1}_{i\geqslant 2} + \mathcal{E}_j\mathbf{1}_{i=1}\right)\\
&\qquad + (1+t)^{2(\lambda+1)}\left\| r\sigma^{\frac{\alpha+i-1}{2}}(\mathcal{A}_1,\mathcal{A}_2)\right\|^2.
\end{aligned}
\tag{3.18}
$$

We will use this estimate to prove the following lemma by mathematical induction.

**Lemma 3.5.** *Under the assumption of (3.2) for suitably small positive number $\epsilon_0 \in (0,1)$, then for $j \geqslant 0, i \geqslant 1$, and $0 \leqslant i+j \leqslant m$*

$$
\mathcal{E}_{j,i}(t) \lesssim \sum_{\ell=0}^{i+j}\mathcal{E}_\ell(t), \quad t\in[0,T]. \tag{3.19}
$$

**Proof.** We prove this lemma by induction on $i+j$. As shown in lemma 3.4, (3.19) holds for $i+j\leqslant 1$. For $1\leqslant k\leqslant m-1$, we make the following induction hypothesis that for all $i\geqslant 1$, $j\geqslant 0$, and $i+j\leqslant k$,

$$
\mathcal{E}_{j,i}(t) \lesssim \sum_{\ell=0}^{i+j}\mathcal{E}_\ell(t), \quad i\geqslant 1,\ j\geqslant 0,\ i+j\leqslant k, \tag{3.20}
$$

it then suffices to prove (3.19) for $i\geqslant 1$, $j\geqslant 0$, and $i+j=k+1$. We will bound $\mathcal{E}_{k+1-\ell,\ell}$ from $\ell=1$ to $k+1$ step by step.

We estimate $\mathcal{A}_1$ and $\mathcal{A}_2$ given by (3.17) as follows. It follows from (2.8), (2.9) and remember that $\sigma = A - Br^2$ and $\sigma_r = -2Br$, after a rearrangement of the index $\ell$, then we have that

$$
\begin{aligned}
|\mathcal{A}_1| &\lesssim \sum_{\ell=1}^{j}(1+t)^{-(\lambda+1)-\ell}\left(r\sigma\left|\partial_t^{j-\ell}\partial_r^{i+1}w\right| + \left|\partial_t^{j-\ell}\partial_r^i w\right|\right)\\
&\quad + \sum_{\ell=0}^{j}(1+t)^{-(\lambda+1)-\ell}\left(r\left|\partial_t^{j-\ell}\partial_r^{i-1}w\right| + \left|\partial_t^{j-\ell}\partial_r^{i-2}w\right|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{A}_2| &\lesssim \left|\partial_t^{j+2}\partial_r^{i-2}w\right| + (1+t)^{-\lambda}\left|\partial_t^{j+1}\partial_r^{i-2}w\right|\\
&\quad + \sum_{\ell=1}^{j+1}(1+t)^{-\lambda-\ell}\left(\left|r\partial_t^{j+1-\ell}\partial_r^{i-1}w\right| + \left|\partial_t^{j+1-\ell}\partial_r^{i-2}w\right|\right).
\end{aligned}
$$

So by the definition of $\mathcal{E}_j$ and $\mathcal{E}_{j,i}$ and boundedness of $\sigma$, we can get

$$
\begin{aligned}
\left\|r\sigma^{\frac{\alpha+i-1}{2}}\mathcal{A}_1\right\|^2 \lesssim & \sum_{\ell=1}^{j}(1+t)^{-2(\lambda+1)-2\ell}\left(\left\|r^2\sigma^{\frac{\alpha+i+1}{2}}\partial_t^{j-\ell}\partial_r^{i+1}w\right\|^2+\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\ell}\partial_r^{i}w\right\|^2\right)\\
&+\sum_{\ell=0}^{j}(1+t)^{-2(\lambda+1)-2\ell}\left(\left\|r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\ell}\partial_r^{i-1}w\right\|^2+\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\ell}\partial_r^{i-2}w\right\|^2\right)\\
\lesssim & (1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\sum_{\ell=1}^{j}\mathcal{E}_{j-\ell,i}\\
&+\sum_{\ell=0}^{j}(1+t)^{-2(\lambda+1)-2\ell}\left(\left\|r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j-\ell}\partial_r^{i-1}w\right\|^2+\left\|r\sigma^{\frac{\alpha+i-3}{2}}\partial_t^{j-\ell}\partial_r^{i-2}w\right\|^2\right)\\
\lesssim & (1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\left(\sum_{\ell=1}^{j}\mathcal{E}_{j-\ell,i}+\sum_{\ell=0}^{j}\left(\mathcal{E}_{j-\ell,i-2}\mathbf{1}_{i\geqslant 2}+\mathcal{E}_{j-\ell}\mathbf{1}_{i=1}\right)\right).
\end{aligned}
$$

And by applying the estimate of $\mathcal{A}_2$, the same as above gives

$$
\begin{aligned}
&\left\|r\sigma^{\frac{\alpha+i-1}{2}}\mathcal{A}_2\right\|^2\\
&\lesssim \left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_r^{i-2}w\right\|^2+(1+t)^{-2\lambda}\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_r^{i-2}w\right\|^2\\
&\quad+\sum_{\ell=1}^{j+1}(1+t)^{-2\lambda-2\ell}\left(\left\|r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1-\ell}\partial_r^{i-1}w\right\|^2\right.\\
&\quad\left.+\left\|r\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1-\ell}\partial_r^{i-2}w\right\|^2\right)\\
&\lesssim \left\|r\sigma^{\frac{\alpha+i-3}{2}}\partial_t^{j+2}\partial_r^{i-2}w\right\|^2+(1+t)^{-2\lambda}\left\|r\sigma^{\frac{\alpha+i-3}{2}}\partial_t^{j+1}\partial_r^{i-2}w\right\|^2\\
&\quad+\sum_{\ell=1}^{j+1}(1+t)^{-2\lambda-2\ell}\left(\left\|r^2\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1-\ell}\partial_r^{i-1}w\right\|^2\right.\\
&\quad\left.+\left\|r\sigma^{\frac{\alpha+i-3}{2}}\partial_t^{j+1-\ell}\partial_r^{i-2}w\right\|^2\right)\\
&\lesssim (1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\left(\mathcal{E}_{j+2,i-2}\mathbf{1}_{i\geqslant 2}+\mathcal{E}_{j+1,i-2}\mathbf{1}_{i\geqslant 2}\right)\\
&\quad+(1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\sum_{\ell=1}^{j+1}\left(\mathcal{E}_{j+1-\ell,i-2}\mathbf{1}_{i\geqslant 2}+\mathcal{E}_{j+1-\ell}\mathbf{1}_{i=1}\right).
\end{aligned}
$$

Now combining all the above estimates for $\mathcal{A}_1$ and $\mathcal{A}_2$, we get

$$\begin{aligned}&\left\|\sigma^{\frac{\alpha+i-1}{2}}(\mathcal{A}_1,\mathcal{A}_2)\right\|^2\\&\lesssim(1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\left(\mathcal{E}_{j+2,i-2}\mathbf{1}_{i\geqslant 2}+\mathcal{E}_{j+1,i-2}\mathbf{1}_{i\geqslant 2}\right)\\&\quad+(1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\left(\sum_{\ell=0}^{j}\mathcal{E}_\ell+\sum_{\substack{0\leqslant\ell\leqslant j\\ \ell+\iota\leqslant i+j-1}}\mathcal{E}_{\ell,\iota}\right).\end{aligned}$$

Substituting this into (3.18), we get

$$\begin{aligned}\mathcal{E}_{j,i}&\lesssim\left(\mathcal{E}_{j+2,i-2}\mathbf{1}_{i\geqslant 2}+\mathcal{E}_{j+1,i-2}\mathbf{1}_{i\geqslant 2}\right)\\&\quad+\sum_{\substack{0\leqslant\ell\leqslant j\\ \ell+\iota\leqslant i+j-1}}\mathcal{E}_{\ell,\iota}+\sum_{\ell=0}^{j}\mathcal{E}_\ell.\end{aligned}\tag{3.21}$$

In particularly, when $i\geqslant 2$, we have

$$\mathcal{E}_{j,i}\lesssim\mathcal{E}_{j+2,i-2}+\mathcal{E}_{j+1,i-2}+\sum_{\substack{0\leqslant\ell\leqslant j\\ \ell+\iota\leqslant i+j-1}}\mathcal{E}_{\ell,\iota}+\sum_{\ell=0}^{j}\mathcal{E}_\ell.\tag{3.22}$$

In what follows, we use (3.22) and the induction hypothesis (3.20) to show that (3.19) holds for $i+j=k+1$. First, choosing $j=k$ and $i=1$ in (3.21) gives

$$\mathcal{E}_{k,1}(t)\lesssim\sum_{\substack{0\leqslant\ell\leqslant k\\ \ell+\iota\leqslant k}}\mathcal{E}_{\ell,\iota}+\sum_{\ell=0}^{k}\mathcal{E}_\ell$$

which, together with (3.20) implies

$$\mathcal{E}_{k,1}(t)\lesssim\sum_{\ell=0}^{k+1}\mathcal{E}_\ell(t).\tag{3.23}$$

Similarly, in (3.22), by choosing $j=k-1, i=2$ and remembering that (3.11) indicates that $\mathcal{E}_{k+1,0}\lesssim\mathcal{E}_{k+1}$ and $\mathcal{E}_{k,0}\lesssim\mathcal{E}_k$, then we have

$$\mathcal{E}_{k-1,2}(t)\lesssim\mathcal{E}_{k+1}(t)+\mathcal{E}_k(t)+\sum_{\substack{0\leqslant\ell\leqslant k-1\\ \ell+\iota\leqslant k}}\mathcal{E}_{\ell,\iota}+\sum_{\ell=0}^{k-1}\mathcal{E}_\ell\lesssim\sum_{\ell=0}^{k+1}\mathcal{E}_\ell(t).$$

For $\mathcal{E}_{k-2,3}$, it follows from (3.22), assumption (3.20) and (3.23) that

$$\mathcal{E}_{k-2,3}(t)\lesssim\mathcal{E}_{k,1}(t)+\mathcal{E}_{k-1,1}(t)+\sum_{\substack{0\leqslant\ell\leqslant k-2\\ \ell+\iota\leqslant k}}\mathcal{E}_{\ell,\iota}+\sum_{\ell=0}^{k-2}\mathcal{E}_\ell\lesssim\sum_{\ell=0}^{k+1}\mathcal{E}_\ell(t).$$

The other cases can be handled similarly. So we have proved (3.19) when $i+j=k+1$. This finishes the proof of lemma 3.5. □

### *3.2. Nonlinear Weighted Energy Estimates*

In this subsection, we prove that the weighted energy $\mathcal{E}_j(t)$ can be bounded by the initial data for $t \in [0, T]$.

**Proposition 3.6.** *Suppose that* (3.2) *holds for a suitably small positive number* $\epsilon_0 \in (0, 1)$, *then for* $t \in [0, T]$

$$\mathcal{E}_j(t) \lesssim \sum_{\ell=0}^{j} \mathcal{E}_\ell(0), \quad j = 0, 1, \ldots, m.$$

The proof of proposition 3.6 contains lemmas 3.7 and 3.8.

**Basic energy estimates**

**Lemma 3.7.** *Suppose that* (3.2) *holds for a suitably small positive number* $\epsilon_0 \in (0, 1)$, *then*

$$\begin{aligned}\mathcal{E}_0(t) + \int_0^t \int &\left[(1+\tau)^{1-\delta \mathbf{1}_{\lambda<1}} r^4 \sigma^\alpha w_\tau^2\right.\\ &\left.+ (1+\tau)^{-1-\delta \mathbf{1}_{\lambda<1}} r^2 \sigma^{\alpha+1}\left(w^2 + (rw_r)^2\right)\right]\\ \lesssim &\mathcal{E}_0(0), \quad t \in [0, T].\end{aligned} \tag{3.24}$$

**Proof.** The proof will be divided into two parts. One is for $0 < \lambda < 1,\ \mu > 0$ and the other is for $\lambda = 1,\ \mu > 2$.

**Case 1: $0 < \boldsymbol{\lambda} < 1, \boldsymbol{\mu} > 0$**

Multiplying (3.9) by $(\kappa + t)^\lambda r^3 w_t$, where $\kappa > 1$ is a suitably large constant, to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\kappa+t)^\lambda r^4 \sigma^\alpha w_t^2 - \frac{\lambda}{2}(\kappa+t)^{\lambda-1}\int r^4\sigma^\alpha w_t^2 + \mu\left(\frac{\kappa+t}{1+t}\right)^\lambda \int r^4 \sigma^\alpha w_t^2\\ &+ (1+o(1))(\kappa+t)^\lambda \tilde{\eta}_r^{1-3\gamma}\int \sigma^{\alpha+1}\left[(3\gamma w + \gamma r w_r)(r^3 w_t)_r - (2r^3 w w_t)_r\right] = 0.\end{aligned} \tag{3.25}$$

A direct computation indicates that

$$\begin{aligned}&(3\gamma w + \gamma r w_r)(r^3 w_t)_r - (2r^3 w w_t)_r\\ &= r^2\left[\frac{3}{2}(3\gamma-2)w^2 + (3\gamma-2)wrw_r + \frac{\gamma}{2}(rw_r)^2\right]_t\\ &:= r^2 \frac{1}{2}[\mathfrak{D}(w)]_t,\end{aligned}$$

where by using $\gamma > 1$, we have

$$\mathfrak{D}(w) := (9\gamma-6)w^2 + (6\gamma-4)wrw_r + \gamma(rw_r)^2 \approx w^2 + (rw_r)^2.$$

Inserting the above two inequalities into (3.25), we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{\lambda}\left[r^4\sigma^{\alpha}w_t^2+(1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right] \\ +\left[\mu\left(\frac{\kappa+t}{1+t}\right)^{\lambda}-\frac{\lambda}{2}(\kappa+t)^{\lambda-1}\right]\int r^4\sigma^{\alpha}w_t^2 \\ -(1+o(1))\frac{1}{2}\partial_t\left((\kappa+t)^{\lambda}\tilde{\eta}_r^{1-3\gamma}\right)\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)=0. \tag{3.26}$$

Using the fact that $\tilde{\eta}_{rt}\geqslant 0$ and throwing the $\tilde{\eta}_{rt}$ term in (3.26), we simplify it to be

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{\lambda}\left[r^4\sigma^{\alpha}w_t^2+(1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right] \\ +\left[\mu\left(\frac{\kappa+t}{1+t}\right)^{\lambda}-\frac{\lambda}{2}(\kappa+t)^{\lambda-1}\right]\int r^4\sigma^{\alpha}w_t^2 \\ -(1+o(1))\frac{\lambda}{2}(\kappa+t)^{\lambda-1}\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\leqslant 0. \tag{3.27}$$

Now multiplying (3.9) by $\nu r^3 w$ for some small $\nu>0$, to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}\int r^4\sigma^{\alpha}w_t w-\nu\int r^4\sigma^{\alpha}w_t^2+\frac{\nu\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{(1+t)^{\lambda}}r^4\sigma^{\alpha}w^2 \\ +\frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int r^4\sigma^{\alpha}w^2 \\ +\nu(1+o(1))\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)=0. \tag{3.28}$$

Adding (3.27) and (3.28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\tilde{\mathfrak{E}}_0(r,t)+\frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int r^4\sigma^{\alpha}w^2 \\ +\left[\mu-\frac{\lambda}{2}(\kappa+t)^{\lambda-1}-\nu\right]\int r^4\sigma^{\alpha}w_t^2 \\ +(1+o(1))\tilde{\eta}_r^{1-3\gamma}\left(\nu-\frac{\lambda}{2}(\kappa+t)^{\lambda-1}\right)\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\leqslant 0. \tag{3.29}$$

Here

$$\tilde{\mathfrak{E}}_0(r,t):=\frac{(\kappa+t)^{\lambda}}{2}\left[r^4\sigma^{\alpha}w_t^2+(1+o(1))\tilde{\eta}_r^{1-3\gamma}\sigma^{\alpha+1}\mathfrak{D}(w)\right] \\ +\nu r^4\sigma^{\alpha}w_t w+\frac{\nu\mu}{2(1+t)^{\lambda}}r^4\sigma^{\alpha}w^2.$$

By using Cauchy–Schwartz inequality, we have

$$\begin{aligned} &\frac{(\kappa+t)^{\lambda}}{4}\left[r^4\sigma^{\alpha}w_t^2+\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right]+\left(\frac{\nu\mu}{2}-\nu^2\right)\frac{r^4}{(1+t)^{\lambda}}\sigma^{\alpha}w^2\\ &\qquad\leqslant \tilde{\mathfrak{E}}_0(r,t)\leqslant\\ &\frac{3(\kappa+t)^{\lambda}}{4}\left[r^4\sigma^{\alpha}w_t^2+\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right]+\left(\frac{\nu\mu}{2}+\nu^2\right)\frac{r^4}{(1+t)^{\lambda}}\sigma^{\alpha}w^2. \end{aligned} \tag{3.30}$$

Since $\lambda<1$, by first choosing small $\nu$ and then large $\kappa$, we can get from (3.29) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int\tilde{\mathfrak{E}}_0(r,t)+\frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int r^4\sigma^{\alpha}w^2\\ &+\frac{\nu}{2}\int r^4\sigma^{\alpha}w_t^2+\frac{\nu}{2}\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\leqslant 0. \end{aligned} \tag{3.31}$$

Now multiplying (3.31) by $(\kappa+t)^{\lambda-\delta}$, we can achieve

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(r,t)-(\lambda-\delta)(\kappa+t)^{\lambda-1-\delta}\tilde{\mathfrak{E}}_0(r,t)\\ &\quad+\frac{\nu\mu\lambda(\kappa+t)^{\lambda-\delta}}{2(1+t)^{\lambda+1}}\int r^4\sigma^{\alpha}w^2\\ &\quad+\frac{\nu(\kappa+t)^{\lambda-\delta}}{2}\left\{\int r^4\sigma^{\alpha}w_t^2+\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right\}\leqslant 0. \end{aligned} \tag{3.32}$$

By inserting the right side of (3.30) into (3.32), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(r,t)\\ &\quad+\frac{\nu\mu(\kappa+t)^{\lambda-\delta}}{2(1+t)^{\lambda+1}}\underbrace{\left(\lambda-(\lambda-\delta)(1+\frac{2\nu}{\mu})\right)}_{L_1}\int r^4\sigma^{\alpha}w^2\\ &\quad+(\kappa+t)^{\lambda-\delta}\underbrace{\left(\frac{\nu}{2}-\frac{3}{4}(\kappa+t)^{\lambda-1}\right)}_{L_2}\left\{\int r^4\sigma^{\alpha}w_t^2+\tilde{\eta}_r^{1-3\gamma}\int\sigma^{\alpha+1}\mathfrak{D}(w)\right\}\leqslant 0. \end{aligned}$$

Again, by choosing small $\nu$ and large $\kappa$, for any $\delta>0$, we can assure that $L_1$ and $L_2$ are positive. Then we have for some constant $c_{\lambda,\mu}$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(r,t)\\ &\quad+c_{\lambda,\mu}(\kappa+t)^{\lambda-\delta}\left\{\int r^4\sigma^{\alpha}w_t^2+\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right\}\leqslant 0. \end{aligned} \tag{3.33}$$

Now we multiply (3.27) by $(\kappa+t)^{1-\delta}$ to achieve that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\kappa+t)^{1+\lambda-\delta}\left[r^4\sigma^\alpha w_t^2+(1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right]\\
&\quad+c_{\lambda,\mu}(\kappa+t)^{1-\delta}\int r^4\sigma^\alpha w_t^2\\
&\quad-c_{\lambda,\mu}(\kappa+t)^{\lambda-\delta}\int\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)+r^4\sigma^\alpha w_t^2\leqslant 0.
\end{aligned}
\tag{3.34}
$$

Multiplying a small number $\nu_1$ to (3.34) and then adding the resulting equation to (3.33), we can get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\mathfrak{E}_0(r,t)+c_{\lambda,\mu}(1+t)^{1-\delta}\int r^4\sigma^\alpha w_t^2\\
&\quad+c_{\lambda,\mu}(1+t)^{-1-\delta}\int r^2\sigma^{\alpha+1}(w^2+(rw_r)^2)\leqslant 0,
\end{aligned}
\tag{3.35}
$$

where we have used the facts that $\mathfrak{D}(w)\approx w^2+(rw_r)^2$ and $\tilde{\eta}^{1-3\gamma}\approx(1+t)^{-(\lambda+1)}$. Here

$$
\begin{aligned}
\mathfrak{E}_0(r,t)&:=(\kappa+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(r,t)\\
&\qquad+\nu_1(\kappa+t)^{1+\lambda-\delta}\left[r^4\sigma^\alpha w_t^2+(1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right]\\
&\approx(1+t)^{1+\lambda-\delta}r^4\sigma^\alpha w_t^2+(1+t)^{-\delta}r^4\sigma^\alpha w^2\\
&\qquad+(1+t)^{-\delta}r^2\sigma^{\alpha+1}(w^2+(rw_r)^2),
\end{aligned}
$$

and $\int\mathfrak{E}_0(r,t)\mathrm{d}r\approx\mathcal{E}_0(t)$.

Now integrating (3.35) with respect to time variable from 0 to $t$. We get (3.24) in case of $0<\lambda<1,\ \mu>0$.

**Case 2: $\boldsymbol{\lambda=1,\mu>2}$**

Multiplying (3.9) by $(1+t)^2r^3w_t$ and integrating the product with respect to the spatial variable, then we can get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^2r^4\sigma^\alpha w_t^2+(\mu-1)(1+t)\int r^4\sigma^\alpha w_t^2\\
&\quad+(1+o(1))(1+t)^2\tilde{\eta}_r^{1-3\gamma}\int\frac{1}{2}r^2\sigma^{\alpha+1}[\mathfrak{D}(w)]_t=0.
\end{aligned}
$$

By using that $\tilde{\eta}_{rt}\geqslant 0$, we then have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}(1+t)^2\left[r^4\sigma^\alpha w_t^2+(1+o(1))r^2\sigma^{\alpha+1}\tilde{\eta}_r^{1-3\gamma}\mathfrak{D}(w)\right]\\
&\quad+(\mu-1)(1+t)\int r^4\sigma^\alpha w_t^2\\
&\quad-(1+o(1))(1+t)\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\leqslant 0.
\end{aligned}
\tag{3.36}
$$

Now multiplying (3.9) by $\nu(1+t)r^3 w$ for some positive $\nu$ to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\nu\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)r^4\sigma^\alpha w_t w - \nu(1+t)\int r^4\sigma^\alpha w_t^2 + \frac{\nu(\mu-1)}{2}\partial_t\int r^4\sigma^\alpha w^2 \\ + \nu(1+o(1))(1+t)\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w) = 0. \tag{3.37}$$

Adding (3.36) and (3.37), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathfrak{E}_0(r,t) + (\mu-1-\nu)(1+t)\int r^4\sigma^\alpha w_t^2 \\ + (\nu-1)(1+o(1))(1+t)\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w) \leqslant 0. \tag{3.38}$$

Here

$$\mathfrak{E}_0(r,t) := \frac{(1+t)^2}{2}\left[r^4\sigma^\alpha w_t^2 + (1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right] \\ + \nu(1+t)r^4\sigma^\alpha w_t w + \frac{\nu(\mu-1)}{2}r^4\sigma^\alpha w^2.$$

Now, since $\mu > 2$, we assume $\mu = 2+2\kappa$ for some positive $\kappa$. Choosing $\nu = 1+\kappa$, we can achieve

$$\mathfrak{E}_0(r,t) := \frac{(1+t)^2}{2}\left[r^4\sigma^\alpha w_t^2 + (1+o(1))\tilde{\eta}_r^{1-3\gamma}r^2\sigma^{\alpha+1}\mathfrak{D}(w)\right] \\ + (1+\kappa)(1+t)r^4\sigma^\alpha w_t w + \frac{(1+\kappa)(1+2\kappa)}{2}r^4\sigma^\alpha w^2.$$

By using Cauchy–Shwartz inequality to absorb the term involving $w_t w$ and remembering that $\tilde{\eta}_r^{1-3\gamma} \approx (1+t)^{-2}$, it is not hard to deduce that

$$\mathfrak{E}_0(r,t) \approx (1+t)^2 r^4\sigma^\alpha w_t^2 + r^2\sigma^{\alpha+1}\left(w^2+(rw_r)^2\right) + r^4\sigma^\alpha w^2, \qquad \int \mathfrak{E}_0(r,t)\mathrm{d}x \approx \mathcal{E}_0(t).$$

Then (3.38) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathfrak{E}_0(r,t)+\kappa(1+t)\int r^4\sigma^\alpha w_t^2 \\ +\kappa(1+t)(1+o(1))\tilde{\eta}_r^{1-3\gamma}\int r^2\sigma^{\alpha+1}\mathfrak{D}(w)\leqslant 0. \tag{3.39}$$

Now integrating (3.39) with respect to time variable from 0 to $t$. We get (3.24) in the case of $\lambda=1,\ \mu>2$. □

**Higher-order energy estimates**

For $k\geqslant 1$, applying $\partial_t^k$ to (3.9) yields that

$$\begin{aligned} &r\sigma^\alpha\partial_t^{k+2}w+\frac{\mu}{(1+t)^\lambda}r\sigma^\alpha\partial_t^{k+1}w+\mu r\sigma^\alpha\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell(1+t)^{-\lambda}\partial_t^{k+1-\ell}w \\ &\quad-(1+o(1))\tilde{\eta}_r^{1-3\gamma}\left\{\left[\sigma^{\alpha+1}(3\gamma\partial_t^k w+\gamma r\partial_t^k w_r)\right]_r-2\partial_t^k w[\sigma^{\alpha+1}]_r\right\} \\ &=(1+o(1))\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell\left(\tilde{\eta}_r^{1-3\gamma}\right)\partial_t^{k-\ell}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma r w_r)\right]_r-2w[\sigma^{\alpha+1}]_r\right\}. \end{aligned}$$

After rearrangement, we rewrite it as

$$\begin{aligned} &r\sigma^\alpha\partial_t^{k+2}w+\frac{\mu}{(1+t)^\lambda}r\sigma^\alpha\partial_t^{k+1}w \\ &\quad-(1+o(1))\tilde{\eta}_r^{1-3\gamma}\left\{\left[\sigma^{\alpha+1}(3\gamma\partial_t^k w+\gamma r\partial_t^k w_r)\right]_r-2\partial_t^k w[\sigma^{\alpha+1}]_r\right\} \\ &:=\mathcal{B}_1+\mathcal{B}_2, \end{aligned}$$

where

$$\begin{aligned} &\mathcal{B}_1:=-\mu r\sigma^\alpha\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell(1+t)^{-\lambda}\partial_t^{k+1-\ell}w, \\ &\mathcal{B}_2:=(1+o(1))\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell\left(\tilde{\eta}_r^{1-3\gamma}\right)\partial_t^{k-\ell}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma r w_r)\right]_r-2w[\sigma^{\alpha+1}]_r\right\}. \end{aligned}$$

**Lemma 3.8.** *Suppose that (3.2) holds for some small positive number $\epsilon_0\in(0,1)$, then for all $j=1,\ldots,m$*

$$\begin{aligned} &\mathcal{E}_j(t)+\int_0^t\int\Big[(1+\tau)^{2j+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha\left(\partial_\tau^{j+1}w\right)^2 \\ &\qquad+(1+\tau)^{2j-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_\tau^j w)^2+(r\partial_\tau^j w_r)^2\right)\Big]\,\mathrm{d}r\,\mathrm{d}\tau \\ &\lesssim\sum_{\ell=0}^{j}\mathcal{E}_\ell(0),\qquad t\in[0,T]. \end{aligned} \tag{3.40}$$

**Proof.** We use induction to prove (3.40). As shown in lemma 3.7 we know that (3.40) holds for $j=0$. For $1\leqslant k\leqslant m$, we make the induction hypothesis that (3.40) holds for all

$j = 0, 1, \ldots, k-1$, i.e.

$$\begin{aligned} &\mathcal{E}_j(t) + \int_0^t \int \Big[ (1+\tau)^{2j+1-\delta \mathbf{1}_{\lambda<1}} r^4 \sigma^\alpha \left(\partial_\tau^{j+1} w\right)^2 \\ &\quad + (1+\tau)^{2j-1-\delta \mathbf{1}_{\lambda<1}} r^2 \sigma^{\alpha+1} \left( (\partial_\tau^j w)^2 + (r \partial_\tau^j w_r)^2 \right) \Big] \, \mathrm{d}\, r \mathrm{d}\tau \\ &\lesssim \sum_{\ell=0}^{j} \mathcal{E}_\ell(0), \quad t \in [0, T], \; 0 \leqslant j \leqslant k-1. \end{aligned} \tag{3.41}$$

It suffices to prove (3.40) holds for $j = k$ under the induction hypothesis (3.41). We divide the proof into three steps.

**Step one: setup of the linearized main term**

If we view $\partial_t^k w$ as $w$ in the proof of lemma 3.7, we can get a similar formula with (3.35) and (3.39) as follows

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int \mathfrak{E}_k(t) \mathrm{d}x + (1+t)^{1-\delta \mathbf{1}_{\lambda<1}} \int r^4 \sigma^\alpha (\partial_t^{k+1} w)^2 \mathrm{d}x \\ &\quad + (1+t)^{-1-\delta \mathbf{1}_{\lambda<1}} \int r^2 \sigma^{\alpha+1} \left( (\partial_t^k w)^2 + (r \partial_t^k w_r)^2 \right) \\ &\lesssim (1+t)^{1+\lambda-\delta \mathbf{1}_{\lambda<1}} \int (\mathcal{B}_1 + \mathcal{B}_2) \, r^3 \partial_t^{k+1} w + (1+t)^{\lambda-\delta \mathbf{1}_{\lambda<1}} \int (\mathcal{B}_1 + \mathcal{B}_2) \, r^3 \partial_t^k w \\ &:= \sum_{i=1}^4 \mathfrak{J}_i \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \mathfrak{E}_k(r,t) \approx\; &(1+t)^{1+\lambda-\delta \mathbf{1}_{\lambda<1}} r^4 \sigma^\alpha (\partial_t^{k+1} w)^2 \\ &+ (1+t)^{-\delta \mathbf{1}_{\lambda<1}} \left[ r^2 \sigma^{\alpha+1} \left( (\partial_t^k w)^2 + (r \partial_t^k w_r)^2 \right) + r^4 \sigma^\alpha (\partial_t^k w)^2 \right], \end{aligned} \tag{3.43}$$

and

$$(1+t)^{2k} \int \mathfrak{E}_k(r,t) \approx \mathcal{E}_k(t).$$

**Step two: estimates of the right hand terms in (3.42)**

Next we show that the right hand terms in (3.42) can be bounded by the left hand of (3.42) and the induction assumption (3.41).

In the process of performing estimates for $\mathfrak{J}_i$, $\mathfrak{J}_1$ and $\mathfrak{J}_3$ are the easiest terms, $\mathfrak{J}_2$ is the most difficult term which will be handled by integration by parts both on the space variable $r$ and the time variable $t$, and estimate of $\mathfrak{J}_4$ is a consequence of that for $\mathfrak{J}_2$.

It is easy to see that

$$|\mathcal{B}_1| \lesssim r\sigma^\alpha \sum_{\ell=1}^{k} (1+t)^{-\lambda-\ell} |\partial_t^{k+1-\ell} w|.$$

Then by using Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\mathfrak{J}_1 &\lesssim \sum_{\ell=1}^{k}(1+t)^{1-\delta\mathbf{1}_{\lambda<1}-\ell}\int r^4\sigma^\alpha|\partial_t^{k+1-\ell}w||\partial_t^{k+1}w| \\
&\lesssim \varepsilon(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k+1}w|^2 \\
&\quad +C_\varepsilon\sum_{\ell=1}^{k}(1+t)^{1-\delta\mathbf{1}_{\lambda<1}-2\ell}\int r^4\sigma^\alpha|\partial_t^{k+1-\ell}w|^2 \\
&\lesssim \varepsilon(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k+1}w|^2 \\
&\quad +C_\varepsilon(1+t)^{-2k}\sum_{\ell=1}^{k}(1+t)^{2(k-\ell)+1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k+1-\ell}w|^2,
\end{aligned}$$

and also the same estimate applied to $\mathfrak{J}_3$ gives

$$\begin{aligned}
\mathfrak{J}_3 &\lesssim \varepsilon(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k+1}w|^2 \\
&\quad +C_\varepsilon(1+t)^{-2k}\sum_{\ell=1}^{k}(1+t)^{2(k-\ell)-1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k-\ell}w|^2.
\end{aligned}$$

Combining the above two estimates, we have

$$\begin{aligned}
|\mathfrak{J}_1|+|\mathfrak{J}_3| &\lesssim \varepsilon(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{k+1}w|^2 \\
&\lesssim (1+t)^{-2k}\sum_{\ell=0}^{k-1}(1+t)^{2\ell+1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{\ell+1}w|^2.
\end{aligned}$$

The first term on the right-hand of the above inequality can be absorbed by the left of (3.42), while the second term on the right-hand can be controlled by the induction assumption.

For $\mathfrak{J}_2$, we need to use integration by parts. First we use integration by parts on $r$ to get

$$\begin{aligned}
\mathfrak{J}_2 &= (1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}(1+o(1))\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell\left(\tilde{\eta}_r^{1-3\gamma}\right) \\
&\quad\times\int\partial_t^{k-\ell}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma r w_r)\right]_r-2w[\sigma^{\alpha+1}]_r\right\}r^3\partial_t^{k+1}w \\
&\lesssim \sum_{\ell=1}^{k}(1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^{\alpha+1}\left\{(3\gamma\partial_t^{k-\ell}w+\gamma r\partial_t^{k-\ell}w_r)\left(r^3\partial_t^{k+1}w\right)_r\right.\right. \\
&\quad\left.\left.-2\left(r^3\partial_t^{k-\ell}w\partial_t^{k+1}w\right)_r\right\}\right| \\
&= \sum_{\ell=1}^{k}(1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int r^2\sigma^{\alpha+1}\left\{(9\gamma-6)\partial_t^{k-\ell}w\partial_t^{k+1}w\right.\right. \\
&\quad+(3\gamma-2)r\partial_t^{k-\ell}w\partial_t^{k+1}w_r+(3\gamma-2)r\partial_t^{k-\ell}w_r\partial_t^{k+1}w \\
&\quad\left.\left.+\gamma r^2\partial_t^{k-\ell}w_r\partial_t^{k+1}w_r\right\}\right|.
\end{aligned}$$

Then, we extract a time derivative term (denoted below by $\mathfrak{J}_{2\text{wave}}$), which can be absorbed by the first term of the left side of (3.42). Continuous calculation of the above inequality implies that

$$
\begin{aligned}
\mathfrak{J}_2 \leq & \sum_{\ell=1}^{k} (1+t)^{-\ell-\delta \mathbf{1}_{\lambda<1}} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int r^2 \sigma^{\alpha+1} \left\{ (9\gamma-6) \partial_t^{k-\ell} w \partial_t^k w \right.\right. \\
& + (3\gamma-2) r \partial_t^{k-\ell} w \partial_t^k w_r + (3\gamma-2) r \partial_t^{k-\ell} w_r \partial_t^k w \\
& \left.\left. + \gamma r^2 \partial_t^{k-\ell} w_r \partial_t^k w_r \right\} \right| \\
& + \sum_{\ell=1}^{k} (1+t)^{-\ell-\delta \mathbf{1}_{\lambda<1}} \left| \int r^2 \sigma^{\alpha+1} \left\{ (9\gamma-6) \partial_t^{k+1-\ell} w \partial_t^k w \right.\right. \\
& + (3\gamma-2) r \partial_t^{k+1-\ell} w \partial_t^k w_r + (3\gamma-2) r \partial_t^{k+1-\ell} w_r \partial_t^k w \\
& \left.\left. + \gamma r^2 \partial_t^{k+1-\ell} w_r \partial_t^k w_r \right\} \right| \\
:= & \mathfrak{J}_{2\text{wave}} + \mathfrak{J}_{2\text{line}}.
\end{aligned}
$$

The same estimates as $\mathfrak{J}_1$ and $\mathfrak{J}_3$ imply that

$$
\begin{aligned}
\mathfrak{J}_{2\text{line}} \lesssim & \, \varepsilon (1+t)^{-1-\delta \mathbf{1}_{\lambda<1}} \int r^2 \sigma^{\alpha+1} \left( (\partial_t^k w)^2 + (r \partial_t^k w_r)^2 \right) \\
& + (1+t)^{-2k} \sum_{\ell=0}^{k-1} (1+t)^{2\ell-1-\delta \mathbf{1}_{\lambda<1}} \int r^2 \sigma^{\alpha+1} \left( (\partial_t^\ell w)^2 + (r \partial_t^\ell w_r)^2 \right).
\end{aligned}
$$

For simplicity, we denote for $\ell \geqslant 1$,

$$
\begin{aligned}
\mathcal{F}_\ell := & \, r^2 \sigma^{\alpha+1} \left\{ (9\gamma-6) \partial_t^{k-\ell} w \partial_t^k w \right. \\
& + (3\gamma-2) r \partial_t^{k-\ell} w \partial_t^k w_r + (3\gamma-2) r \partial_t^{k-\ell} w_r \partial_t^k w \\
& \left. + \gamma r^2 \partial_t^{k-\ell} w_r \partial_t^k w_r \right\}.
\end{aligned}
$$

Then $\mathfrak{J}_{2\text{wave}}$ can be estimated as follows.

$$
\begin{aligned}
\mathfrak{J}_{2\text{wave}} \leqslant & \left| \frac{\mathrm{d}}{\mathrm{d}t} \int (1+t)^{-\ell-\delta \mathbf{1}_{\lambda<1}} \mathcal{F}_\ell \right| \\
& + (\ell + \delta \mathbf{1}_{\lambda<1}) (1+t)^{-1-\ell-\delta \mathbf{1}_{\lambda<1}} \left| \int \mathcal{F}_\ell \right|.
\end{aligned}
$$

The second term in the above have the same estimate as $\mathfrak{J}_1$, $\mathfrak{J}_3$ and $\mathfrak{J}_{2\mathrm{line}}$. So at last we get

$$\begin{aligned}\mathfrak{J}_2 \lesssim & \left|\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\ell=1}^{k}\int (1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right| \\ & + \varepsilon(1+t)^{-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right) \\ & + (1+t)^{-2k}\sum_{\ell=0}^{k-1}(1+t)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right).\end{aligned}$$

For $\mathfrak{J}_4$, after integration by parts on $r$, it will have the same estimates as $\mathfrak{J}_1$, $\mathfrak{J}_3$ and $\mathfrak{J}_{2\mathrm{line}}$. We proceed it as follows.

$$\begin{aligned}\mathfrak{J}_4 =& (1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}(1+o(1))\sum_{\ell=1}^{k}\partial_t^\ell\left(\tilde{\eta}_r^{1-3\gamma}\right) \\ & \times\int \partial_t^{k-\ell}\left\{\left[\sigma^{\alpha+1}(3\gamma w+\gamma r w_r)\right]_r - 2w[\sigma^{\alpha+1}]_r\right\} r^3\partial_t^k w \\ \lesssim & \sum_{\ell=1}^{k}(1+t)^{-1-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int \sigma^{\alpha+1}\left\{(3\gamma\partial_t^{k-\ell}w+\gamma r\partial_t^{k-\ell}w_r)\left(r^3\partial_t^k w\right)_r\right.\right. \\ & \left.\left. - 2\left(r^3\partial_t^{k-\ell}w\partial_t^k w\right)_r\right\}\right| \\ =& \sum_{\ell=1}^{k}(1+t)^{-1-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int r^2\sigma^{\alpha+1}\left\{(9\gamma-6)\partial_t^{k-\ell}w\partial_t^k w\right.\right. \\ & + (3\gamma-2)r\partial_t^{k-\ell}w\partial_t^k w_r + (3\gamma-2)r\partial_t^{k-\ell}w_r\partial_t^k w \\ & \left.\left. + \gamma r^2\partial_t^{k-\ell}w_r\partial_t^k w_r\right\}\right|.\end{aligned}$$

$\mathfrak{J}_4$ has the same estimates as $\mathfrak{J}_{2\mathrm{line}}$.

Combining all the above estimates for terms $\mathfrak{J}_i$, we can arrive from (3.42) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int \mathfrak{E}_k(t) \pm \sum_{\ell=1}^{k}\int (1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right\} + (1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha(\partial_t^{k+1}w)^2\mathrm{d}x \\ &+ (1+t)^{-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right) \\ &\lesssim (1+t)^{-2k}\sum_{\ell=0}^{k-1}\left\{(1+t)^{2\ell+1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{\ell+1}w|^2\right. \\ &\left. + (1+t)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right)\right\}.\end{aligned}$$

**Step three: finishing proof of lemma 3.8**

Multiplying the above inequality by $(1+t)^{2k}$, we can get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2k}\int \mathfrak{E}_k(t) \pm \sum_{\ell=1}^{k}\int (1+t)^{2k-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right\}\\
&\quad -2k(1+t)^{2k-1}\int\left[\mathfrak{E}_k(t)+\sum_{\ell=1}^{k}\int (1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right]\\
&\quad +N\int\Big[(1+t)^{2k+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha(\partial_t^{k+1}w)^2\\
&\quad +(1+t)^{2k-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right)\Big]\\
&\lesssim \sum_{\ell=0}^{k-1}\left\{(1+t)^{2\ell+1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{\ell+1}w|^2\right.\\
&\quad \left.+(1+t)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right)\right\},
\end{aligned}\tag{3.44}$$

where $N$ is a suitable large constant.

For the term containing $\mathcal{F}_\ell$, using Cauchy–Schwartz inequality, for a small $\nu$, we have

$$\begin{aligned}
&\left|(1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\sum_{\ell=1}^{k}\int\mathcal{F}_\ell\right|\\
&\lesssim \nu(1+t)^{-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right)\\
&\quad +(1+t)^{-2k}\sum_{\ell=0}^{k-1}(1+t)^{2\ell-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right).
\end{aligned}\tag{3.45}$$

By choosing sufficiently small $\nu$, we have

$$\begin{aligned}
&(1+t)^{2k}\int\mathfrak{E}_k(t)\pm\sum_{\ell=1}^{k}\int(1+t)^{2k-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\\
&\gtrsim \mathcal{E}_k-\sum_{\ell=0}^{k-1}\mathcal{E}_\ell.
\end{aligned}\tag{3.46}$$

Also from (3.43) and (3.45), we have

$$\begin{aligned}
&\left|2k(1+t)^{2k-1}\int\left[\mathfrak{E}_k(t)+\sum_{\ell=1}^{k}\int(1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right]\right|\\
&\lesssim (1+t)^{2k+\lambda-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha(\partial_t^{k+1}w)^2\\
&\quad +(1+t)^{2k-1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha(\partial_t^k w)^2\\
&\quad +\sum_{\ell=0}^{k}(1+t)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right).
\end{aligned}\tag{3.47}$$

Inserting (3.47) into (3.44) and by choosing sufficiently large $N$ and using induction assumption (3.41), we can get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)^{2k}\int \mathfrak{E}_k(t) \pm \sum_{\ell=1}^{k}\int (1+t)^{2k-\ell-\delta\mathbf{1}_{\lambda<1}}\mathcal{F}_\ell\right\}\\
&\quad+\int\left[(1+t)^{2k+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha(\partial_t^{k+1}w)^2\right.\\
&\quad\left.+(1+t)^{2k-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right)\right]\\
&\lesssim\sum_{\ell=0}^{k-1}\left\{(1+t)^{2\ell+1-\delta\mathbf{1}_{\lambda<1}}\int r^4\sigma^\alpha|\partial_t^{\ell+1}w|^2\right.\\
&\quad\left.+(1+t)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}\int r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right)\right\}.
\end{aligned}$$

Integrating the above inequality from 0 to $t$ and remembering (3.46) and (3.41), we can get

$$\begin{aligned}
&\mathcal{E}_k(t)-\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(t)+\int_0^t\int\left[(1+\tau)^{2k+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha(\partial_t^{k+1}w)^2\right.\\
&\quad\left.+(1+\tau)^{2k-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right)\right]\\
&\lesssim\sum_{\ell=0}^{k}\mathcal{E}_\ell(0)+\sum_{\ell=0}^{k-1}\int_0^t\int(1+\tau)^{2\ell+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha\left(\partial_t^{\ell+1}w\right)^2\\
&\quad+\sum_{\ell=0}^{k-1}\int_0^t\int(1+\tau)^{2\ell-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_t^\ell w)^2+(r\partial_t^\ell w_r)^2\right)\\
&\lesssim\sum_{\ell=0}^{k}\mathcal{E}_\ell(0).
\end{aligned}$$

Again using (3.41), we can get from the above inequality that

$$\begin{aligned}
&\mathcal{E}_k(t)+\int_0^t\int\left[(1+\tau)^{2k+1-\delta\mathbf{1}_{\lambda<1}}r^4\sigma^\alpha(\partial_t^{k+1}w)^2\right.\\
&\quad\left.+(1+\tau)^{2k-1-\delta\mathbf{1}_{\lambda<1}}r^2\sigma^{\alpha+1}\left((\partial_t^k w)^2+(r\partial_t^k w_r)^2\right)\right]\\
&\lesssim\sum_{\ell=0}^{k}\mathcal{E}_\ell(0).
\end{aligned}$$

This finishes the proof of lemma 3.8. □

Then propositions 3.3 and 3.6 together imply (3.6), which proves theorem 2.1 by continuation argument.

**Remark 3.9.** In the proof of theorem 2.1, there are two points we need to give an explanation. One is that it seems that we seldom use the $L^\infty$ estimate in lemma 3.1 and the other is that when performing the nonlinear energy estimates in proposition 3.6, it seems that it doesn't involve in the elliptic estimates in proposition 3.3 and can be self contained. But the fact is not so, because lemma 3.1 and proposition 3.3 mainly play a role in the nonlinear lower order derivative quadratic or multi-power's terms' estimates when we use Taylor formula to expand the nonlinear term in (3.7). Since we only use the simplified equation (3.9) which only involves in the linear highest order derivative term to catch the essential structure of equation (3.7), the effectiveness of lemma 3.1 and proposition 3.3 can not be reflected well in the proof. Interesting readers can refer to Zeng [48] for more detailed calculation on how they work.

## 4. Proof of theorem 2.4

**Proof.** In this section, we prove theorem 2.4. First, it follows from (2.3), (2.6), (2.10), and that for $(r,t) \in \mathcal{I} \times [0,\infty)$

$$\rho(\eta(r,t),t) - \bar{\rho}(\bar{\eta}(r,t),t) = \frac{r^2\bar{\rho}_0(r)}{\eta^2(r,t)\eta_r(r,t)} - \frac{r^2\bar{\rho}_0(r)}{\bar{\eta}^2(r,t)\bar{\eta}_r(r,t)},$$

and

$$u(\eta(r,t),t) - \bar{u}(\bar{\eta}(r,t),t) = (rw + rh)_t(r,t).$$

Hence, by virtue of (3.1), (2.8), (2.9) and the boundedness of $h$, we have, for $(r,t) \in \mathcal{I} \times [0,\infty)$,

$$\begin{aligned}&\left|\rho(\eta(x,t),t) - \bar{\rho}(\bar{\eta}(x,t),t)\right| \\ \lesssim&\left(A - Br^2\right)^{\frac{1}{\gamma-1}}(1+t)^{-\frac{4(\lambda+1)}{3\gamma-1}}\left((1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}}\sqrt{\mathcal{E}(0)}+1\right)\end{aligned}$$

and

$$\begin{aligned}&\left|u(\eta(x,t),t) - \bar{u}(\bar{\eta}(x,t),t)\right| \\ \lesssim&r(1+t)^{-1}\left((1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}}\sqrt{\mathcal{E}(0)}+1\right).\end{aligned}$$

Then (2.15) and (2.16) follow. It follows from (2.5), (2.6) and (2.10) that

$$\begin{aligned}R(t) = \eta\left(\sqrt{A/B},t\right) &= (\tilde{\eta} + rw)\left(\sqrt{A/B},t\right) \\ &= (\bar{\eta} + rh + rw)\left(\sqrt{A/B},t\right) \\ &= \sqrt{A/B}\left((1+t)^{\frac{\lambda+1}{3\gamma-1}} + h(t) + w(\sqrt{AB^{-1}},t)\right).\end{aligned}$$

Again using the boundedness of $h$ and (3.1), we have

$$\begin{aligned}&\sqrt{A/B}(1+t)^{\frac{\lambda+1}{3\gamma-1}} - C(1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}}\sqrt{\mathcal{E}(0)} \\ &\leqslant R(t) \leqslant \\ &\sqrt{A/B}(1+t)^{\frac{\lambda+1}{3\gamma-1}} + C\left((1+t)^{\frac{\delta}{2}\mathbf{1}_{\lambda<1}}\sqrt{\mathcal{E}(0)}+1\right),\end{aligned}$$

which implies (2.17). For $k = 1, 2, 3$

$$\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} = \partial_t^k \tilde{\eta}(\sqrt{A/B}, t) + (r\partial_t^k w)(\sqrt{A/B}, t).$$

So using (2.8), (2.9) and (3.1), we get (2.18). □

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