J. Math. Fluid Mech. (2022) 24:16

1422-6928/22/010001-19
https://doi.org/10.1007/s00021-021-00651-6

Journal of Mathematical Fluid Mechanics

A Single-Component BKM-Type Regularity Criterion for the Inviscid Axially Symmetric Hall-MHD System

Zijin Li and Xinghong Pan

Communicated by D. Chae

Abstract. In this paper, we prove a single-component BKM-type regularity criterion for the inviscid axially symmetric Hall-MHD system. More precisely, we show that if the current density is swirl-free, then the $L^1_t L^\infty_x$ boundedness of partial vorticity (corresponding to curl of the swirl part of the velocity) implies regularity of the solution. The novelty of our result is that we only impose a critical regularity criterion on one component of the velocity which is new even if we ignore the magnetic field and consider only the 3D axially symmetric Euler equations. Our results can not be easily extended to the classical $L^1_t BMO$ norm regularity criterion at present, which will be pursued in our further works.

Mathematics Subject Classification. 35Q35, 76D03.

Keywords. Incompressible, Inviscid, Hall-MHD system, Axially symmetric, Regularity criterion, One direction.

Contents

1. Introduction

The Hall-MHD (HMHD) system has been widely used to describe plasma phenomena over decades, e.g. the structuring of sub-Alfvénic plasma expansion and rapid magnetic field transport in plasma opening switches [18]. It has become more evident that Hall physics plays a critical role in magnetic reconnection processes and this has spurred renewed interest in this subject these years. The theory of Hall-MHD is applicable to phenomena occurring on length scales shorter than an ion inertial length, and time scales shorter than an ion cyclotron period.

In this paper, we consider the 3D inviscid and resistive Hall-MHD system (Euler-HMHD), which reads

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \dfrac{1}{\mu_0} h \cdot \nabla h, \\ \partial_t h + u \cdot \nabla h + \nu_0 \nabla \times \big[(\nabla \times h) \times h \big] = h \cdot \nabla u + \nu \Delta h, \\ \nabla \cdot u = 0, \\ \nabla \cdot h = 0. \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity and $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic field. $p : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure. $\mu_0, \nu, \nu_0 > 0$ stand for the constant vacuum permeability, ohmic resistivity and ratio for the Hall effect.

Let us first briefly introduce the Hall-MHD system (1.1). Physically the first equation in (1.1) is the conservation of momentum, which is originally expressed by

$$\partial_t u + u \cdot \nabla u + \nabla P = j \times h. \tag{1.2}$$

Here the left hand side is identical with Euler equations, while $j \times h$ is the term of *Lorentz force*, with j stands for the current density. Consider the Ampère's circuital law (with Maxwell's addition):

$$\nabla \times h = \mu_0 j + \frac{1}{c^2} \partial_t E. \tag{1.3}$$

Here E is the electronic field and the ratio μ_0 is the vacuum permeability. If the characteristic speed of the process is far more smaller than the speed of light c, we can approximate (1.3) by neglecting $\frac{1}{c^2}\partial_t E$. Thus

$$j = \frac{1}{\mu_0} \nabla \times h. \tag{1.4}$$

Substituting (1.4) in (1.2), using the identity

$$(\nabla \times h) \times h = h \cdot \nabla h - \frac{1}{2} \nabla |h|^2,$$

and denoting $p := P + \frac{1}{2\mu_0}|h|^2$, one arrives $(1.1)_1$.

Now we are ready to introduce $(1.1)_2$. For a completely ionized plasma containing only n electrons and one type of ions of charge e, the *generalized Ohm's law* can be written as [3]

$$\frac{m_e}{ne^2} \partial_t j - \frac{1}{ne} \nabla \cdot \mathcal{P}_e = E + u \times h - \frac{1}{ne} j \times h - j/\sigma. \tag{1.5}$$

Here m_e is the electron mass. For the cases that the charge current density j does not vary with time, we have $\partial_t j \equiv 0$. If we further consider the pressure term $\nabla \cdot \mathcal{P}_e$ is negligible, then (1.5) can be simplified to

$$j = \sigma(E + u \times h) - \underbrace{\frac{\sigma}{ne} j \times h}_{\text{Hall effect term}}. \tag{1.6}$$

Here the last term in equation (1.6) is related to a phenomenon called the *Hall effect* in magnetohydrodynamic flow, and thus normally called the *Hall effect term*. For the case of perfect conductivity ($\sigma \to \infty$), it follows that

$$E + u \times h - \frac{1}{ne} j \times h = 0.$$

Using (1.4), one can express E by

$$E = -u \times h + \frac{1}{\mu_0 ne} (\nabla \times h) \times h.$$

Substituting the above equality in the resistive Maxwell-Faraday equation

$$\partial_t h = -\nabla \times E + \nu \Delta h,$$

we arrive

$$\partial_t h = \nabla \times (u \times h) - \frac{1}{\mu_0 ne} \nabla \times \big[(\nabla \times h) \times h\big] + \nu \Delta h.$$

Thus $(1.1)_2$ is derived by denoting $\nu_0 = \frac{1}{\mu_0 ne}$.

The third line of (1.1) is the conservation of mass, which is reduced to the divergence free of velocity for incompressible fluid. Finally the fourth line is the Gauss's law for magnetism.

In our paper, the coefficients μ_0, ν, ν_0 play no essential roles in the proof, so without loss of generality, we set $\mu_0 = \nu = \nu_0 = 1$.

We will consider axially symmetric solutions of system (1.1) and most of the proof is carried out in the cylindrical coordinates $(r,\, \theta,\, z)$, i.e., for $x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$,

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3.$$

A solution of (1.1) is called an axially symmetric solution, if and only if

$$\begin{cases} u = u^r(t,r,z)e_r + u^\theta(t,r,z)e_\theta + u^z(t,r,z)e_z, \\ h = h^r(t,r,z)e_r + h^\theta(t,r,z)e_\theta + h^z(t,r,z)e_z, \end{cases}$$

satisfy the system (1.1). Here the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0,0,1).$$

From the local existence and uniqueness results, it is clear that if the swirl component of the initial current density $j_0^\theta e_\theta = \nabla \times (h_0^r e_r + h_0^z e_z)$ vanishes, then j^θ will vanish for all time. That is: one only needs to assume $h_0^r = h_0^z \equiv 0$, then vanishing of h^r and h^z holds for any time. In this case and in cylindrical coordinates, (1.1) can be written into the following

$$\begin{cases} \partial_t u^r + (u^r\partial_r + u^z\partial_z)u^r - \dfrac{(u^\theta)^2}{r} + \partial_r p = -\dfrac{(h^\theta)^2}{r}, \\ \partial_t u^\theta + (u^r\partial_r + u^z\partial_z)u^\theta + \dfrac{u^\theta u^r}{r} = 0, \\ \partial_t u^z + (u^r\partial_r + u^z\partial_z)u^z + \partial_z p = 0, \\ \partial_t h^\theta + (u^r\partial_r + u^z\partial_z)h^\theta - \dfrac{h^\theta u^r}{r} = \left(\Delta - \dfrac{1}{r^2}\right) h^\theta + \dfrac{\partial_z (h^\theta)^2}{r}, \\ \nabla \cdot u = \partial_r u^r + \dfrac{u^r}{r} + \partial_z u^z = 0. \end{cases} \tag{1.7}$$

The vorticity w of the axially symmetric velocity u is given by

$$w = \nabla \times u = w^r(t,r,z)e_r + w^\theta(t,r,z)e_\theta + w^z(t,r,z)e_z,$$

where

$$w^r = -\partial_z u^\theta, \quad w^\theta = \partial_z u^r - \partial_r u^z, \quad w^z = \partial_r u^\theta + \frac{u^\theta}{r}.$$

By the first three equations of (1.7), (w^r, w^θ, w^z) satisfies

$$\begin{cases} \partial_t w^r + (u^r\partial_r + u^z\partial_z)w^r = (w^r\partial_r + w^z\partial_z)u^r, \\ \partial_t w^\theta + (u^r\partial_r + u^z\partial_z)w^\theta = \dfrac{u^r}{r} w^\theta + \dfrac{1}{r}\partial_z (u^\theta)^2 - \dfrac{1}{r}\partial_z (h^\theta)^2, \\ \partial_t w^z + (u^r\partial_r + u^z\partial_z)w^z = (w^r\partial_r + w^z\partial_z)u^z. \end{cases} \tag{1.8}$$

Our main theorem in this paper is that for the axially symmetric solution of system (1.1), if the current density is swirl-free, then the $L^1_t L^\infty_x$ boundedness of partial vorticity (corresponding to curl of the swirl part of the velocity) implies the regularity of the solution, which is a Beale–Kato–Majda-type regularity criterion. See [1] for the original BKM criterion for 3D Euler equations.

Theorem 1.1. *For any $0 < T_* < \infty$, let $(u, h) \in C\left([0, T_*); H^m(\mathbb{R}^3)\right)$ $(m \geq 3)$ be the unique solution of (1.1) with the initial data $(u_0, h_0) \in H^m(\mathbb{R}^3)$, being axially symmetric and satisfying $\nabla \cdot u_0 = h_0^r = h_0^z \equiv 0$. Then $(u, h)(t, \cdot)$ keeps in $H^m(\mathbb{R}^3)$ at $t = T_*$ if and only if*

$$\int_0^{T_*} \left\| \nabla \times (u^\theta e_\theta)(t, \cdot) \right\|_{L^\infty} dt \leq C_* < \infty. \tag{1.9}$$

□

In particular, it is evident that (1.9) holds automatically if $u^\theta \equiv 0$, for any $T_* > 0$. Therefore concerning the global well-posedness, we have the following direct corollary. Compared to the paper [14], our system is equipped with an inviscid velocity field.

Corollary 1.2. *Under the same conditions as Theorem 1.1, if we further assume $u_0^\theta \equiv 0$, then the solution (u, h) of (1.7) keeps in $H^m(\mathbb{R}^3)$ globally in time.* □

Remark 1.3. If $h \equiv 0$, system (1.1) is reduced to the Euler equations. The regularity criterion in Theorem 1.1 is new even for the axially symmetric Euler equations to the best of the authors' knowledge. If the spacial domain Ω is away from the axis of symmetry, authors in the pioneer work [6] showed the following criterion for the 3D axisymmetric Euler equations: For any $m \geq 3$, $\limsup_{t \nearrow T_*} \|u(t, \cdot)\|_{H^m(\Omega)} < \infty$ provided

$$\int_0^{T_*} \|\nabla u^\theta(t, \cdot)\|_{L^\infty(\Omega)} dt < \infty.$$

See also [4] for a BKM-type blow-up criterion of 3D axially symmetric Euler equations in terms of one component of the vorticity field. □

There already have been many studies and fruitful results related to the well-posedness and regularity of the Hall-MHD system, so it is impossible to include all the important and interesting results in our introduction. We only present some results that in the authors' interest. Chae et al. [5] established the global existence of weak solutions and the local well-posedness of smooth solutions in Sobolev space $H^s(\mathbb{R}^3)$ with $s > 5/2$. Later on, Benvenutti–Ferreira in [2] proved the local-in-time well-posedness of H^2 strong solutions. Recently, Dai [13] improved the local well-posedness theory in $H^s(\mathbb{R}^n)$ with $s > n/2$. Chae–Wan–Wu [10] proved local well-posedness for the Hall-MHD equations with fractional magnetic diffusion. Chae-Weng [11] showed the non-resistive Hall-MHD system is not globally in time well-posed in any Sobolev space $H^s(\mathbb{R}^3)$ with $s > 7/2$. Some regularity criteria for large data solutions, together with global well-posedness and asymptotic behavior of small data solutions can be found in [8,9,15,25,27,28] and references therein.

Throughout the paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on a, b, $c, \ldots$ which may be different from line to line. Moreover, we denote $C_{0,a,b,c,\ldots}$ if the positive constant also depends on initial data. We also apply $A \lesssim B$ to denote $A \leq CB$. Meanwhile, $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$. L stands for a multi-index such that $L = (l_1, l_2, l_3)$ where $l_1, l_2, l_3 \in \mathbb{N} \cup \{0\}$ and $|L| = l_1 + l_2 + l_3$, $\nabla^L = \partial_{x_1}^{l_1} \partial_{x_2}^{l_2} \partial_{x_3}^{l_3}$. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, L^p denotes the usual Lebesgue space with norm

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^3} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \underset{x \in \mathbb{R}^3}{esssup} |f(x)|, & p = \infty, \end{cases}$$

while $W^{k,p}$ denotes the usual Sobolev space and $\dot{W}^{k,p}$ denotes the usual homogeneous Sobolev space with their norm and semi-norm

$$\|f\|_{W^{k,p}} := \sum_{0\le|L|\le k} \|\nabla^L f\|_{L^p},$$

$$|f|_{\dot{W}^{k,p}} := \sum_{|L|=k} \|\nabla^L f\|_{L^p},$$

respectively. We also simply denote $W^{k,p}$ and $\dot{W}^{k,p}$ by H^k and $\dot{H}^k$ provided $p=2$. For any Banach space X, we say $v: [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ belongs to the Bochner space $L^p(0,T;X)$, if

$$\|v(t,\cdot)\|_X \in L^p(0,T),$$

and we usually use $L^p_T X$ for short notation of $L^p(0,T;X)$.

Let us briefly present our idea of proof. We mainly focus on deriving the $L^1(0,T_*,L^\infty)$ boundedness of w^θ and $L^\infty(0,T_*,L^2)$ boundedness of ∇h. Then by combining Condition 1.9, the BKM-type criterion on u and $L^p_{T^*}L^q$ $(2/p+3/q\le 1)$ criterion on ∇h are satisfied. Then by applying the result in [20,26], we can obtain the validity of Theorem 1.1. Define

$$\Omega := \frac{w^\theta}{r}, \quad J := \frac{w^r}{r}, \quad H := \frac{h^\theta}{r}, \quad b = u^r e_r + u^z e_z.$$

The $L^\infty_{T_*}L^\infty$ boundedness of w^θ and $L^\infty_{T_*}L^2$ boundedness of ∇h will be presented by using the bootstrap argument as the following order:

$$\|H\|_{L^\infty_{T_*}L^\infty} \to \|(\Omega,J)\|_{L^\infty_{T_*}(L^2\cap L^6)} \to \|\nabla b\|_{L^\infty_{T_*}(L^2\cap L^6)} \to \|\partial_z H\|_{L^1_{T_*}H^2} \to \|w^\theta\|_{L^\infty_{T_*}L^\infty} \to \|\nabla h\|_{L^\infty_{T_*}L^2}.$$

The remaining of this paper is organized as follows. In Sect. 2, we provide some useful Lemmas concerning interpolation inequalities, L^p boundedness of the gradient of velocity by the vorticity coming from Biot–Savart law, L^p boundedness of the gradient of u^r/r by w^θ/r, and a Hardy type inequality which indicates the L^∞ boundedness of u^θ/r by w^z. Finally, in Sect. 3, we provide the proof of Theorem 1.1.

2. Preliminaries

At the beginning, let us introduce some useful lemmas which will be frequently used in the proof of the main theorem. First is the well-known *Gagliardo – Nirenberg* interpolation inequality in $\mathbb{R}^3$. We list here without proof.

Lemma 2.1 (Gagliardo–Nirenberg). *Fix $q,r\in[1,\infty]$ and $j,m\in\mathbb{N}\cup\{0\}$ with $j\le m$. Suppose that $f\in L^q\cap\dot{W}^{m,r}$ and there exists a real number $\alpha\in[j/m,1]$ such that*

$$\frac{1}{p} = \frac{j}{3} + \alpha\left(\frac{1}{r}-\frac{m}{3}\right) + \frac{1-\alpha}{q}.$$

Then $f\in\dot{W}^{j,p}$ and there exists a constant $C>0$ such that

$$\|\nabla^j f\|_{L^p} \le C\|\nabla^m f\|^\alpha_{L^r}\|f\|^{1-\alpha}_{L^q},$$

except the following two cases:

(i) $j=0$, $mr<d$ and $q=\infty$; (In this case it is necessary to assume also that either $u\to 0$ at infinity, or $u\in L^s$ for some $s<\infty$.)

(ii) $1<r<\infty$ and $m-j-3/r\in\mathbb{N}$. (In this case it is necessary to assume also that $\alpha<1$.) □

Using the Biot–Savart law and the L^p boundedness of Calderon–Zygmund singular integral operators, we have the following lemma whose detailed proof can be found in [7,12].

Lemma 2.2. *Let $u = u^r e_r + u^\theta e_\theta + u^z e_z$ be an axially symmetric vector field, $w = \nabla \times u = w^r e_r + w^\theta e_\theta + w^z e_z$ and $b = u^r e_r + u^z e_z$. Then we have*

$$\|\nabla u\|_{L^p} \le C_p \|w\|_{L^p}$$

and

$$\|\nabla b\|_{L^p} \le C_p \|w^\theta\|_{L^p}$$

for all $1 < p < \infty$. □

Next we state that the L^p $(1 < p < \infty)$ bound of $\nabla \frac{u^r}{r}$ could be controlled by the L^p bound of $\frac{w^\theta}{r}$, while the L^∞ norm of $\frac{u^\theta}{r}$ could be dominated by L^∞ norm of w^z. Here goes two lemmas:

Lemma 2.3. *Define $\Omega := \frac{w^\theta}{r}$. For $1 < p < +\infty$, there exists an absolute constant $C_p > 0$ such that*

$$\left\| \nabla \frac{u^r}{r}(t, \cdot) \right\|_{L^p} \le C_p \|\Omega(t, \cdot)\|_{L^p}.$$

The proof of this lemma can be founded in many literatures, such as [19] (equation (A.5)) and [23] (Proposition 2.5). □

Lemma 2.4.

$$\left\| \frac{u^\theta}{r}(t, \cdot) \right\|_{L^\infty} \le \frac{1}{2} \|w^z(t, \cdot)\|_{L^\infty} \tag{2.1}$$

for any $t > 0$.

To prove Lemma 2.4, one needs the following one dimensional Hardy inequality, which could be found in [16] (Theorem 330).

Proposition 2.5. *If $p > 1$, $\sigma \ne 1$, f is a nonnegative measurable function and F is defined by*

$$F(x) = \int_0^x f(t)dt, \quad (\sigma > 1), \quad F(x) = \int_x^\infty f(t)dt, \quad (\sigma < 1).$$

Then

$$\int_0^\infty x^{-\sigma} F^p dx < \left(\frac{p}{|\sigma - 1|} \right)^p \int_0^\infty x^{-\sigma} (xf)^p dx,$$

unless $f \equiv 0$. □

Proof of Lemma 2.4. Choosing $\sigma = 2p - 1 > 1$, and $f(r) = r|w^z|$, Proposition 2.5 indicates

$$\int_0^\infty r^{-2p+1} \left(\int_0^r s|w^z(t, s, z)|ds \right)^p dr \le \left(\frac{1}{2} \right)^p \int_0^\infty |w^z(t, r, z)|^p r dr. \tag{2.2}$$

Noting that

$$|ru^\theta(t, r, z)| = \left| \int_0^r s w^z(t, s, z) ds \right| \le \int_0^r s|w^z(t, s, z)|ds,$$

(2.2) implies

$$\int_0^\infty \left| \frac{u^\theta}{r}(t, r, z) \right|^p r dr \le \left(\frac{1}{2} \right)^p \int_0^\infty |w^z(t, r, z)|^p r dr. \tag{2.3}$$

Integrating (2.3) with z on $\mathbb{R}$, we know that

$$\left\| \frac{u^\theta}{r}(t, \cdot) \right\|_{L^p} \le \frac{1}{2} \|w^z(t, \cdot)\|_{L^p}.$$

Therefore (2.1) follows by choosing $p \to \infty$. □

3. Proof of Theorem 1.1

First we give some initial data explanations. Actually from $u_0, h_0 \in H^3(\mathbb{R}^3)$, we can deduce

$$\frac{u_0^r}{r}, \quad \frac{u_0^\theta}{r} \quad \text{and} \quad \frac{h_0^\theta}{r} \in H^2(\mathbb{R}^3).$$

The proof is presented in Appendix. Moreover, by Sobolev imbedding,

$$\frac{u_0^r}{r}, \frac{u_0^\theta}{r}, \frac{h_0^\theta}{r} \in L^p(\mathbb{R}^3), \quad \forall p \in [2, \infty].$$

In the cylindrical coordinates, we note that for the axially symmetric initial vorticity $w_0 = w_0^r(r,z)\boldsymbol{e_r} + w_0^\theta(r,z)\boldsymbol{e_\theta} + w_0^z(r,z)\boldsymbol{e_z}$,

$$|\nabla w_0|^p \simeq_p |\bar{\nabla} w_0^r|^p + |\bar{\nabla} w_0^\theta|^p + |\bar{\nabla} w_0^z|^p + \left|\frac{w_0^r}{r}\right|^p + \left|\frac{w_0^\theta}{r}\right|^p,$$

where $\bar{\nabla} = (\partial_r, \partial_z)$. Since $\nabla w_0 \in H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, we conclude that $\frac{w_0^r}{r}, \frac{w_0^\theta}{r} \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. These embedding results above guarantee the validity of all the initial data in the proof of Theorem 1.1 below.

Since $\nabla \times (u^\theta e_\theta) = w^r e_r + w^z e_z$, Condition (1.9) implies the $L^1(0, T_*, L^\infty)$ boundedness of both w^r and w^z. Therefore our first task in the proof is to derive the $L^1(0, T_*, L^\infty)$ boundedness for the rest component w^θ. This indicates a BKM-type criterion for the total vorticity. In normal sense, we still need a $L_t^p L^q$ $(2/p + 3/q \leq 1)$ criterion for ∇h to derive the regularity for invicid Hall-MHD system (1.1). See, for example, [26, Theorem 1.1]. However, under the framework of axial symmetry and vanishing of $h^r e_r + h^z e_z$, we can conclude that $\nabla h \in L_t^2 L^\infty$ by the energy method with no extra conditions. Thus a direct application of [26, Theorem 1.1] indicates validity of Theorem 1.1.

3.1. Fundamental Estimates

At the beginning, the following Lemma states fundamental estimates of the system (1.7):

Lemma 3.1 *(Fundamental Energy Estimates). Define $H := \frac{h^\theta}{r}$. Let $(u, h) \in H^m$ be the solution of (1.7), then we have*

(i) for $p \in [2, \infty)$ and $t \in (0, \infty)$,

$$\|H(t,\cdot)\|_{L^p}^p + p(p-1)\int_0^t \int_{\mathbb{R}^3} |\nabla H(s,x)|^2 |H(s,x)|^{p-2} dx ds \leq \|H_0\|_{L^p}^p; \tag{3.1}$$

$$\|H(t,\cdot)\|_{L^\infty} \leq \|H_0\|_{L^\infty}; \tag{3.2}$$

(ii) for $u_0, h_0 \in L^2$ and $t \in (0, \infty)$,

$$\|(u, h^\theta)(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla h^\theta(s,\cdot)\|_{L^2}^2 ds + \int_0^t \left\|\frac{h^\theta}{r}(s,\cdot)\right\|_{L^2}^2 ds \leq C_0, \tag{3.3}$$

where C_0 depends only on $\|(u_0, h_0)\|_{L^2}$.

Proof. By $(1.7)_4$, we have H satisfies

$$\partial_t H + (u^r \partial_r + u^z \partial_z) H - \left(\Delta + \frac{2}{r}\partial_r\right) H - 2H\partial_z H = 0. \tag{3.4}$$

Performing the L^p estimate of (3.4)

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|H(t,\cdot)\|_{L^p}^p+(p-1)\int_{\mathbb{R}^3}|H|^{p-2}|\nabla H|^2dx&=\frac{2}{p}\int_{\mathbb{R}^3}\frac{1}{r}\partial_r|H|^pdx+\frac{2}{p+1}\int_{\mathbb{R}^3}\partial_z\big(H|H|^p\big)dx\\&=\frac{4\pi}{p}\int_{\mathbb{R}}\int_0^\infty\partial_r|H|^pdrdz\\&=-\frac{4\pi}{p}\int_{\mathbb{R}}|H(t,0,z)|^pdz\le 0.\end{aligned}$$

Integrating over $(0,t)$, one derives (3.1). (3.2) follows by letting $p\to\infty$. Meanwhile, (3.3) follows from the standard L^2 estimate of the system (1.1). □

3.2. $L^\infty_{T_*}\left(L^2\cap L^6\right)$ Bound of Ω and $L^\infty_t\dot H^1\cap L^2_t\dot H^2$ Estimate of H

Starting with the fundamental estimates (3.1) (3.2) and (3.3) from the previous subsection, our second step is to obtain an a priori bound for $\|\Omega\|_{L^\infty_{T_*}(L^2\cap L^6)}$, together with bound for the magnetic quantity $\|H\|_{L^\infty_{T_*}\dot H^1\cap L^2_{T_*}\dot H^2}$. We will apply the L^p energy estimate on the equations of the couple $\Omega=\frac{w^\theta}{r}$, $J=\frac{w^r}{r}$:

$$\begin{cases}\partial_t\Omega+b\cdot\nabla\Omega=-\partial_zH^2-2\dfrac{u^\theta}{r}J,\\[2mm] \partial_tJ+b\cdot\nabla J=\left(w^r\partial_r+w^z\partial_z\right)\dfrac{u^r}{r},\end{cases}\tag{3.5}$$

which can be derived from $(1.8)_{1,2}$ via direct calculations. First we show that $\|(\Omega,J)\|_{L^\infty_{T_*}(L^2\cap L^6)}$ can be controlled by $\|H\|_{L^\infty_{T_*}\dot H^1\cap L^2_{T_*}\dot H^2}$. On the other hand, an energy estimate of the equation of H

$$\partial_tH+(u^r\partial_r+u^z\partial_z)H-\left(\Delta+\frac{2}{r}\partial_r\right)H-2H\partial_zH=0$$

indicates $\|H\|_{L^\infty_{T_*}\dot H^1\cap L^2_{T_*}\dot H^2}$ can be bounded by $\|\Omega\|_{L^\infty_{T_*}(L^2\cap L^6)}$. Thus we consequently derive a self-closed a priori estimate for the couple of quantities (Ω,H). The detailed result is stated as follows:

Proposition 3.2. *Define $\Omega:=\frac{w^\theta}{r}$ and $H:=\frac{h^\theta}{r}$. Assume that $\nabla\cdot u_0=h_0^r=h_0^z\equiv 0$. Let (u,h), satisfying (1.9), be the unique local axially symmetric solution of (1.1) with the initial data $(u_0,h_0)\in H^m(\mathbb{R}^3)$ $(m\ge 3)$. The following $(L^\infty_{T_*}(L^2\cap L^6))\times(L^\infty_{T_*}\dot H^1\cap L^2_{T_*}\dot H^2)$ estimate of (Ω,H) holds*

$$\sup_{0\le t\le T_*}\left(\|\Omega(t,\cdot)\|^2_{L^2\cap L^6}+\|\nabla H(t,\cdot)\|^2_{L^2}\right)+\int_0^{T_*}\|\nabla^2H(s,\cdot)\|^2_{L^2}ds\le C_{0,T_*}<\infty.$$

Here $C_{0,T_*}>0$ is a constant depends only on the prescribed initial data and $T_*<\infty$.

Proof. We first bound Ω in terms of H. For any $p\in[1,\infty]$, performing the L^p estimates for $(3.5)_1$. Using (3.2), one arrives

$$\begin{aligned}\|\Omega(t,\cdot)\|_{L^p}&\lesssim\|\Omega_0\|_{L^p}+\int_0^t\|\partial_zH^2(s,\cdot)\|_{L^p}ds+\int_0^t\left\|\frac{u^\theta}{r}(s,\cdot)\right\|_{L^\infty}\|J(s,\cdot)\|_{L^p}ds\\&\lesssim\|\Omega_0\|_{L^p}+\|H\|_{L^\infty_TL^\infty}\int_0^t\|\partial_zH(s,\cdot)\|_{L^p}ds+\int_0^t\|w^z(s,\cdot)\|_{L^\infty}\|J(s,\cdot)\|_{L^p}ds\\&\lesssim\|\Omega_0\|_{L^p}+\|H_0\|_{L^\infty}\int_0^t\|\partial_zH(s,\cdot)\|_{L^p}ds+\int_0^t\left\|\nabla\times(u^\theta e_\theta)(s,\cdot)\right\|_{L^\infty}\|J(s,\cdot)\|_{L^p}ds;\end{aligned}\tag{3.6}$$

Here the second inequality of (3.6) follows from Corollary 2.4, and the third inequality is deduced by the identity

$$\nabla\times(u^\theta e_\theta)=w^re_r+w^ze_z.$$

Meanwhile, the same estimate for $(3.5)_2$ indicates

$$\begin{aligned}\|J(t,\cdot)\|_{L^p} &\lesssim \|J_0\|_{L^p} + \int_0^t \|(w^r,w^z)(s,\cdot)\|_{L^\infty}\|\nabla\frac{u^r}{r}(s,\cdot)\|_{L^p}ds\\ &\lesssim \|J_0\|_{L^p} + \int_0^t \|\nabla\times(u^\theta e_\theta)(s,\cdot)\|_{L^\infty}\|\Omega(s,\cdot)\|_{L^p}ds.\end{aligned}\tag{3.7}$$

Here the second line of (3.7) follows from Lemma 2.3. Combining (3.6) and (3.7), and using Gronwall inequality, one derives that

$$\begin{aligned}\|(\Omega,J)(t,\cdot)\|_{L^p} &\lesssim \left(\|(\Omega_0,J_0)\|_{L^p} + \|H_0\|_{L^\infty}\int_0^t \|\partial_z H(s,\cdot)\|_{L^p}ds\right)\exp\left(\int_0^t \|\nabla\times(u_\theta e_\theta)(s,\cdot)\|_{L^\infty}ds\right)\\ &\lesssim \|(\Omega_0,J_0)\|_{L^p} + \|H_0\|_{L^\infty}\int_0^t \|\partial_z H(s,\cdot)\|_{L^p}ds\end{aligned}\tag{3.8}$$

for any $t\le T_*$. Specially, for $p=2$, we find (3.8) together with the fundamental estimate (3.1) indicates

$$\begin{aligned}\|(\Omega,J)(t,\cdot)\|_{L^2}^2 &\lesssim \|(\Omega_0,J_0)\|_{L^2}^2 + t\|H_0\|_{L^\infty}^2\int_0^t \|\partial_z H(s,\cdot)\|_{L^2}^2ds\\ &\lesssim \|(\Omega_0,J_0)\|_{L^2}^2 + T_*\|H_0\|_{L^2}^2\|H_0\|_{L^\infty}^2 < \infty\end{aligned}\tag{3.9}$$

holds for any $t\le T_*$, which is already a self-closed a priori estimate.

Nevertheless, to handle the last term on the far right of (3.8) for $p>2$, one needs to derive a higher order estimate of $H=h^\theta/r$. Applying (∂_r,∂_z) on

$$\partial_t H + b\cdot\nabla H - \left(\Delta + \frac{2}{r}\partial_r\right)H - 2H\partial_z H = 0,$$

one finds

$$\partial_t\partial_r H + b\cdot\nabla\partial_r H + \partial_r b\cdot\nabla H - \left(\Delta+\frac{2}{r}\partial_r\right)\partial_r H + \frac{3\partial_r H}{r^2} - 2\partial_r H\partial_z H - 2H\partial_{rz}^2 H = 0,\tag{3.10}$$

and

$$\partial_t\partial_z H + b\cdot\nabla\partial_z H + \partial_z b\cdot\nabla H - \left(\Delta+\frac{2}{r}\partial_r\right)\partial_z H - 2(\partial_z H)^2 - 2H\partial_z^2 H = 0.\tag{3.11}$$

Taking the L^2 energy estimate for (3.10) and (3.11) respectively, and integrating on $\mathbb{R}^3$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla H(t,)\|_{L^2}^2 + \left\|\nabla^2 H(t,\cdot)\right\|_{L^2}^2 + 3\left\|\frac{\partial_r H}{r}(t,\cdot)\right\|_{L^2}^2 - \underbrace{\int_{\mathbb{R}^3}\frac{2}{r}\partial_r\nabla H\cdot\nabla H dx}_{I_1}\\ &= -\underbrace{\int_{\mathbb{R}^3}\left[\partial_r b\cdot\nabla H\partial_r H + \partial_z b\cdot\nabla H\partial_z H\right]dx}_{I_2}\\ &\quad + \underbrace{2\int_{\mathbb{R}^3}\left[(\partial_r H)^2\partial_z H + H\partial_r H\partial_{rz}^2 H + (\partial_z H)^3 + H\partial_z H\partial_z^2 H\right]dx}_{I_3}.\end{aligned}\tag{3.12}$$

Here $\nabla = e_r\partial_r + e_z\partial_z$ which equals to $(\partial_1,\partial_2,\partial_3)$ for an axially symmetric scalar function. The last term on the left hand side of (3.12) follows that

$$I_1 = 2\pi\int_{\mathbb{R}}\int_0^\infty \frac{1}{r}\partial_r|\nabla H|^2 r dr dz = -2\pi\int_{\mathbb{R}}|\nabla H(t,0,z)|^2 dz.\tag{3.13}$$

Then, after a direct calculation, it shows that

$$I_2 = \underbrace{\int_{\mathbb{R}^3} (\partial_r H)^2 \partial_r u^r dx}_{I_{21}} + \underbrace{\int_{\mathbb{R}^3} (\partial_z H)^2 \partial_z u^z dx}_{I_{22}} + \underbrace{\int_{\mathbb{R}^3} \partial_z H \partial_r H \left(\partial_z u^r + \partial_r u^z\right) dx}_{I_{23}} \tag{3.14}$$

In the following we estimate I_{21}–I_{23} of (3.14) term by term. First we see

$$(\partial_r H)^2 \partial_r u^r = \partial_r H \left(\partial_r h^\theta - \frac{h^\theta}{r}\right) \partial_r \frac{u^r}{r} + (\partial_r H)^2 \frac{u^r}{r}.$$

Since $\|\nabla h(t,\cdot)\|_{L^2} \simeq \|\nabla h^\theta(t,\cdot)\|_{L^2} + \left\|\frac{h^\theta}{r}(t,\cdot)\right\|_{L^2}$, using Hölder inequality, Young inequality, interpolation inequality (Lemma 2.1) and Lemma 2.3, we derive

$$\begin{aligned}
|I_{21}| &\lesssim \|\nabla H(t,\cdot)\|_{L^3} \|\nabla h(t,\cdot)\|_{L^2} \left\|\nabla \frac{u^r}{r}(t,\cdot)\right\|_{L^6} + \|\nabla H(t,\cdot)\|^2_{L^{\frac{12}{5}}} \left\|\frac{u^r}{r}(t,\cdot)\right\|_{L^6} \\
&\lesssim \|\nabla H(t,\cdot)\|^{1/2}_{L^2} \left\|\nabla^2 H(t,\cdot)\right\|^{1/2}_{L^2} \|\nabla h(t,\cdot)\|_{L^2} \left\|\nabla \frac{u^r}{r}(t,\cdot)\right\|_{L^6} \\
&\quad + \|\nabla H(t,\cdot)\|^{\frac{3}{2}}_{L^2} \left\|\nabla^2 H(t,\cdot)\right\|^{\frac{1}{2}}_{L^2} \left\|\nabla \frac{u^r}{r}(t,\cdot)\right\|_{L^2} \\
&\le \frac{1}{4} \left\|\nabla^2 H(t,\cdot)\right\|^2_{L^2} + C\|\nabla H(t,\cdot)\|^2_{L^2} \left(1 + \|\Omega(t,\cdot)\|^{\frac{4}{3}}_{L^2}\right) + \|\nabla h(t,\cdot)\|^2_{L^2} \|\Omega(t,\cdot)\|^2_{L^6}.
\end{aligned} \tag{3.15}$$

Second, using the divergence free property of u, one notes that

$$(\partial_z H)^2 \partial_z u^z = -\partial_z h^\theta \partial_z H \partial_r \frac{u^r}{r} - 2 (\partial_z H)^2 \frac{u^r}{r}.$$

Similarly as the estimate of I_{21} in (3.15), one finds

$$|I_{22}| \le \frac{1}{4} \left\|\nabla^2 H(t,\cdot)\right\|^2_{L^2} + C\|\nabla H(t,\cdot)\|^2_{L^2} \left(1 + \|\Omega(t,\cdot)\|^{\frac{4}{3}}_{L^2}\right) + \|\nabla h(t,\cdot)\|^2_{L^2} \|\Omega(t,\cdot)\|^2_{L^6}. \tag{3.16}$$

Third, direct calculation shows

$$\partial_z H \partial_r H (\partial_z u^r + \partial_r u^z) = \partial_z h^\theta \partial_r H \left(2\partial_z \frac{u^r}{r} - \frac{w^\theta}{r}\right).$$

It indicates that

$$\begin{aligned}
|I_{23}| &\lesssim \|\partial_z h^\theta(t,\cdot)\|_{L^2} \|\partial_r H(t,\cdot)\|_{L^3} \left(\left\|\partial_z \frac{u^r}{r}(t,\cdot)\right\|_{L^6} + \|\Omega(t,\cdot)\|_{L^6}\right) \\
&\lesssim \|\Omega(t,\cdot)\|_{L^6} \|\nabla h^\theta(t,\cdot)\|_{L^2} \|\nabla H(t,\cdot)\|^{1/2}_{L^2} \|\nabla^2 H(t,\cdot)\|^{1/2}_{L^2} \\
&\le \frac{1}{4} \|\nabla^2 H(t,\cdot)\|^2_{L^2} + C\|\nabla H(t,\cdot)\|^2_{L^2} + \|\Omega(t,\cdot)\|^2_{L^6} \|\nabla h(t,\cdot)\|^2_{L^2}.
\end{aligned} \tag{3.17}$$

Finally, we focus on the last line of (3.12). We denote

$$I_3 = \underbrace{\int_{\mathbb{R}^3} (\partial_r H)^2 \partial_z H dx}_{I_{31}} + \underbrace{\int_{\mathbb{R}^3} H \partial_r H \partial^2_{rz} H dx}_{I_{32}} + \underbrace{\int_{\mathbb{R}^3} (\partial_z H)^3 dx}_{I_{33}} + \underbrace{\int_{\mathbb{R}^3} H \partial_z H \partial^2_z H dx}_{I_{34}}.$$

Using Young inequality and the fundamental estimate (3.2), we find

$$|I_{32}| + |I_{34}| \le \frac{1}{8} \|\nabla^2 H(t,\cdot)\|^2_{L^2} + C\|H_0\|^2_{L^\infty} \|\nabla H(t,\cdot)\|^2_{L^2}. \tag{3.18}$$

Meanwhile, applying integration by parts, $|I_{31}|+|I_{33}|$ enjoys the same estimate as (3.18). Thus we substitute (3.13)–(3.18) in (3.12) to derive

$$\begin{aligned}&\frac{d}{dt}\|\nabla H(t,)\|_{L^2}^2+\left\|\nabla^2 H(t,\cdot)\right\|_{L^2}^2\\ \lesssim&\|\nabla H(t,\cdot)\|_{L^2}^2\left(1+\|H_0\|_{L^2}^2+\|\Omega(t,\cdot)\|_{L^2}^{\frac{4}{3}}\right)+\|\nabla h(t,\cdot)\|_{L^2}^2\|\Omega(t,\cdot)\|_{L^6}^2.\end{aligned}\tag{3.19}$$

Now we are ready to finish the proof of the proposition. A combination of (3.19) and (3.8) shows that

$$\begin{aligned}&\frac{d}{dt}\|\nabla H(t,)\|_{L^2}^2+\|\nabla^2 H(t,\cdot)\|_{L^2}^2\\ \leq&C_0\|\nabla H(t,\cdot)\|_{L^2}^2+C_{0,C_*,T_*}\|\nabla h(t,\cdot)\|_{L^2}^2\left(1+\int_0^t\|\partial_z H(s,\cdot)\|_{L^6}^2 ds\right)\\ \lesssim&\left(1+\|\nabla h(t,\cdot)\|_{L^2}^2\right)\left(1+\|\nabla H(t,\cdot)\|_{L^2}^2+\int_0^t\|\nabla^2 H(s,\cdot)\|_{L^2}^2 ds\right),\end{aligned}\tag{3.20}$$

for any $t\leq T_*$. Here

$$\|\nabla h(t,\cdot)\|_{L^2}^2\simeq\|\nabla h^\theta(t,\cdot)\|_{L^2}^2+\left\|\frac{h^\theta}{r}(t,\cdot)\right\|_{L^2}^2.$$

By denoting

$$\mathcal{G}(t):=1+\|\nabla H(t,\cdot)\|_{L^2}^2+\int_0^t\|\nabla^2 H(s,\cdot)\|_{L^2}^2 ds,$$

then (3.20) follows that

$$\mathcal{G}'(t)\leq C_{0,C_*,T_*}\left(1+\|\nabla h(t,\cdot)\|_{L^2}^2\right)\mathcal{G}(t).$$

Gronwall inequality and the fundamental energy estimate (3.3) indicate that

$$\mathcal{G}(t)\leq C_0\exp\left(C_{0,C_*,T_*}\int_0^t\|\nabla h(s,\cdot)\|_{L^2}^2 ds+C_{0,C_*,T_*}t\right)\leq C_{0,C_*,T_*}.\tag{3.21}$$

Therefore, (3.21) together with (3.9) and (3.8) for $p=6$, implies

$$\|\Omega(t,\cdot)\|_{L^2\cap L^6}^2+\|\nabla H(t,\cdot)\|_{L^2}^2+\int_0^t\|\nabla^2 H(s,\cdot)\|_{L^2}^2 ds\leq C_{0,C_*,T_*}<\infty,$$

for any $t\leq T_*$. This completes the proof of Proposition 3.2. □

3.3. $L^\infty_{T_*}\left(L^2\cap L^6\right)$ Boundedness of ∇b

Starting from estimates of (Ω,H) in the previous subsection, our next step focuses on $L^\infty_{T_*}L^p$ $(2\leq p\leq 6)$ norm of the gradient of the velocity field $b=u^r e_r+u^z e_z$. Noting that ∇b is given by the following 3×3 matrix

$$\nabla b=\begin{pmatrix}\partial_r u^r & 0 & \partial_z u^r\\ 0 & \frac{u^r}{r} & 0\\ \partial_r u^z & 0 & \partial_z u^z\end{pmatrix}.$$

Using Biot–Savart law (Lemma 2.2), it is sufficient to provide the same estimate for w^θ. Our proof can be divided into following steps:

$$\left\|\frac{u^r}{r}\right\|_{L^\infty_{T_*}L^\infty}\to\|h^\theta\|_{L^\infty_{T_*}(L^2\cap L^\infty)}\to\left\|\frac{u^\theta}{r}\right\|_{L^\infty_{T_*}(L^2\cap L^\infty)}\to\|w^\theta\|_{L^\infty_{T_*}(L^2\cap L^6)}.$$

Here goes the result:

Proposition 3.3. *Assume that $\nabla\cdot u_0 = h_0^r = h_0^z \equiv 0$. Let (u,h), satisfying (1.9), be the unique local axially symmetric solution of (1.1) with the initial data $(u_0,h_0)\in H^m(\mathbb{R}^3)$ $(m\geq 3)$. The following $L_t^\infty L^p$ estimate of ∇b holds*

$$\|\nabla b(t,\cdot)\|_{L^p} < C_{0,C_*,T_*}, \quad \textit{for} \quad 2\leq p\leq 6, \quad \forall t\leq T_*.$$

Here $C_{0,C_*,T_*}>0$ is a constant depending only on the prescribed initial data and C_*, $T_*<\infty$ in Condition (1.9).

Proof. We first pay attention to the following estimates of u^r/r, h^θ and u^θ/r, respectively. □

$L^\infty_{T_*}L^\infty$ Estimate of $\frac{u^r}{r}$

Using interpolation (Lemma 2.1) and Lemma 2.3, $\frac{u^r}{r}$ satisfies

$$\begin{aligned}\left\|\frac{u^r}{r}(t,\cdot)\right\|_{L^\infty} &\lesssim \left\|\frac{u^r}{r}(t,\cdot)\right\|_{L^6}^{1/2}\left\|\nabla\frac{u^r}{r}(t,\cdot)\right\|_{L^6}^{1/2} \lesssim \left\|\nabla\frac{u^r}{r}(t,\cdot)\right\|_{L^2}^{1/2}\left\|\nabla\frac{u^r}{r}(t,\cdot)\right\|_{L^6}^{1/2} \\ &\lesssim \|\Omega(t,\cdot)\|_{L^2}^{1/2}\|\Omega(t,\cdot)\|_{L^6}^{1/2}\lesssim C_{0,C_*,T_*}\end{aligned} \tag{3.22}$$

for any $t\leq T_*$. □

$L^\infty_{T_*}L^p$ Estimate of h^θ for $2\leq p\leq\infty$

For any $p\geq 1$, multiplying $h^\theta|h^\theta|^{p-2}$ on $(1.7)_4$, one derives

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|h^\theta(t,\cdot)\|_{L^p}^p &\leq \left\|\frac{u^r}{r}(t,\cdot)\right\|_{L^\infty}\|h^\theta(t,\cdot)\|_{L^p}^p - \int_{\mathbb{R}^3}\frac{|h^\theta|^p}{r^2}dx - (p-1)\int_{\mathbb{R}^3}|\nabla h^\theta|^2|h^\theta|^{p-2}dx \\ &\quad + \int_{\mathbb{R}^3}\frac{1}{r}\partial_z(h^\theta)^2h^\theta|h^\theta|^{p-2}dx \\ &\leq \left\|\frac{u^r}{r}(t,\cdot)\right\|_{L^\infty}\|h^\theta(t,\cdot)\|_{L^p}^p.\end{aligned}$$

Here, we have applied the identity

$$\int_{\mathbb{R}^3}\frac{1}{r}\partial_z(h^\theta)^2h^\theta|h^\theta|^{p-2}dx = \frac{2}{p+1}\int_{\mathbb{R}^3}\partial_z\left(\frac{1}{r}h^\theta|h^\theta|^p\right)dx = 0.$$

Canceling $\|h^\theta(t,\cdot)\|_{L^p}^{p-1}$ on each sides and using (3.22) and Gronwall inequality, one finds

$$\|h^\theta(t,\cdot)\|_{L^p}\leq C_0\exp\left(\int_0^t\left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}ds\right) < C_{0,C_*,T_*}, \quad \text{uniformly for} \quad p\in[2,\infty), \tag{3.23}$$

and the $L^\infty_{T_*}L^\infty$ estimate of h^θ is achieved by choosing $p\to\infty$ in (3.23) since the far right above is independent of p. □

$L^\infty_{T_*}L^p$ Estimate of $\frac{u^\theta}{r}$ for $2\leq p\leq\infty$

Due to $(1.7)_2$, $\frac{u^\theta}{r}$ satisfies

$$\partial_t\frac{u^\theta}{r} + (b\cdot\nabla)\frac{u^\theta}{r} + 2\frac{u^r}{r}\cdot\frac{u^\theta}{r} = 0.$$

Performing the L^p estimate and using Gronwall inequality, it follows that

$$\begin{aligned}\left\|\frac{u^\theta}{r}(t,\cdot)\right\|_{L^p} &\le \left\|\frac{u_0^\theta}{r}\right\|_{L^p}\exp\left(2\int_0^t\left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}ds\right)\\ &\le C_{0,C_*,T_*},\quad \text{for any}\quad p\in[2,\infty],\quad t\le T_*.\end{aligned}$$

□ Finally, we perform the L^p estimate of equation

$$\partial_t w^\theta + (u^r\partial_r + u^z\partial_z)w^\theta = \frac{u^r}{r}w^\theta - \frac{2}{r}u^\theta w^r - \partial_z H h^\theta$$

with $2\le p\le 6$ to derive

$$\|w^\theta(t,\cdot)\|_{L^p} \lesssim \underbrace{\|w_0^\theta\|_{L^p} + \|w^r\|_{L_t^1L^\infty}\left\|\frac{u^\theta}{r}\right\|_{L_t^\infty L^p} + \|\partial_z H\|_{L_t^1L^p}\|h^\theta\|_{L_t^\infty L^\infty}}_{G_1} + \int_0^t\|w^\theta(s,\cdot)\|_{L^p}\left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}ds. \tag{3.24}$$

Based on the aforementioned estimates of u^r/r, h^θ and u^θ/r, the estimate of ∇H in Proposition 3.2, together with the assumption of initial data in Theorem 1.1, noting that Condition (1.9) leading to

$$\|w^r\|_{L_t^1L^\infty}\le\|\nabla\times(u^\theta e_\theta)\|_{L_t^1L^\infty}\le C_*<\infty,\quad \forall t\le T_*,$$

we find

$$G_1\le C_{0,C_*,T_*}<\infty.$$

Thus (3.24) and Gronwall inequality indicate that

$$\|w^\theta(t,\cdot)\|_{L^p}\le C_{0,C_*,T_*}\exp\left(\int_0^t\left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}ds\right)<C_{0,C_*,T_*},\quad \forall p\in[2,6],$$

for any $t\le T_*$. Furthermore, by Lemma 2.2, we finishes the $L^\infty_{T_*}L^p$ estimate of ∇b, i.e.

$$\|\nabla b(t,\cdot)\|_{L^p}\le C_{0,C_*,T_*}<\infty,\quad 2\le p\le 6,\quad \forall t\le T_*.$$

□

3.4. $L^\infty_{T_*}\dot H^1\cap L^2_{T_*}\dot H^2$ Boundedness of $\partial_z H$

Let us recall the equation of w^θ:

$$\partial_t w^\theta + (u^r\partial_r + u^z\partial_z)w^\theta = \frac{u^r}{r}w^\theta + \frac{1}{r}\partial_z(u^\theta)^2 - \frac{1}{r}\partial_z(h^\theta)^2.$$

One observes that, aiming at obtaining the $L^\infty_{T_*}L^\infty$ estimate of w^θ, one still needs to show the $L^1_{T_*}L^\infty$ boundedness of $\partial_z H$, since the last term above satisfies

$$\frac{1}{r}\partial_z(h^\theta)^2 = 2h^\theta\partial_z H,$$

and the $L^\infty_{T_*}L^\infty$ bound of h^θ is already given in (3.23). Thus we derive the following proposition.

Proposition 3.4. *Assume that $\nabla\cdot u_0 = h_0^r = h_0^z \equiv 0$. Let (u,h), satisfying (1.9), be the unique local axially symmetric solution of (1.1) with the initial data $(u_0,h_0)\in H^m(\mathbb{R}^3)$ $(m\ge 3)$, the following estimate of $\partial_z H$ holds*

$$\|\nabla\partial_z H(t,)\|_{L^2}^2 + \int_0^t\left\|\nabla^2\partial_z H(s,\cdot)\right\|_{L^2}^2 ds < C_{0,C_*,T_*},\quad \forall t\le T_*.$$

Here $C_{0,C_*,T_*}>0$ is a constant depending only on the prescribed initial data and C_*, $T_*<\infty$ in Condition 1.9.

Proof. Applying ∂_z on (3.10), (3.11) and performing the L^2 energy estimates for the resulting equations with $\partial^2_{rz}H$ and $\partial^2_z H$ respectively. Adding them together, we conclude that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\partial_z H(t,)\|^2_{L^2} &+ \left\|\nabla^2\partial_z H(t,\cdot)\right\|^2_{L^2} + 2\pi\int_{\mathbb{R}}|\nabla\partial_z H(t,0,z)|^2 dz + 3\left\|\frac{\partial^2_{rz}H}{r}(t,\cdot)\right\|^2_{L^2}\\
&= -\underbrace{\int_{\mathbb{R}^3}\left[\partial_z b\cdot\nabla\partial_r H\partial^2_{rz}H + \partial_z b\cdot\nabla\partial_z H\partial^2_z H\right]dx}_{J_1}\\
&\quad -\underbrace{\int_{\mathbb{R}^3}\left[\partial_z(\partial_r b\cdot\nabla H)\partial^2_{rz}H + \partial_z(\partial_z b\cdot\nabla H)\partial^2_z H\right]dx}_{J_2}\\
&\quad +2\underbrace{\int_{\mathbb{R}^3}\left[H\left(\left(\partial^2_{rz}H\right)^2 + \left(\partial^2_z H\right)\right) + \partial^2_{rz}H\partial_r H\partial_z H + \partial^2_z H\left(\partial_z H\right)^2\right]dx}_{J_3}.\end{aligned} \tag{3.25}$$

Using Proposition 3.3, Hölder inequality, Sobolev embedding, and Young inequality, J_1 satisfies

$$\begin{aligned}|J_1| &\lesssim \int_{\mathbb{R}^3}|\nabla b||\nabla^2 H||\nabla\partial_z H|dx\\
&\lesssim \|\nabla b(t,\cdot)\|_{L^3}\|\nabla^2 H(t,\cdot)\|_{L^2}\|\nabla\partial_z H(t,\cdot)\|_{L^6}\\
&\le C\|\nabla b(t,\cdot)\|_{L^2}\|\nabla b(t,\cdot)\|_{L^6}\|\nabla^2 H(t,\cdot)\|^2_{L^2} + \frac{1}{4}\|\nabla^2\partial_z H(t,\cdot)\|^2_{L^2}.\end{aligned}$$

Similarly, it follows that for J_2

$$\begin{aligned}|J_2| &= \left|\int_{\mathbb{R}^3}\left[(\partial_r b\cdot\nabla H)\partial^3_{rzz}H + (\partial_z b\cdot\nabla H)\partial^3_z H\right]dx\right|\\
&\lesssim \|\nabla b(t,\cdot)\|_{L^3}\|\nabla H(t,\cdot)\|_{L^6}\|\nabla^2\partial_z H(t,\cdot)\|_{L^2}\\
&\le C\|\nabla b(t,\cdot)\|_{L^2}\|\nabla b(t,\cdot)\|_{L^6}\|\nabla^2 H(t,\cdot)\|^2_{L^2} + \frac{1}{4}\|\nabla^2\partial_z H(t,\cdot)\|^2_{L^2}.\end{aligned}$$

Now we are ready for the estimate related to Hall-effect term. Using integration by parts, it follows that

$$\begin{aligned}|J_3| &= \left|\int_{\mathbb{R}^3}\left[H\left(\left(\partial^2_{rz}H\right)^2 + \left(\partial^2_z H\right)\right) - H\partial^3_{rzz}H\partial_r H - H\partial^3_z H\partial_z H\right]dx\right|\\
&\le \|H_0\|_{L^\infty}\|\nabla\partial_z H(t,\cdot)\|^2_{L^2} + \|H_0\|_{L^\infty}\|\nabla^2\partial_z H(t,\cdot)\|_{L^2}\|\nabla H(t,\cdot)\|_{L^2}\\
&\le \|H_0\|_{L^\infty}\|\nabla^2\partial_z H(t,\cdot)\|_{L^2}\|\partial_z H(t,\cdot)\ |_{L^2} + \|H_0\|_{L^\infty}\|\nabla^2\partial_z H(t,\cdot)\|_{L^2}\|\nabla H(t,\cdot)\|_{L^2}\\
&\le \frac{1}{4}\|\nabla^2\partial_z H(t,\cdot)\|^2_{L^2} + C_{0,C_*,T_*}.\end{aligned}$$

Here in the third line above, we have applied the interpolation

$$\|\nabla f\|_{L^2} \le C\|f\|^{1/2}_{L^2}\|\nabla^2 f\|^{1/2}_{L^2}$$

and the boundedness of $\|\nabla H(t,\cdot)\|_{L^2}$, which follows from Proposition 3.2. Therefore combining the above estimates, (3.25) indicates

$$\begin{aligned}&\|\nabla\partial_z H(t,)\|^2_{L^2} + \int_0^t\left\|\nabla^2\partial_z H(s,\cdot)\right\|^2_{L^2}ds\\
&\lesssim \|\nabla\partial_z H_0\|^2_{L^2} + \|\nabla b\|_{L^\infty(0,T_*,\,L^2)}\|\nabla b\|_{L^\infty(0,T_*,\,L^6)}\int_0^t\left\|\nabla^2 H(s,\cdot)\right\|^2_{L^2}ds + C_{0,C_*,T_*}\\
&\le C_{0,C_*,T_*},\quad \text{for all}\quad t\le T_*.\end{aligned}$$

□

Following the result from Proposition 3.4, using interpolation (Lemma 2.1), we arrive that

$$\|\partial_z H(t,\cdot)\|_{L^1(0,T_*,\,L^\infty)} \lesssim \int_0^{T_*} \|\nabla\partial_z H(t,\cdot)\|_{L^2}^{1/4}\|\nabla^2\partial_z H(t,\cdot)\|_{L^2}^{3/4}dt \le C_{0,C_*,T_*}. \tag{3.26}$$

3.5. $L^\infty_{T_*}L^\infty$ Boundedness of w^θ

Consequently, we can arrive the $L^1_{T_*}L^\infty$ estimate of w^θ. Taking the L^∞ estimate of $(1.8)_2$, applying (1.9) (3.23) and (3.26), one finds

$$\begin{aligned}
\|w^\theta(t,\cdot)\|_{L^\infty} \lesssim & \|w_0^\theta\|_{L^\infty} + \int_0^t \left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}\|w^\theta(s,\cdot)\|_{L^\infty}ds \\
&+ \|w^r\|_{L^1(0,T_*,\,L^\infty)}\left\|\frac{u^\theta}{r}\right\|_{L^\infty(0,T_*,\,L^\infty)} + \|\partial_z H\|_{L^1(0,T_*,\,L^\infty)}\|h^\theta\|_{L^\infty(0,T_*,\,L^\infty)} \\
\le & C_{0,C_*,T_*} + \int_0^t \left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}\|w^\theta(s,\cdot)\|_{L^\infty}ds.
\end{aligned}$$

Gronwall inequality indicates that

$$\|w^\theta(t,\cdot)\|_{L^\infty} \le C_{0,C_*,T_*}\exp\left(\int_0^t \left\|\frac{u^r}{r}(s,\cdot)\right\|_{L^\infty}ds\right) < C_{0,C_*,T_*}, \quad \text{for all} \quad t \le T_*. \tag{3.27}$$

3.6. $L^\infty_{T_*}L^2 \cap L^2_{T_*}\dot{H}^1$ Boundedness of ∇h

Estimate (3.27) at the end of Sect. 3.5, together with the assumption (1.9) in our main Theorem, indicate the full Beale–Kato–Majda criterion

$$\int_0^{T_*} \|\nabla\times u(t,\cdot)\|_{L^\infty}dt < \infty. \tag{3.28}$$

For the general 3D inviscid and resistive Hall-MHD system, one needs another criterion of the magnetic field to guarantee the regularity up to T_*, say

$$\int_0^{T_*} \|\nabla h(t,\cdot)\|_{L^p}^q dt < \infty, \quad \text{where } \frac{3}{p}+\frac{2}{q} \le 1, \quad 3 < p \le \infty. \tag{3.29}$$

The detailed proof could be found in [26]. However, due to the special structure of axisymmetric velocity and magnetic fields, together with the vanishing of $h^r e_r + h^z e_z$, other than proposing an extra condition (3.29), one can derive an estimate of ∇h in (3.29)'s type. Here goes the result:

Proposition 3.5. *Assume that $\nabla\cdot u_0 = h_0^r = h_0^z \equiv 0$. Let (u,h), satisfying (1.9), be the unique local axially symmetric solution of (1.1) with the initial data $(u_0,h_0)\in H^m(\mathbb{R}^3)$ $(m\ge 3)$. Then the gradient of magnetic field h enjoys the following estimate for $T_* < \infty$:*

$$\int_0^{T_*} \|\nabla h(t,\cdot)\|_{L^\infty}^2 dt < \infty.$$

Proof. First we perform L^2 inner product of $(1.1)_2$ with Δh and integrate by parts to obtain

$$\begin{aligned}
&\frac{d}{dt}\|\nabla h(t,\cdot)\|_{L^2}^2 + \|\nabla^2 h(t,\cdot)\|_{L^2}^2 \\
\le & \underbrace{\int_{\mathbb{R}^3}\left|\left(\frac{h^\theta u^r}{r} - (u^r\partial_r + u^z\partial_z)h^\theta\right)e_\theta\right||\Delta h|dx}_{K_1} + \underbrace{\int_{\mathbb{R}^3}\left|\frac{\partial_z(h^\theta)^2}{r}e_\theta\right||\Delta h|dx}_{K_2}.
\end{aligned} \tag{3.30}$$

Because of the previous L^2 bound of b (3.3) and ∇b (Prop 3.3), Cauchy–Schwarz inequality and interpolation tell us

$$
\begin{aligned}
K_1 &\leq \left(\left\|\frac{h^\theta u^r}{r}(t,\cdot)\right\|_{L^2} + \|b\cdot\nabla h^\theta(t,\cdot)\|_{L^2}\right)\|\Delta h(t,\cdot)\|_{L^2}\\
&\lesssim \|H_0\|_{L^\infty}\|b(t,\cdot)\|_{L^2}\|\nabla^2 h(t,\cdot)\|_{L^2} + \|\nabla b(t,\cdot)\|_{L^2}\|\nabla h(t,\cdot)\|_{L^2}^{1/2}\|\nabla^2 h(t,\cdot)\|_{L^2}^{3/2}\\
&\leq \frac{1}{4}\|\nabla^2 h(t,\cdot)\|_{L^2}^2 + C_{0,C_*,T_*}\left(1+\|\nabla h(t,\cdot)\|_{L^2}^2\right).
\end{aligned}
\tag{3.31}
$$

Meanwhile, the term T_2 can be estimates exactly as T_1, which follows that

$$
\begin{aligned}
K_2 &\leq 2\|H_0\|_{L^\infty}\|\nabla h(t,\cdot)\|_{L^2}\|\Delta h(t,\cdot)\|_{L^2}\\
&\leq \frac{1}{4}\|\nabla^2 h(t,\cdot)\|_{L^2}^2 + C_{0,C_*,T_*}\|\nabla h(t,\cdot)\|_{L^2}^2.
\end{aligned}
\tag{3.32}
$$

After substituting (3.31) and (3.32) into (3.30) and using Gronwall inequality, one arrives at

$$
\sup_{0\leq t\leq T_*}\|\nabla h(t,\cdot)\|_{L^2}^2 + \int_0^{T_*}\|\nabla^2 h(t,\cdot)\|_{L^2}^2 dt < C_{0,C_*,T_*}. \tag{3.33}
$$

□

Based on this $L_t^\infty L^2 \cap L_t^2 \dot{H}^1$ bound of ∇h, now we are in a position to derive a higher order regularity of h by the maximal regularity of heat flow:

Lemma 3.6 [21, Theorem 7.3]. *Let us define the operator $\mathcal{A}$ by the formula*

$$
\mathcal{A}:\quad f \longmapsto \int_0^t \nabla^2 e^{(t-s)\Delta} f(s,\cdot)ds.
$$

Then $\mathcal{A}$ is bounded from $L^q\left(0,T;L^p(\mathbb{R}^d)\right)$ to itself for every $T\in(0,\infty]$ and $1<p,q<\infty$. Moreover, there holds:

$$
\|\mathcal{A}f\|_{L^q(0,T;L^p)} \leq C\|f\|_{L^q(0,T;L^p)}.
$$

□

By direct computation from $(1.1)_2$ and $(1.7)_4$,

$$
\partial_t\nabla h - \Delta\nabla h = \nabla(f(r,z)e_\theta),
$$

where

$$
f(t,r,z) = \frac{h^\theta u^r}{r} - (u^r\partial_r + u^z\partial_z)h^\theta + \frac{\partial_z(h^\theta)^2}{r}.
$$

Using tensor notations under the cylindric system, one obtains

$$
\nabla(f(r,z)e_\theta) = \partial_r f e_\theta\otimes e_r - \frac{f}{r}e_r\otimes e_\theta + \partial_z f e_\theta \otimes e_z.
$$

For the first term on the right hand side, direct calculation shows

$$
\partial_r f = \underbrace{\partial_r h^\theta\frac{u^r}{r} - \partial_r b\cdot\nabla h^\theta}_{|\nabla b||\nabla h|} + \underbrace{H\partial_r u^r - H\frac{u^r}{r}}_{|H||\nabla b|} - \underbrace{b\cdot\nabla\partial_r h^\theta}_{|b||\nabla^2 h|} + 2\underbrace{\partial_{rz}^2 h^\theta H}_{|H||\nabla^2 h|} + 2\underbrace{\partial_z H\partial_r h^\theta}_{|\partial_z H||\nabla h|} - 2\underbrace{H\partial_z H}_{|H||\partial_z H|},
$$

which indicates

$$
|\partial_r f| \simeq |\nabla b||\nabla h| + |H||\nabla b| + |b||\nabla^2 h| + |H||\nabla^2 h| + |\partial_z H||\nabla h| + |H||\partial_z H|.
$$

Previous results, including Lemma 3.1, Proposition 3.2, Proposition 3.3, Proposition 3.4, together with (3.33), indicate that

$$
\begin{aligned}
&H \in L^\infty(0,T_*;L^2\cap L^\infty), \quad \nabla H, \nabla\partial_z H \in L^\infty(0,T_*;L^2),\\
&\nabla b \in L^\infty(0,T_*;L^2\cap L^6), \quad \nabla h \in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;\dot{H}^1).
\end{aligned}
$$

Using interpolation and Hölder inequality, it is clear that

$$\partial_r f \in L^2(0, T_*; L^2). \tag{3.34}$$

Moreover, by a similar and simpler argument, we also have

$$\frac{f}{r},\ \partial_z f \in L^2(0, T_*; L^2). \tag{3.35}$$

Thus, (3.34) and (3.35) infer that

$$\nabla(f(t,z)e_\theta) \in L^2(0, T_*; L^2).$$

Now the maximal regularity of the heat flow (Lemma 3.6) infers that

$$\nabla^3 h \in L^2(0, T_*; L^2),$$

which indicates

$$\int_0^{T_*} \|\nabla h(t,\cdot)\|_{L^\infty}^2 dt < \infty, \tag{3.36}$$

by the Sobolev imbedding. This completes the proof.

3.7. End Proof of Theorem 1.1

Based on estimates (3.28) and (3.36), we can actually derive the solution can be smoothly extended to T_* by applying strategies in [20] and [26]. Actually Theorem 1.1 of [26] can be applied directly just by remembering $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$.

Acknowledgements. Z. Li is supported by Natural Science Foundation of Jiangsu Province (No. BK20200803), National Natural Science Foundation of China (No. 12001285) and the Startup Foundation for Introducing Talent of NUIST (No. 2019r033). X. Pan is supported by Natural Science Foundation of Jiangsu Province (No. BK20180414) and National Natural Science Foundation of China (Nos. 11801268, 12031006).

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix: Initial Data Regularity Analysis

Here are some derivations about the regularity of initial data. We have the following Lemma.

Lemma A.1. *Suppose a vector function $v(x) = v^r(r,z)e_r + v^\theta(r,z)e_\theta + v^z(r,z)e_z$ is axially symmetric and belongs to $H^3(\mathbb{R}^3)$, then we have*

$$\frac{v^r}{r},\ \frac{v^\theta}{r} \in H^2(\mathbb{R}^3).$$

Proof. Starting from $v \in H^3(\mathbb{R}^3)$, one can directly derive that

$$\omega := \nabla \times v = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z \in H^2(\mathbb{R}^3),$$
$$\tilde{\omega} := \nabla \times \omega = \tilde{\omega}^r e_r + \tilde{\omega}^\theta e_\theta + \tilde{\omega}^z e_z \in H^1(\mathbb{R}^3).$$

Noting that for an axially symmetric vector function $f = f^r(r,z)e_r + f^\theta(r,z)e_\theta + f^z(r,z)e_z$ and $0 < p < +\infty$

$$|\nabla f|^p \simeq_p |\nabla f^r|^p + |\nabla f^\theta|^p + |\nabla f^z|^p + \left|\frac{f^r}{r}\right|^p + \left|\frac{f^\theta}{r}\right|^p. \tag{A.1}$$

First from ∇v and $\nabla\tilde{\omega}$ belongs to $L^2(\mathbb{R}^3)$, we have

$$\frac{v^r}{r},\ \frac{v^\theta}{r} \in L^2(\mathbb{R}^3) \text{ and } \frac{\tilde{\omega}^r}{r},\ \frac{\tilde{\omega}^\theta}{r} \in L^2(\mathbb{R}^3). \tag{A.2}$$

Meanwhile, direct calculation shows

$$-\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\frac{v^r}{r} = \frac{\tilde{\omega}^r}{r}, \quad -\left(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\right)\frac{v^\theta}{r} = \frac{\tilde{\omega}^\theta}{r}.$$

Following the idea in [17,19] (see also [22], Sect. 3), we treat the operator $(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)$ as a 5D Laplacian Δ_5, by formally writing $\tilde{x} = (x_1, x_2, x_3, x_4, z)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. Since r^{-2} is a legal A_p weight for Riesz operator in $\mathbb{R}^5$, (See [24, Chapter V] for more details about A_p weight.) one has

$$\begin{aligned}\int_{\mathbb{R}^3}\left|\nabla^2\frac{v^r}{r}\right|^2 dx =& 2\pi\int_{-\infty}^{\infty}\int_0^{\infty}\left|\nabla^2\frac{v^r}{r}\right|^2 r^{-2}r^3 dr dz \simeq \int_{\mathbb{R}^5}\left|\nabla^2(-\Delta_5)^{-1}\frac{\tilde{\omega}^r}{r}\right|^2 r^{-2}d\tilde{x}\\ &\lesssim \int_{\mathbb{R}^5}\left|\frac{\tilde{\omega}^r}{r}\right|^2 r^{-2}d\tilde{x} \simeq \int_{\mathbb{R}^3}\left|\frac{\tilde{\omega}^r}{r}\right|^2 dx.\end{aligned}$$

Together with (A.2), we find $\frac{v^r}{r}, \frac{v^\theta}{r} \in H^2(\mathbb{R}^3)$, which proves Lemma A.1. □

References

[1] Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-d Euler equations. Comm. Math. Phys. **94**, 61–66 (1984)

[2] Benvenutti, M.J., Ferreira, L.C.F.: Existence and stability of global large strong solutions for the Hall-MHD system. Differ. Integr. Equ. **29**(9–10), 977–1000 (2016)

[3] Bittencourt, J.A.: Fundamentals of Plasma Physics, 3rd edn. Springer, New York (2004)

[4] Chae, D.: Remarks on the blow-up criterion of the three-dimensional Euler equations. Nonlinearity **18**, 1021–1029 (2005)

[5] Chae, D., Degond, P., Liu, J.: Well-posedness for Hall-magnetohydrodynamics. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(3), 555–565 (2014)

[6] Chae, D., Imanuvilov, O.: Generic solvability of the axisymmetric 3-D Euler equations and the 2-D Boussinesq equations. J. Differ. Equ. **156**, 1–17 (1999)

[7] Chae, D., Lee, J.: On the regularity of the axisymmetric solutions of the Navier–Stokes equations. Math. Z. **239**(4), 645–671 (2002)

[8] Chae, D., Lee, J.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. J. Differ. Equ. **256**(11), 3835–3858 (2014)

[9] Chae, D., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equations. J. Differ. Equ. **255**(11), 3971–3982 (2013)

[10] Chae, D., Wan, R., Wu, J.: Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion. J. Math. Fluid Mech. **17**(4), 627–638 (2015)

[11] Chae, D., Weng, S.: Singularity formation for the incompressible Hall-MHD equations without resistivity. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**(4), 1009–1022 (2016)

[12] Chen, Q., Zhang, Z.: Regularity criterion of axisymmetric weak solutions to the 3D Navier–Stokes equations. J. Math. Anal. Appl. **331**, 1384–1395 (2007)

[13] Dai, M.: Local well-posedness of the Hall-MHD system in $H^s(\mathbb{R}^n)$ with $s{>}n/2$. Math. Nachr. **293**(1), 67–78 (2020)

[14] Fan, J., Huang, S., Nakamura, G.: Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations. Appl. Math. Lett. **26**(9), 963–967 (2013)

[15] Fan, J., Li, F., Nakamura, G.: Regularity criteria for the incompressible Hall-magnetohydrodynamic equations. Nonlinear Anal. **109**, 173–179 (2014)

[16] Hardy, G., Littlewood, J., Pólya, G.: Inequalities, 2nd edn. at the University Press, Cambridge (1952)

[17] Hou, T., Lei, Z., Li, C.: Global regularity of the 3D axi-symmetiric Naiver–Stokes equations with anisotropic data. Comm. Partial Differ. Equ. **33**, 1622–1637 (2008)

[18] Huba, J.: Hall magnetohydrodynamics—a tutorial. In: Büchner, J., Scholer, M., Dum, C.T. (Eds.) Space Plasma Simulation. Lecture Notes in Physics, vol. 615, pp. 166–192. Springer, Berlin (2003)

[19] Lei, Z.: On axially symmetric incompressible magnetohydrodynamics in three dimensions. J. Differ. Equ. **259**(7), 3202–3215 (2015)
[20] Lei, Z., Zhou, Y.: BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity. Discrete Contin. Dyn. Syst. **25**(2), 575–583 (2009)
[21] Lemarié-Rieusset, P.G.: Recent developments in the Navier–Stokes problem. Chapman & Hall/CRC Res. Notes Math. **431**, 66 (2002)
[22] Li, Z., Pan, X.: Some remarks on regularity criteria of axially symmetric Navier–Stokes equations. Comm. Pure Appl. Anal. **18**(3), 1333–1350 (2019)
[23] Miao, C., Zheng, X.: On the global well-posedness for the Boussinesq system with horizontal dissipation. Comm. Math. Phys. **321**(1), 33–67 (2013)
[24] Stein, E., Murphy, T.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals (PMS-43). Princeton University Press, Princeton (1993)
[25] Wan, R., Zhou, Y.: On global existence, energy decay and blow-up criteria for the Hall-MHD system. J. Differ. Equ. **259**(11), 5982–6008 (2015)
[26] Wang, Y., Zou, W.: On the blow-up criterion of smooth solutions for Hall-magnetohydrodynamics system with partial viscosity. Commun. Pure Appl. Anal. **13**(3), 1327–1336 (2014)
[27] Weng, S.: On analyticity and temporal decay rates of solutions to the viscous resistive Hall-MHD system. J. Differ. Equ. **260**(8), 6504–6524 (2016)
[28] Weng, S.: Space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations. J. Funct. Anal. **270**(6), 2168–2187 (2016)

Zijin Li
School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044
China
e-mail: zijinli@nuist.edu.cn

Xinghong Pan
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016
China

and

Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA)
MIIT
Nanjing 210016
China
e-mail: xinghong_87@nuaa.edu.cn

(accepted: December 13, 2021; published online: January 7, 2022)