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Global existence in critical spaces for non Newtonian compressible viscoelastic flows

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Abstract

We are interested in the multi-dimensional compressible viscoelastic flows of Oldroyd type, which is one of non-Newtonian fluids exhibiting the elastic behavior. In order to capture the damping effect of the additional deformation tensor, to the best of our knowledge, the "div-curl" structural condition plays a key role in previous efforts. Our aim of this paper is to remove the structural condition and prove a global existence of strong solutions to compressible viscoelastic flows in critical spaces. In absence of compatible conditions, the new effective flux is introduced, which enables us to capture the dissipation arising from *combination* of density and deformation tensor. The partial dissipation in non-Newtonian compressible fluids, is weaker than that of classical Navier-Stokes equations.

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1. Introduction

In the Eulerian description, a general compressible fluid evolving in some open set Ω of $\mathbb{R}^n\,(n \geq 2)$ is characterized at every material point x in Ω and time $t \in \mathbb{R}$ by its *velocity field* $u =$

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$u(t,x) \in \mathbb{R}^n$, *density* $\rho = \rho(t,x) \in \mathbb{R}_+$, *pressure* $\Pi = \Pi(t,x) \in \mathbb{R}$. In the absence of external forces and heat diffusion, those physical quantities are governed by

- The mass conservation:

$$\partial_t \rho + \text{div}\,(\rho u) = 0.$$

- The momentum conservation:

$$\partial_t(\rho u) + \text{div}\,(\rho u \otimes u) = \text{div}\,\mathbb{S} - \nabla \Pi.$$

In the regime of *Newtonian fluids*, $\mathbb{S}$ stands for the *viscous stress tensor*, which is given by

$$\mathbb{S} \triangleq \lambda \text{div}\, u\, \text{Id} + 2\mu D(u).$$

Here λ and μ are the *viscosity coefficients* and $D(u) \triangleq \frac{1}{2}(\nabla u + {}^T\nabla u)$ is the *deformation tensor.* So one can get the baratropic Navier-Stokes equations of compressible fluids:

$$\begin{cases} \partial_t \rho + \text{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}\,(\rho u \otimes u) - \text{div}\,\big(2\mu D(u) + \lambda\, \text{div}\, u\, \text{Id}\big) + \nabla \Pi = 0. \end{cases} \tag{1.1}$$

In the last several decades, there have been many attempts to capture different phenomena for *non-Newtonian fluids* such as those in the Ericksen-Rivlin models, the high-grade fluid models, the Ladyzhenskaya models and so on. One particular subclass of non-Newtonian fluids is of Oldroyd type, that is, $\mathbb{S} \triangleq \lambda \text{div}\, u\, \text{Id} + 2\mu D(u) + \big(\frac{W_F(F)F^T}{\det F}\big)$, where the deformation tensor F satisfies the transport equation

$$\partial_t F + u \cdot \nabla F = \nabla u F.$$

Formulations about viscoelastic flows of Oldroyd-B type are first introduced by Oldroyd [38] and are extensively discussed in [1,30]. Consequently, we are concerned with the following compressible viscoelastic flow of Oldroyd type

$$\begin{cases} \partial_t \rho + \text{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}\,(\rho u \otimes u) - \text{div}\,\big(2\mu D(u) + \lambda\, \text{div}\, u\, \text{Id}\big) + \nabla \Pi \\ = \text{div}\big(\frac{W_F(F)F^T}{\det F}\big), \\ \partial_t F + u \cdot \nabla F = \nabla u F, \end{cases} \tag{1.2}$$

where $W(F)$ is the elastic energy. $W_F(F)$ takes the Piola-Kirchhoff form and $\big(\frac{W_F(F)F^T}{\det F}\big)$ is the Cauchy-Green tensor, respectively. For simplicity, a special form of the Hookean linear elasticity has been taken:

$$W(F) = \frac{\alpha}{2}|F|^2,$$

where $\alpha > 0$ is elastic parameter. The initial data are supplemented by

$$(\rho, F; u)|_{t=0} = (\rho_0(x), F_0(x); u_0(x)), \quad x \in \mathbb{R}^n. \tag{1.3}$$

In the present paper, we shall investigate the existence of global solutions to the Cauchy problem (1.2)-(1.3), as initial data are the perturbation of constant equilibrium state $(1, I, 0)$. First of all, let us recall those previous efforts for incompressible viscoelastic flow, which reads as

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu\Delta u + \nabla\Pi = \operatorname{div}(FF^T), \\ \partial_t F + u \cdot \nabla F = \nabla u F, \\ \operatorname{div} u = 0. \end{cases} \tag{1.4}$$

For incompressible Oldroyd models, Renardy [42] in 1985 investigated the existence and uniqueness of slow steady flows of viscoelastic fluids. The global existence of a small smooth solution was firstly established by Guillopé and Saut [19]. Later, they [20] investigated shearing motions and Poiseuille flows of Oldroyd fluids with retardation time, which exist for arbitrary time and arbitrary initial data. The case of L^s-L^r solutions has been treated by Fernandez Cara, Guillén and Ortega in [17]. In higher dimensions, Lions and Masmoudi [37] constructed global weak solutions for general initial conditions. Chemin and Masmoudi [8] in the critical Besov space proved the existence and uniqueness of local and global solutions. Constantin and Kliegel [14] established the global regularity of strong solutions for 2D Oldroyd-B fluids with additional diffusive stress. Elgindi and Rousset [16] proved the global regularity of smooth solutions for 2D generalized Oldroyd-B type models without diffusive velocity. If the damping is absent in the classical Oldroyd case, the velocity viscosity alone may not be sufficient to guarantee the regularity of (1.4). The "div-curl" structure is full explored by Lin, Liu and Zhang [35], Lei, Liu and Zhou [32] and it was shown that the Cauchy problem of (1.4) admitted the global classical solution in Sobolev spaces. Since then, there are a number of efforts available under the assumption of "div-curl" structure, see for example [11,36,44,47]. In three dimensions, the third author [45] proved the global existence of small solutions to the incompressible Oldroyd-B model without damping mechanism. Actually, her results can be also applied to the system (1.4), where the "div-curl" compatible condition is no longer needed. Recently, Chen and Hao [9] proved the global critical regularity in the Besov space based on the observation of Green's matrix. The reader is also referred to [23,34] for the research summary of (1.4).

In this paper, we are concerned with the compressible viscoelastic flows. The mathematical modeling of compressible viscoelastic fluids was proposed in the earlier paper [15] due to Beris and Edwards (see also their book [3] or [5] and references therein). Fixed some positive time, Lei and Zhou [33] established the global existence of classical solutions to two-dimensional case, when initial data are subjected to incompressible constraints. Furthermore, the incompressible limit to (1.4) was rigorously justified. The existence and uniqueness of local-in-time strong solutions with large initial data for the three-dimensional compressible viscoelastic flow was established by Hu and Wang [24]. Compared to the study of (1.4), the major difficulty for proving the global existence of (1.2) lies in the lacking of the dissipative estimates for the deformation and density. Inspired by the investigation of (1.4) (see [32,35]), Hu-Wang [25] and Qian-Zhang [41] independently explored intrinsic properties of (1.2) such that the desired dissipation can be available. Indeed, their compatible conditions are listed as follows

$$\rho_0 \det F_0 = 1, \quad \nabla \cdot (\rho_0 F_0^T) = 0 \tag{1.5}$$

and

$$F_0^{lk} \partial_l F_0^{ij} - F_0^{lj} \partial_l F_0^{ik} = 0. \tag{1.6}$$

The divergence constraint (1.5) makes sure that the gradient of F behaves well in the elementary energy method, and (1.6) is used to control the quantity $\nabla \times F$. The conditional equivalence of (1.5)-(1.6) was shown by [27].

On the other hand, as in many works dedicated to compressible Navier-Stokes equations, *scaling invariance* plays a fundamental role. The reason is that whenever such an invariance exists, suitable critical quantities (that is, having the same scaling invariance as the system under consideration) control the possible finite time blow-up and achieve the global existence of strong solutions. Danchin [12] firstly solved (1.1) globally in critical homogeneous Besov spaces of L^2 type. Later, the result has been extended to those critical Besov spaces that are not related to L^2, by Charve-Danchin [6] and Chen-Miao-Zhang [10] independently. Recently, Danchin and the second author [13,43] showed the optimal decay rates in general L^p critical spaces. A natural (non Newtonian) extension in analysis is to consider (1.2). Notice that (1.2) is invariant by the transformation

$$\rho(t,x) \rightsquigarrow \rho(\ell^2 t, \ell x), \quad u(t,x) \rightsquigarrow \ell u(\ell^2 t, \ell x), \quad F(t,x) \rightsquigarrow F(\ell^2 t, \ell x) \qquad \ell > 0,$$

up to a change of the pressure term Π into $\ell^2 \Pi$ and the constant α into $\ell^2 \alpha$. Under the assumptions (1.5)-(1.6), Hu-Wang [25] and Qian-Zhang [41] independently deduced *a priori* dissipation estimates for complicated hyperbolic-parabolic systems, which lead to the existence of global solutions in the critical L^2 Besov space. Hu-Wu [26] proved the global existence of strong solutions to (1.2) as initial data are the small perturbation $(1, I; 0)$ in $H^2(\mathbb{R}^3)$. Furthermore, it was shown that those solutions converged to equilibrium state at the decay rates of heat kernel. Barrett, Lu and E. Süli [4] investigated 2D compressible Oldroyd-B type model which is derived from the compressible Navier-Stokes-Fokker-Planck system and proved the existence of large data global-in-time finite-energy weak solutions. Huo and Yong [28] studied the structural stability of a 1D compressible viscoelastic fluid model which was proposed by Öttinger [39] and established the global existence of smooth solutions near equilibrium.

Based on [25,41], the first two authors [40] established the global existence and time-decay estimates of solutions to (1.2) in the general L^p Besov space. The argument of effective velocities developed by Haspot [21] was mainly employed, which is analogue of Hoff's viscous effective flux in [22]. Let us point out that the dissipation of (1.2) with constraints (1.5)-(1.6) is standard, which is similar to that of the compressible Navier-Stokes equations (1.1). A question thus follows. *Is it possible to find any new dissipative ingredients on non Newtonian compressible viscoelastic flows without* (1.5)-(1.6)*?* Here we aim at recasting a global-in-time existence of strong solutions in the framework of spatially Besov spaces with critical regularity without (1.5)-(1.6) that has been playing the key role in previous efforts.

Before writing out the main statement of our paper, let us introduce some notations and definitions first. To begin with, we need a Littlewood-Paley decomposition. There exist two radial smooth functions $\varphi(x)$, $\chi(x)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$ and the ball $B = \{\xi \in \mathbb{R}^n : |\xi| \le 4/3\}$, respectively such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)f, \quad \dot{S}_j f = \sum_{k\le j-1} \dot{\Delta}_k f = \chi(2^{-j}D)f.$$

We denote by $\mathcal{Z}'(\mathbb{R}^n)$ the dual space of

$$\mathcal{Z}(\mathbb{R}^n) \triangleq \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in (\mathbb{N} \cup 0)^n\}.$$

Let us now turn to the definition of the main functional spaces and norms that will come into play in our paper.

Definition 1.1. Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \Big(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} < \infty.$$

Also, we introduce the hybrid Besov space since our analysis will be performed at different frequencies.

Definition 1.2. Let $s, \sigma \in \mathbb{R}$. The hybrid Besov space $\dot{\mathcal{B}}^{s,\sigma}$ is defined by

$$\dot{\mathcal{B}}^{s,\sigma} \triangleq \{f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{B}}^{s,\sigma}} < \infty\},$$

with

$$\|f\|_{\dot{\mathcal{B}}^{s,\sigma}} \triangleq \sum_{2^k\le R_0} 2^{ks}\|\dot{\Delta}_k f\|_{L^2} + \sum_{2^k> R_0} 2^{k\sigma}\|\dot{\Delta}_k f\|_{L^2},$$

where R_0 is a fixed constant to be defined. $\dot{\mathcal{B}}^{s,s}$ is the usual Besov space $\dot{B}^s_{2,1}$ if $\sigma = s$. In the case where u depends on the time variable, we consider the space-time mixed spaces as follows

$$\|u\|_{L^q_T\dot{\mathcal{B}}^{s,\sigma}} := \big\| \|u(t,\cdot)\|_{\dot{\mathcal{B}}^{s,\sigma}} \big\|_{L^q(0,T)}.$$

In addition, we introduce another space-time mixed spaces, which is usually referred to Chemin-Lerner's spaces. The definition is given by

$$\|u\|_{\tilde{L}^q_T\dot{\mathcal{B}}^{s,\sigma}} \triangleq \sum_{2^k\le R_0} 2^{ks}\|\dot{\Delta}_k u\|_{L^q(0,T)L^2} + \sum_{2^k> R_0} 2^{k\sigma}\|\dot{\Delta}_k u\|_{L^q(0,T)L^2}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\tilde{\mathcal{C}}_b(\dot{\mathcal{B}}^{s,\sigma})$ the subset of functions $\tilde{L}^\infty(\dot{\mathcal{B}}^{s,\sigma})$ which are continuous from $\mathbb{R}_+$ to $\dot{\mathcal{B}}^{s,\sigma}$. It is easy to check that $\tilde{L}^1_T\dot{\mathcal{B}}^{s,\sigma} = L^1_T\dot{\mathcal{B}}^{s,\sigma}$ and $\tilde{L}^q_T\dot{\mathcal{B}}^{s,\sigma} \subseteq L^q_T\dot{\mathcal{B}}^{s,\sigma}$ for $q > 1$.

Define $\tau_0 \triangleq \frac{F_0F_0^T}{\det F_0} - I$ and $\tau \triangleq \frac{FF^T}{\det F} - I$, where I is the unit matrix of order n. Our result is stated as follows.

Theorem 1.1. *Let $\Pi'(1) = 1$. There exists a constant $\eta > 0$ such that if*

$$(\rho_0 - 1, \tau_0; u_0) \in \dot{\mathcal{B}}^{n/2-1,n/2} \times \dot{\mathcal{B}}^{n/2-1,n/2} \times \dot{B}^{n/2-1}_{2,1}, \quad F_0 - I \in \dot{B}^{n/2}_{2,1},$$

and

$$\|(\rho_0 - 1, \tau_0)\|_{\dot{\mathcal{B}}^{n/2-1,n/2}} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}} \leq \eta,$$

then the Cauchy problem (1.2)-(1.3) *has a global unique solution* $(\rho, F; u)$ *such that*

$$(\rho - 1, \tau) \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{\mathcal{B}}^{n/2-1,n/2}), \quad (F - I) \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/2}_{2,1})$$
$$u \in \tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1}_{2,1}).$$

Moreover, the following estimate holds:

$$\|(\rho - 1, \tau)\|_{\tilde{L}^\infty\dot{\mathcal{B}}^{n/2-1,n/2}} + \|u\|_{\tilde{L}^\infty\dot{B}^{n/2-1}_{2,1}\cap L^1\dot{B}^{n/2+1}_{2,1}} \leq M\eta, \tag{1.7}$$

where M is some positive constant.

Remark 1.1. In comparison with [12] for compressible Navier-Stokes equations of Newtonian fluids, the dissipation of the density and deformation is absent and thus they might grow in time. Theorem 1.1 is actually established by the smallness of $\tau_0 = \frac{F_0F_0^T}{\det F_0} - I$ (linked with the initial strain part in deformation) rather than the total deformation $F_0 - I$, which implies that for the rotation strain decomposition of deformation, the initial rotation does not need to be small. The rotation strain decomposition is an old problem due to Friedrichs [18] and John [29], similar results have been shown in the incompressible non Newtonian fluids [31,45].

Remark 1.2. In absence of initial compatible conditions (1.5)-(1.6), it seems impossible to obtain the damping mechanism of the density and deformation. In order to eliminate the major difficulty, as in [46], one can view $\frac{FF^T}{\det F}$ as a variable rather than the nonlinear term. Furthermore, the new effective flux $(\theta, \mathcal{G})$ (see below) is found, which allows to capture the partial dissipation arising from the combination of density and deformation. Consequently, it is shown that the evolution of critical regularity of perturbation variable (ρ, u, τ) satisfying (1.7) can be established, which indicates that there is no regularity loss for the quantities ρ, u and F.

Remark 1.3. It is possible to develop the analogue of Theorem 1.1 in more general L^p framework like those efforts for compressible Navier-Stokes equations (see for example [6,10,21]). This is beyond our primary interest in the present paper, since we focus on the basic dissipative structure of non-Newtonian fluids. In fact, the L^2 orthogonal property of projection operator

$(\mathcal{P}, \mathcal{P}^{\perp})$ is employed in the proof of Proposition 3.1 (see (3.16)-(3.17)). Last not but least, it is worth pointing out that our method can be used to establish the global-in-time existence for the compressible Oldroyd-B model without damping.

We end this section with a strategy in the proof of Theorem 1.1. The starting point is to rewrite (1.2) as the linearized compressible viscoelastic flows about $(1, I, 0)$. In order to avoid those initial compatible conditions, one can view $\frac{FF^T}{\det F}$ as a new variable rather than the nonlinear term. Without loss of generality, we set $\Pi'(1) = 1$. Denote

$$\rho = 1 + a, \quad p(t,x) = \Pi(1+a) - \Pi(1), \quad \tau(t,x) = \frac{FF^T}{\det F}(t,x) - I.$$

It is shown that by the direct computation

$$\begin{cases} \partial_t p + u \cdot \nabla p + \nabla \cdot u = F_1, \\ \partial_t u - \mathcal{A}u + (\nabla p - \alpha \nabla \cdot \tau) = F_2, \\ \partial_t \tau + u \cdot \nabla \tau + \nabla \cdot u \mathrm{Id} - 2D(u) = F_3, \end{cases} \tag{1.8}$$

with

$$F_1 \triangleq -K(a)\nabla \cdot u,$$

$$F_2 \triangleq -I(a)\mathcal{A}u - u \cdot \nabla u + I(a)(\nabla p - \alpha \nabla \cdot \tau) + \frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(u) + \lambda(a)\mathrm{div} u \mathrm{Id}\big)$$

and

$$F_3 \triangleq \nabla u \tau + \tau(\nabla u)^T - \nabla \cdot u \tau,$$

where

$$I(a) \triangleq \frac{a}{1+a}, \ K(a) \triangleq \Pi'(1+a)(1+a) - 1, \ \mathcal{A} = \mu(1)\Delta + (\lambda(1) + \mu(1))\nabla \mathrm{div},$$

and

$$\tilde{\mu}(a) = \mu(1+a) - \mu(1), \ \tilde{\lambda}(a) = \lambda(1+a) - \lambda(1).$$

For simplicity, we denote $\lambda(1) = \lambda_0$, $\mu(1) = \mu_0$. In order to capture the dissipation arising from the complicated coupling between p and τ, let us introduce the *new effective flux* $\theta = \nabla p - \alpha \nabla \cdot \tau$. By employing the operator ∇ to $(1.8)_1$ and the operator $\alpha\mathrm{div}$ to $(1.8)_3$, respectively, one can get

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \alpha \Delta u + \nabla(\nabla \cdot u) = \tilde{F}_1 - \alpha \tilde{F}_3, \\ \partial_t u - \mathcal{A}u + \theta = F_2, \end{cases} \tag{1.9}$$

with $\tilde{F}_1 = -\nabla(K(a)\nabla \cdot u) - (\nabla u)^T \nabla p$, $\tilde{F}_3 = -((\nabla u)^T \cdot \nabla) \cdot \tau + \nabla \cdot F_3$. The corresponding linear system reads

$$\begin{cases} \partial_t \theta + \alpha \Delta u + \nabla(\nabla \cdot u) = 0, \\ \partial_t u - \mathcal{A} u + \theta = 0. \end{cases} \tag{1.10}$$

As shown by the formal spectral analysis in Section 2, we see that (1.10) admits the similar dissipation structure as that of usual compressible Navier-Stokes equations. The observation on the combination of density and deformation tensor (*without any compatible conditions*) is new in compressible non-Newtonian fluids, which enables us to establish a global existence in the critical Besov space.

2. Formal spectral analysis and energy functionals

In order to understand the proof of Theorem 1.1, it is convenient to give the formal spectral analysis for (1.10). For $s \in \mathbb{R}$, we denote $\Lambda^s f \triangleq \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f))$. Also we use $\mathcal{P}$ to denote the projection operator $I + (-\Delta)^{-1}\nabla\nabla\cdot$ and $\mathcal{P}^\perp = -(-\Delta)^{-1}\nabla\nabla\cdot$ on the divergence-free vector and potential vector, respectively. By applying $\Lambda^{-1}\mathcal{P}$, $\Lambda^{-1}\mathcal{P}^\perp$ to the first equation of (1.10) and $\mathcal{P}$, $\mathcal{P}^\perp$ to the second equation of (1.10), we get

$$\begin{cases} \partial_t \Lambda^{-1}\mathcal{P}^\perp \theta - (1+\alpha)\Lambda \mathcal{P}^\perp u = 0, \\ \partial_t \mathcal{P}^\perp u - (\lambda_0 + 2\mu_0)\Delta \mathcal{P}^\perp u + \mathcal{P}^\perp \theta = 0, \\ \partial_t \Lambda^{-1}\mathcal{P}\theta - \alpha \Lambda \mathcal{P} u = 0, \\ \partial_t \mathcal{P} u - \mu_0 \Delta \mathcal{P} u + \mathcal{P}\theta = 0. \end{cases}$$

Clearly, we see that there are *two hyperbolic-parabolic coupled systems* for $(\Lambda^{-1}\mathcal{P}^\perp\theta, \mathcal{P}^\perp u)$ and $(\Lambda^{-1}\mathcal{P}\theta, \mathcal{P}u)$ available, which are similar with the case of compressible Navier-Stokes equations (see [12]). For example, we investigate the 2×2 subsystem for $(\Phi, \Psi) \triangleq (\Lambda^{-1}\mathcal{P}\theta, \mathcal{P}u)$:

$$\begin{cases} \partial_t \Phi - \alpha \Lambda \Psi = 0, \\ \partial_t \Psi - \mu_0 \Delta \Psi + \Lambda \Phi = 0. \end{cases}$$

The Green matrix is given by $G(D) = \begin{pmatrix} 0 & \alpha\Lambda \\ -\Lambda & \mu_0 \Delta \end{pmatrix}$. Let $\lambda_\pm$ be the eigenvalues of $G(\xi)$. For low frequencies ($\mu_0|\xi| < 2\sqrt{\alpha}$), the eigenvalues are

$$\lambda_\pm = -\frac{\mu_0|\xi|^2}{2}\Big(1 \pm \mathrm{i}\sqrt{\frac{4\alpha}{\mu_0^2|\xi|^2} - 1}\Big)$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The situation of high frequencies ($\mu_0|\xi| > 2\sqrt{\alpha}$) is quite different. The eigenvalues are

$$\lambda_\pm = -\frac{\mu_0|\xi|^2}{2}\Big(1 \pm \sqrt{1 - \frac{4\alpha}{\mu_0^2|\xi|^2}}\Big).$$

Consequently, as in [12], one can expect a parabolic smoothing for low frequencies of (Φ, Ψ), a damping for high frequencies of Φ and a parabolic smoothing for high frequencies of Ψ. Similar analysis can be performed for another hyperbolic-parabolic system. So it is reasonable to define the energy at low frequencies as

$$\tilde{\mathcal{E}}_T^{\ell} \triangleq \sup_{t\in[0,T)} \|(u,\Lambda^{-1}\theta)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \int_0^T \|(u,\Lambda^{-1}\theta)\|^{\ell}_{\dot{B}^{n/2+1}_{2,1}} dt,$$

and the energy at high frequencies

$$\tilde{\mathcal{E}}_T^{h} \triangleq \sup_{t\in[0,T)} \left(\|u\|^{h}_{\dot{B}^{n/2-1}_{2,1}} + \|\Lambda^{-1}\theta\|^{h}_{\dot{B}^{n/2}_{2,1}} \right) + \int_0^T \|u\|^{h}_{\dot{B}^{n/2+1}_{2,1}} dt + \int_0^T \|\Lambda^{-1}\theta\|^{h}_{\dot{B}^{n/2}_{2,1}} dt.$$

The above analysis looks so standard, however, keep in mind that the partial dissipation of density and deformation tensor is captured only. In subsequent analysis, we have to meet those nonlinear terms (see F_1, F_2 and F_3) with respect to the variable (a,τ) itself. In order to close the energy method, we need additional L^∞ estimates for a and τ in time. For that end, we introduce another *new effective flux* $\mathcal{G} \triangleq \tau - p\mathrm{Id}$. It follows from $(1.8)_1\mathrm{Id}$ and $(1.8)_3$ that

$$\begin{cases} \partial_t p + u\cdot\nabla p + \nabla\cdot u = F_1, \\ \partial_t u - \mathcal{A}u + (1-\alpha)\nabla p - \alpha\nabla\cdot\mathcal{G} = F_2, \\ \partial_t \mathcal{G} + u\cdot\nabla\mathcal{G} - 2D(u) = F_3 - F_1\mathrm{Id}. \end{cases} \tag{2.1}$$

Furthermore, we revise our energy functionals (2.3)-(2,4) a little bit, which are given by

$$\mathcal{E}_T^{\ell} \triangleq \sup_{t\in[0,T)} \|(p,\tau,\ u,\Lambda^{-1}\theta)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \int_0^T \|(u,\Lambda^{-1}\theta)\|^{\ell}_{\dot{B}^{n/2+1}_{2,1}} dt, \tag{2.2}$$

and

$$\mathcal{E}_T^{h} \triangleq \sup_{t\in[0,T)} \left(\|u\|^{h}_{\dot{B}^{n/2-1}_{2,1}} + \|(p,\ \tau,\ \Lambda^{-1}\theta)\|^{h}_{\dot{B}^{n/2}_{2,1}} \right) + \int_0^T \|u\|^{h}_{\dot{B}^{n/2+1}_{2,1}} dt + \int_0^T \|\Lambda^{-1}\theta\|^{h}_{\dot{B}^{n/2}_{2,1}} dt. \tag{2.3}$$

Indeed, by combining the L^∞ estimates of (p,τ) (see (3.22), (3.24) and (3.25) for details), it is sufficient to establish the global existence of strong solutions to the Cauchy problem (1.2)-(1.3). Finally, it's worth noting that our analysis holds true for non-small coupling parameter α.

3. A priori energy estimates

Following from the spectral analysis in Section 2, we shall prove crucial *a priori* estimates for the energy functionals in (2.2) and (2.3).

Let $T > 0$. We by $\mathcal{E}_T$ denote the functional space

$$\begin{aligned}\mathcal{E}_T \triangleq \big\{(p,\tau;u,\theta)|(p,\tau) \in \tilde{L}^\infty(0,T;\dot{\mathcal{B}}^{n/2-1,n/2}),\\ u \in \tilde{L}^\infty(0,T;\dot{B}^{n/2-1}_{2,1}) \cap L^1(0,T;\dot{B}^{n/2+1}_{2,1}),\\ \Lambda^{-1}\theta \in \tilde{L}^\infty(0,T;\dot{\mathcal{B}}^{n/2-1,n/2}) \cap L^1(0,T;\dot{\mathcal{B}}^{n/2+1,n/2})\big\}\end{aligned}$$

and the corresponding norm is given by

$$\begin{aligned}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T} \triangleq \|(p,\tau)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/2}} + \|u\|_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}\cap L^1_T\dot{B}^{n/2+1}_{2,1}}\\ +\|\Lambda^{-1}\theta\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/2}\cap L^1_T\dot{\mathcal{B}}^{n/2+1,n/2}}.\end{aligned}$$

Note that $p = \Pi(1+a) - \Pi(1)$, $\Pi'(1) = 1$. There exists a small number $\eta_0 > 0$ such that $\|a\|_{L^\infty([0,T]\times\mathbb{R}^n)} \leq \eta_0$. Consequently, a can be expressed by a smooth function of p. Set $a = h(p)$.

Proposition 3.1. *Assume that $(p,\tau;u)$ is a strong solution of System (1.8) on $[0,T]$ with*

$$\|a\|_{L^\infty([0,T]\times\mathbb{R}^n)} \leq \eta_0.$$

Then it holds that

$$\begin{aligned}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T} \leq C\Big\{\|(p,\tau;u)(0)\|_{\mathcal{E}_0}\\ +\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}\big(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T}\big)^{n+3}\Big\},\end{aligned} \tag{3.1}$$

where $\|(p,\tau;u)(0)\|_{\mathcal{E}_0} \triangleq \|(p,\tau)(0)\|_{\dot{\mathcal{B}}^{n/2-1,n/2}} + \|u(0)\|_{\dot{B}^{n/2-1}_{2,1}}$.

We divide the proof of Proposition 3.1 into three parts for clarity. The first part is devoted to dissipative estimates for variables (θ, u). More precisely, the parabolic smoothing effect for low frequencies of (θ, u), the damping for high frequencies of θ and the parabolic smoothing effect for high frequencies of u will be addressed. With the help of the new effective flux $\mathcal{G}$, in the second part, we give the additional L^∞ estimates for full variables p and τ in time. The last part is dedicated to bounding of those nonlinear terms.

3.1. Dissipative estimates of $(\boldsymbol{\theta}, \boldsymbol{u})$

In this subsection, we derive the parabolic smoothing effect for low frequencies of (θ, u), the damping for high frequencies of θ and the parabolic smoothing effect for high frequencies of u.

Set $(\Phi,\Psi) \triangleq (\Lambda^{-1}\mathcal{P}\theta, \mathcal{P}u)$ and $(\Phi^\perp,\Psi^\perp) \triangleq (\Lambda^{-1}\mathcal{P}^\perp\theta, \mathcal{P}^\perp u)$. We use the notation $f_k = \dot{\Delta}_k f$ for any scalar (vector or matrix, respectively) function f.

Step 1: Low-frequency estimates ($2^k \le R_0$)
By applying $\Lambda^{-1}\mathcal{P}\dot{\Delta}_k$, $\Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k$ to the first equation of (1.9), $\mathcal{P}\dot{\Delta}_k$, $\mathcal{P}^\perp\dot{\Delta}_k$ to the second equation of (1.9), we have

$$\begin{cases} \partial_t \Phi_k^\perp + u\cdot\nabla\Phi_k^\perp - (1+\alpha)\Lambda\Psi_k^\perp = \Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^1, \\ \partial_t\Psi_k^\perp - (\lambda_0 + 2\mu_0)\Delta\Psi_k^\perp + \Lambda\Phi_k^\perp = \dot{\Delta}_k\mathcal{P}^\perp F_2, \\ \partial_t\Psi_k - \mu_0\Delta\Psi_k + \Lambda\Phi_k = \dot{\Delta}_k\mathcal{P}F_2, \\ \partial_t\Phi_k + u\cdot\nabla\Phi_k - \alpha\Lambda\Psi_k = \Lambda^{-1}\mathcal{P}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^2, \end{cases} \tag{3.2}$$

where commutators are given by $\mathcal{R}_k^1 = [u\cdot\nabla, \Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k]\theta$, $\mathcal{R}_k^2 = [u\cdot\nabla, \Lambda^{-1}\mathcal{P}\dot{\Delta}_k]\theta$.

Taking L^2 inner product of $(3.2)_1$ with $\frac{1}{1+\alpha}\Phi_k^\perp$, $(3.2)_2$ with $\Psi_k^\perp$, $(3.2)_3$ with Ψ_k and $(3.2)_4$ with $\frac{1}{\alpha}\Phi_k$ respectively, and then adding the resulting equations together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|(\Psi_k, \Psi_k^\perp)\|_{L^2}^2 + \frac{1}{\alpha}\|\Phi_k\|_{L^2}^2 + \frac{1}{1+\alpha}\|\Phi_k^\perp\|_{L^2}^2\Big) \\ &\quad + \mu_0\|\Lambda\Psi_k\|_{L^2}^2 + (\lambda_0+2\mu_0)\|\Lambda\Psi_k^\perp\|_{L^2}^2 \\ &= (u_k|\dot{\Delta}_k F_2) + \Big(\frac{1}{2(1+\alpha)}|\Phi_k^\perp|^2 + \frac{1}{2\alpha}|\Phi_k|^2\Big|\nabla\cdot u\Big) \\ &\quad + \frac{1}{\alpha}\big(\Phi_k|\Lambda^{-1}\mathcal{P}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^2\big) \\ &\quad + \frac{1}{1+\alpha}\big(\Phi_k^\perp|\Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^1\big). \end{aligned} \tag{3.3}$$

To capture the dissipation arising from θ, we take the L^2 inner product of $(3.2)_1$ with $\Lambda\Psi_k^\perp$, $(3.2)_2$ with $\Lambda\Phi_k^\perp$, $(3.2)_3$ with $\Lambda\Phi_k$ and $(3.2)_4$ with $\Lambda\Psi_k$ respectively. Then we add these resulting equations together and get

$$\begin{aligned} &\frac{d}{dt}\big[(\Psi_k^\perp|\Lambda\Phi_k^\perp) + (\Psi_k|\Lambda\Phi_k)\big] + \|\Lambda(\Phi_k, \Phi_k^\perp)\|_{L^2}^2 \\ &\quad - \alpha\|\Lambda\Psi_k\|_{L^2}^2 - (1+\alpha)\|\Lambda\Psi_k^\perp\|_{L^2}^2 \\ &\quad - \mu_0(\Delta\Psi_k|\Lambda\Phi_k) - (\lambda_0+2\mu_0)(\Delta\Psi_k^\perp|\Lambda\Phi_k^\perp) \\ &= \big(\Lambda\Psi_k^\perp|\Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^1\big) + (\dot{\Delta}_k F_2|\theta_k) \\ &\big(\Lambda\Psi_k|\Lambda^{-1}\mathcal{P}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^2\big) - (\Lambda u_k\big|u\cdot\nabla\Lambda^{-1}\theta_k). \end{aligned} \tag{3.4}$$

Now, we multiply a small constant ν (to be determined) to (3.4) and then add the resulting equation with (3.3) together. Consequently, we are led to the following inequality

$$\frac{d}{dt}f_{\ell,k}^2 + \tilde{f}_{\ell,k}^2 \lesssim |F(t)|,$$

where

$$
\begin{aligned}
f_{\ell,k}^2 \triangleq\ & \|(\Psi_k, \Psi_k^{\perp})\|_{L^2}^2 + \frac{1}{\alpha}\|\Phi_k\|_{L^2}^2 + \frac{1}{1+\alpha}\|\Phi_k^{\perp}\|_{L^2}^2 \\
& + 2\nu(\Psi_k^{\perp}|\Lambda\Phi_k^{\perp}) + 2\nu(\Psi_k|\Lambda\Phi_k), \\
\tilde{f}_{\ell,k}^2 \triangleq\ & (\mu_0 - \alpha\nu)\|\Lambda\Psi_k\|_{L^2}^2 + (\lambda_0 + 2\mu_0 - (1+\alpha)\nu)\|\Lambda\Psi_k^{\perp}\|_{L^2}^2 \\
& + \nu\|\Lambda(\Phi_k, \Phi_k^{\perp})\|_{L^2}^2 - \mu_0\nu(\Delta\mathcal{P}u_k|\mathcal{P}\theta_k) \\
& - (\lambda_0 + 2\mu_0)\nu(\Delta\mathcal{P}^{\perp}u_k|\mathcal{P}^{\perp}\theta_k),
\end{aligned}
$$

and

$$
\begin{aligned}
F(t) \triangleq\ & (u_k|\dot{\Delta}_k F_2) + \Big(\frac{1}{2(1+\alpha)}|\Phi_k^{\perp}|^2 + \frac{1}{2\alpha}|\Phi_k|^2\Big|\nabla\cdot u\Big) \\
& + \frac{1}{\alpha}\big(\Phi_k|\Lambda^{-1}\mathcal{P}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^2\big) \\
& + \frac{1}{1+\alpha}\big(\Phi_k^{\perp}|\Lambda^{-1}\mathcal{P}^{\perp}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^1\big) \\
& + \nu\big(\Lambda\Psi_k^{\perp}|\Lambda^{-1}\mathcal{P}^{\perp}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^1\big) \\
& + \nu(\dot{\Delta}_k F_2|\theta_k) + \nu\big(\Lambda\Psi_k|\Lambda^{-1}\mathcal{P}\dot{\Delta}_k(\tilde{F}_1 - \alpha\tilde{F}_3) + \mathcal{R}_k^2\big) \\
& - \nu(\Lambda u_k\big|u\cdot\nabla\Lambda^{-1}\theta_k).
\end{aligned}
$$

For any fixed R_0, we choose $\nu \sim \nu(\lambda_0, \mu_0, R_0)$ sufficiently small such that

$$
\begin{aligned}
& f_{\ell,k}^2 \sim \|u_k\|_{L^2}^2 + \|\Lambda^{-1}\theta_k\|_{L^2}^2, \\
& \tilde{f}_{\ell,k}^2 \sim 2^{2k}\big(\|u_k\|_{L^2}^2 + \|\Lambda^{-1}\theta_k\|_{L^2}^2\big).
\end{aligned}
$$

By using Cauchy-Schwarz inequality, furthermore, one can get owing to $2^k \le R_0$,

$$
\begin{aligned}
\frac{d}{dt}f_{\ell,k} + 2^{2k}f_{\ell,k} \lesssim\ & \|\dot{\Delta}_k(\Lambda^{-1}\tilde{F}_1, \Lambda^{-1}\tilde{F}_3, F_2), \mathcal{R}_k^1, \mathcal{R}_k^2\|_{L^2} \\
& + \|\Lambda^{-1}\theta_k\nabla\cdot u, u\cdot\nabla\Lambda u_k, \nabla\cdot u\Lambda u_k\|_{L^2},
\end{aligned}
$$

which indicates that

$$
\begin{aligned}
& \|(u, \Lambda^{-1}\theta)\|^{\ell}_{\tilde{L}_T^{\infty}\dot{B}_{2,1}^{\frac{n}{2}-1}\cap\tilde{L}_T^1\dot{B}_{2,1}^{\frac{n}{2}+1}} \\
& \lesssim \|(u, \Lambda^{-1}\theta)(0)\|^{\ell}_{\dot{B}_{2,1}^{\frac{n}{2}-1}} + \|(\Lambda^{-1}\tilde{F}_1, F_2, \Lambda^{-1}\tilde{F}_3)\|^{\ell}_{\tilde{L}_T^1\dot{B}_{2,1}^{\frac{n}{2}-1}} \\
& \quad + \sum_{2^j\le R_0} 2^{j(n/2-1)}\int_0^T \Big\|\mathcal{R}_k^1, \mathcal{R}_k^1, \Lambda^{-1}\theta_k\nabla\cdot u, u\cdot\nabla\Lambda u_k, \nabla\cdot u\Lambda u_k\Big\|_{L^2}dt
\end{aligned}
\tag{3.5}
$$

$$\lesssim \|(p,\tau;u)(0)\|^{\ell}_{\dot B^{\frac{n}{2}-1}_{2,1}}+\|(\Lambda^{-1}\tilde F_1,F_2,\Lambda^{-1}\tilde F_3)\|^{\ell}_{\tilde L^1_T\dot B^{\frac{n}{2}-1}_{2,1}}$$
$$+\sum_{2^j\le R_0}2^{j(n/2-1)}\int_0^T\Big\|\mathcal{R}^1_k,\mathcal{R}^2_k,\Lambda^{-1}\theta_k\nabla\cdot u,u\cdot\nabla\Lambda u_k,\nabla\cdot u\Lambda u_k\Big\|_{L^2}dt.$$

It follows from those commutator estimates in [2] that

$$\sum_{k\in\mathbb{Z}}2^{ks}\int_0^T\|\mathcal{R}^i_k\|_{L^2}dt\lesssim\|\nabla u\|_{\tilde L^{r_1}_T\dot B^{n/2}_{2,1}}\|\Lambda^{-1}\theta\|_{\tilde L^{r_2}_T\dot B^{s}_{2,1}}.\quad\text{for}\quad i=1,2.$$

Here $\frac{1}{r_1}+\frac{1}{r_2}=1$. In particular, one can get

$$\sum_{2^k\le R_0}2^{k(n/2-1)}\int_0^T\big\|\mathcal{R}^1_k,\mathcal{R}^2_k\big\|_{L^2}dt\lesssim\|\nabla u\|_{\tilde L^1_T\dot B^{n/2}_{2,1}}\|\Lambda^{-1}\theta\|_{\tilde L^\infty_T\dot B^{n/2}_{2,1}}\lesssim\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\tag{3.6}$$

Similarly, regarding other terms in the last integral of (3.5), we arrive at

$$\sum_{2^k\le R_0}2^{k(n/2-1)}\int_0^T\big\|\Lambda^{-1}\theta_k\nabla\cdot u\big\|_{L^2}dt\lesssim\int_0^T\|\nabla u\|_{L^\infty}dt\|\Lambda^{-1}\theta\|^{\ell}_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}$$
$$\lesssim\|\nabla u\|_{\tilde L^1_T\dot B^{n/2}_{2,1}}\|\Lambda^{-1}\theta\|_{\tilde L^\infty_T\dot B^{n/2}_{2,1}}\lesssim\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T},$$
$$\sum_{2^k\le R_0}2^{k(n/2-1)}\int_0^T\big\|u\cdot\nabla\Lambda u_k\big\|_{L^2}dt\lesssim\Big(\int_0^T\|u\|^2_{L^\infty}dt\Big)^{1/2}\|\Lambda^2u_k\|^{\ell}_{\tilde L^2_T\dot B^{n/2-1}_{2,1}}$$
$$\lesssim\|u\|_{\tilde L^2_T\dot B^{n/2}_{2,1}}\|u\|^{\ell}_{\tilde L^2_T\dot B^{n/2}_{2,1}}\lesssim\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}$$

and

$$\begin{aligned}&\sum_{2^k\le R_0}2^{k(n/2-1)}\int_0^T\big\|\nabla\cdot u\Lambda u_k\big\|_{L^2}dt\lesssim\int_0^T\|\nabla u\|_{L^\infty}dt\|\Lambda u\|^{\ell}_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}\\&\lesssim\|u\|_{\tilde L^1_T\dot B^{n/2+1}_{2,1}}\|u\|^{\ell}_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}\lesssim\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}\tag{3.7}$$

Together with (3.5)-(3.7), we deduce that

$$\begin{aligned}\|(u,\Lambda^{-1}\theta)\|^{\ell}_{\tilde L^\infty_T\dot B^{\frac{n}{2}-1}_{2,1}\cap\tilde L^1_T\dot B^{\frac{n}{2}+1}_{2,1}}\lesssim&\|(p,\tau;u)(0)\|^{\ell}_{\dot B^{\frac{n}{2}-1}_{2,1}}+\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}\\&+\|(\Lambda^{-1}\tilde F_1,F_2,\Lambda^{-1}\tilde F_3)\|^{\ell}_{\tilde L^1_T\dot B^{\frac{n}{2}-1}_{2,1}}.\end{aligned}\tag{3.8}$$

Step 2: High-frequency estimates ($2^k > R_0$)

At high frequencies, let us perform the effective velocity argument as in [21] that was originated from Hoff's viscous effective flux in [22], and overcome the loss of one derivative of $(\Phi, \Phi^\perp)$. By applying $\mathcal{P}$ and $\mathcal{P}^\perp$ to (1.9), we get

$$\begin{cases} \partial_t \mathcal{P}\theta + \alpha\Delta\mathcal{P}u = \mathcal{P}\big(\tilde{F}_1 - \alpha\tilde{F}_3 - u\cdot\nabla\theta\big), \\ \partial_t\mathcal{P}u - \mu_0\Delta\mathcal{P}u + \mathcal{P}\theta = \mathcal{P}F_2, \\ \partial_t\mathcal{P}^\perp\theta + (1+\alpha)\Delta\mathcal{P}^\perp u = \mathcal{P}^\perp\big(\tilde{F}_1 - \alpha\tilde{F}_3 - u\cdot\nabla\theta\big), \\ \partial_t\mathcal{P}^\perp u - (\lambda_0 + 2\mu_0)\Delta\mathcal{P}^\perp u + \mathcal{P}^\perp\theta = \mathcal{P}^\perp F_2. \end{cases}$$

We define the effective velocities $w, w^\perp$ such that $-\mu_0\Delta\mathcal{P}u + \mathcal{P}\theta = -\mu_0\Delta w$ and $-(\lambda_0 + 2\mu_0)\Delta\mathcal{P}^\perp u + \mathcal{P}^\perp\theta = -(\lambda_0+2\mu_0)\Delta w^\perp$. It follows that

$$w = \mathcal{P}u + \frac{1}{\mu_0}(-\Delta)^{-1}\mathcal{P}\theta,\ w^\perp = \mathcal{P}^\perp u + \frac{1}{\lambda_0 + 2\mu_0}(-\Delta)^{-1}\mathcal{P}^\perp\theta. \tag{3.9}$$

Firstly, we do some estimates for effective velocities w and $w^\perp$. It is easy to check that

$$\begin{cases} \partial_t w - \mu_0\Delta w + \dfrac{\alpha}{\mu_0^2}(-\Delta)^{-1}\mathcal{P}\theta - \dfrac{\alpha}{\mu_0}w \\ = \mathcal{P}F_2 + \dfrac{1}{\mu_0}(-\Delta)^{-1}\mathcal{P}\big(\tilde{F}_1 - \alpha\tilde{F}_3 - u\cdot\nabla\theta\big), \\ \partial_t w^\perp - (\lambda_0+2\mu_0)\Delta w^\perp + \dfrac{1+\alpha}{(\lambda_0+2\mu_0)^2}(-\Delta)^{-1}\mathcal{P}^\perp\theta - \dfrac{1+\alpha}{\lambda_0+2\mu_0}w^\perp \\ = \mathcal{P}^\perp F_2 + \dfrac{1}{\lambda_0+2\mu_0}(-\Delta)^{-1}\mathcal{P}^\perp\big(\tilde{F}_1 - \alpha\tilde{F}_3 - u\cdot\nabla\theta\big). \end{cases} \tag{3.10}$$

Note that (5.2) to (3.10), the parabolic smooth estimate (See Lemma 5.3) enables us to obtain

$$\begin{aligned} &\|(w, w^\perp)\|^h_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}\cap L^1_T\dot{B}^{n/2+1}_{2,1}} \\ &\lesssim \|(w_0, w_0^\perp)\|^h_{\dot{B}^{n/2-1}_{2,1}} + \|(w, w^\perp)\|^h_{L^1_T\dot{B}^{n/2-1}_{2,1}} + \|\theta\|^h_{L^1_T\dot{B}^{n/2-3}_{2,1}} \\ &\quad + \|(\tilde{F}_1, \tilde{F}_3)\|^h_{L^1(\dot{B}^{n/2-3}_{2,1})} + \|F_2\|^h_{L^1(\dot{B}^{n/2-1}_{2,1})} + \|u\cdot\nabla\theta\|^h_{L^1(\dot{B}^{n/2-3}_{2,1})}. \end{aligned} \tag{3.11}$$

Owing to the high frequency cut-off $2^k > R_0$, we have

$$\|(w, w^\perp)\|^h_{L^1_T\dot{B}^{n/2-1}_{2,1}} \lesssim R_0^{-2}\|(w, w^\perp)\|^h_{L^1_T\dot{B}^{n/2+1}_{2,1}},$$

$$\|\theta\|^h_{L^1_T\dot{B}^{n/2-3}_{2,1}} \lesssim R_0^{-2}\|\theta\|^h_{L^1_T\dot{B}^{n/2-1}_{1,1}}.$$

Choosing $R_0 > 0$ sufficiently large, the terms $\|(w, w^\perp)\|^h_{L^1_T \dot{B}^{n/2-1}_{2,1}}$ on right-side of (3.11) can be absorbed by the corresponding parts on left-hand side of (3.11). Consequently, we conclude that

$$\begin{aligned} &\|(w, w^\perp)\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1} \cap L^1_T \dot{B}^{n/2+1}_{2,1}} \\ &\lesssim \|(w_0, w_0^\perp)\|^h_{\dot{B}^{n/2-1}_{2,1}} + R_0^{-2}\|\theta\|^h_{L^1_T \dot{B}^{n/2-1}_{2,1}} \\ &\quad + R_0^{-2}\|(\tilde{F}_1, \tilde{F}_3)\|^h_{L^1(\dot{B}^{n/2-1}_{2,1})} + \|F_2\|^h_{L^1(\dot{B}^{n/2-1}_{2,1})} + \|u \cdot \nabla\theta\|^h_{L^1(\dot{B}^{n/2-3}_{2,1})}. \end{aligned} \tag{3.12}$$

Next, we intend to obtain the damping estimate for θ at high frequencies. Indeed, it follows from the first equation of (1.9) that

$$\partial_t\theta + u \cdot \nabla\theta + \alpha\Delta u + \nabla\nabla \cdot u = \tilde{F}_1 - \alpha\tilde{F}_3. \tag{3.13}$$

Applying $\dot{\Delta}_k$ to (3.13), we can get

$$\begin{aligned} &\partial_t\theta_k + u \cdot \nabla\theta_k + \alpha\Delta u_k + \nabla\nabla \cdot u_k \\ &= \dot{\Delta}_k\tilde{F}_1 - \alpha\dot{\Delta}_k\tilde{F}_3 + \mathcal{R}_k, \end{aligned} \tag{3.14}$$

where $\mathcal{R}_k = [u \cdot \nabla, \dot{\Delta}_k]\theta$. The last two terms on left-hand side of (3.14) can be written as

$$\begin{aligned} &\alpha\Delta u_k + \nabla\nabla \cdot u_k \\ &= \alpha\Delta(\mathcal{P}u_k + \mathcal{P}^\perp u_k) + \Delta\mathcal{P}^\perp u_k = \alpha\Delta\mathcal{P}u_k + (1+\alpha)\Delta\mathcal{P}^\perp u_k. \end{aligned} \tag{3.15}$$

Now inserting (3.9) into (3.15) and substituting the resulting equation into (3.14), we can get

$$\begin{aligned} &\partial_t\theta_k + u \cdot \nabla\theta_k + \frac{\alpha}{\mu_0}\mathcal{P}\theta_k + \frac{1+\alpha}{\lambda_0 + 2\mu_0}\mathcal{P}^\perp\theta_k \\ &= -\alpha\Delta w_k - (1+\alpha)\Delta w_k^\perp + \dot{\Delta}_k\tilde{F}_1 - \alpha\dot{\Delta}_k\tilde{F}_3 + \mathcal{R}_k. \end{aligned} \tag{3.16}$$

Recalling the fact that $(\mathcal{P}\theta_k|\mathcal{P}^\perp\theta_k) = 0$ and $\theta_k = \mathcal{P}\theta_k + \mathcal{P}^\perp\theta_k$. A routine procedure shows that after multiplying (3.16) by $\mathcal{P}\theta_k$ and $\mathcal{P}^\perp\theta_k$, respectively,

$$\begin{aligned} &\|\theta_k(t)\|_{L^2} + \int_0^t \|\theta_k\|_{L^2} d\tau \\ &\lesssim \|\theta_k(0)\|_{L^2} + \int_0^t \|\nabla u\|_{L^\infty}\|\theta_k\|_{L^2} d\tau \\ &\quad + \int_0^t \|(\Lambda^2 w_k + \Lambda^2 w_k^\perp)\|_{L^2} d\tau + \int_0^t \|(\dot{\Delta}_k\tilde{F}_1, \dot{\Delta}_k\tilde{F}_3, \mathcal{R}_k)\|_{L^2} d\tau. \end{aligned} \tag{3.17}$$

Employing commutator estimates in [2] again enable us to get

$$\sum_{k\in\mathbb{Z}} 2^{ks}\|\mathcal{R}_k\|_{L^2} \lesssim \|\nabla u\|_{\dot{B}^{n/2}_{2,1}}\|\theta\|_{\dot{B}^{s}_{2,1}}.$$

Consequently, multiplying (3.17) by $2^{k(\frac{n}{2}-1)}$ and then summing over the index k satisfying $2^k > R_0$, we are led to

$$\begin{aligned}
&\|\theta\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap \tilde{L}^1_T \dot{B}^{n/2-1}_{2,1}} \\
&\lesssim \|\theta(0)\|^h_{\dot{B}^{n/2-1}_{2,1}} + \|\nabla u\|_{\tilde{L}^1_T \dot{B}^{n/2}_{2,1}}\|\theta\|_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}} \\
&\quad + \|(w,w^\perp)\|_{\tilde{L}^1_T \dot{B}^{n/2+1}_{2,1}} + \|(\tilde{F}_1,\tilde{F}_3)\|^h_{\tilde{L}^1_T \dot{B}^{n/2-1}_{2,1}}.
\end{aligned} \tag{3.18}$$

Multiply (3.18) by a constant $\delta > 0$ and then add the resulting inequality to (3.12) together. By choosing R_0 sufficiently large and $\delta > 0$ suitably small, we arrive at

$$\begin{aligned}
&\|\theta\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2-1}_{2,1}} + \|(w,w^\perp)\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{p,1}\cap L^1_T \dot{B}^{n/2+1}_{p,1}} \\
&\lesssim \|(p,\tau)(0)\|^h_{\dot{B}^{n/2}_{2,1}} + \|(w,w^\perp)(0)\|^h_{\dot{B}^{n/2-1}_{2,1}} + \|\nabla u\|_{\tilde{L}^1_T \dot{B}^{n/2}_{2,1}}\|\theta\|_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}} \\
&\quad + \|u\cdot\nabla\theta\|^h_{L^1(\dot{B}^{n/2-3}_{2,1})} + \|(\tilde{F}_1,F_2,\tilde{F}_3)\|^h_{L^1_T \dot{B}^{n/2-1}_{2,1}}.
\end{aligned} \tag{3.19}$$

Clearly, the third term on the right-hand side of (3.19) is easily bounded by $\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}$. The fourth term can be estimated as

$$\begin{aligned}
&\|u\cdot\nabla\theta\|^h_{L^1(\dot{B}^{n/2-3}_{2,1})} \lesssim \|u\cdot\nabla\theta\|^h_{L^1(\dot{B}^{n/2-2}_{2,1})} \\
&\lesssim \|u\|_{\tilde{L}^2(\dot{B}^{n/2}_{2,1})}\|\nabla\theta\|_{\tilde{L}^2(\dot{B}^{n/2-2}_{2,1})} \lesssim \|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.
\end{aligned}$$

Finally, keep in mind (3.9), we can conclude that

$$\begin{aligned}
&\|\theta\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2-1}_{2,1}} + \|u\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2+1}_{2,1}} \\
&\lesssim \|(p,\tau)(0)\|^h_{\dot{B}^{n/2}_{2,1}} + \|u(0)\|^h_{\dot{B}^{n/2-1}_{2,1}} + \|(p,\tau;u)\|^2_{\mathcal{E}_T} + \|(\tilde{F}_1,F_2,\tilde{F}_3)\|^h_{L^1_T \dot{B}^{n/2-1}_{2,1}}.
\end{aligned} \tag{3.20}$$

3.2. $\boldsymbol{L^\infty}$ *estimate of* $(\boldsymbol{p},\boldsymbol{\tau})$

In this part, we see that the effective flux $\mathcal{G}$ mentioned plays a key role in deducing the L^∞ estimate of (p,τ) at low frequencies.

Step 1: Low-frequency estimates ($2^k > R_0$)
Apply $\dot{\Delta}_k$ to (2.1) to get

$$\begin{cases}\partial_t p_k + u\cdot\nabla p_k + \nabla\cdot u_k = \dot{\Delta}_k F_1 + \mathcal{R}_k^3,\\ \partial_t u_k + (1-\alpha)\nabla p_k - \alpha\nabla\cdot\mathcal{G}_k - \mathcal{A}v_k = \dot{\Delta}_k F_2,\\ \partial_t\mathcal{G}_k + u\cdot\nabla\mathcal{G}_k - 2D(u_k) = \dot{\Delta}_k(F_3 - F_1\mathrm{Id}) + \mathcal{R}_k^5,\end{cases} \tag{3.21}$$

where $\mathcal{R}_k^5 = [u\cdot\nabla, \dot{\Delta}_k]\mathcal{G}$.

By taking L^2 inner product of $(3.21)_3$ with $\mathcal{G}_k$, $(3.21)_2$ with $\frac{2}{\alpha}u_k$, $(3.21)_1$ with $\frac{2(1-\alpha)}{\alpha}p_k$ and then summing up the resulting equations, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\mathcal{G}_k\|_{L^2}^2 + \frac{2}{\alpha}\|u_k\|_{L^2}^2 + \frac{2(1-\alpha)}{\alpha}\|p_k\|_{L^2}^2\Big) - \frac{2}{\alpha}(\mathcal{A}u_k|u_k)\\ =&\frac{2(1-\alpha)}{\alpha}(p_k|\dot{\Delta}_k F_1 + \mathcal{R}_k^3) + \frac{2}{\alpha}(u_k|\dot{\Delta}_k F_2) + \big(\mathcal{G}_k|\dot{\Delta}_k(F_3 - F_1\mathrm{Id})\\ &+\mathcal{R}_k^5\big) + \frac{1}{2}(|p_k|^2|\nabla\cdot u) + \frac{1}{2}(|\mathcal{G}_k|^2|\nabla\cdot u) \triangleq \hat{F}(t).\end{aligned} \tag{3.22}$$

Notice that the coefficient of $\|p_k\|_{L^2}^2$ might be non-positive if $\alpha \geq 1$. In that case, we need to give an auxiliary estimate. Set $\Omega^\perp \triangleq \Lambda^{-1}\mathcal{P}^\perp\nabla\cdot\tau$. Applying $\dot{\Delta}_k$ to $(1.8)_1$, $\dot{\Delta}_k\mathcal{P}^\perp$ to $(1.8)_2$ and $\dot{\Delta}_k\Lambda^{-1}\mathcal{P}^\perp\nabla\cdot$ to $(1.8)_3$, we have

$$\begin{cases}\partial_t p_k + u\cdot\nabla p_k + \nabla\cdot\Psi_k^\perp = \dot{\Delta}_k F_1 + \mathcal{R}_k^3,\\ \partial_t\Psi_k^\perp - (\lambda_0+2\mu_0)\Delta\Psi_k^\perp + \nabla p_k - \alpha\Lambda\Omega_k^\perp = \dot{\Delta}_k\mathcal{P}^\perp F_2,\\ \partial_t\Omega_k^\perp + u\cdot\nabla\Omega_k^\perp + \Lambda\Psi_k^\perp = \dot{\Delta}_k\Lambda^{-1}\mathcal{P}^\perp\nabla\cdot F_3 + \mathcal{R}_k^4,\end{cases} \tag{3.23}$$

where $\mathcal{R}_k^3 = [u\cdot\nabla, \dot{\Delta}_k]p$, $\mathcal{R}_k^4 = [u\cdot\nabla, \dot{\Delta}_k\Lambda^{-1}\mathcal{P}^\perp\nabla\cdot]\tau$.

Taking L^2 inner product of $(3.23)_1$ with p_k, $(3.23)_2$ with $\Psi_k^\perp$, $(3.23)_3$ with $\alpha\Omega_k^\perp$ and then summing up the resulting equations, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|p_k\|_{L^2}^2 + \|\Psi_k^\perp\|_{L^2}^2 + \alpha\|\Omega_k^\perp\|_{L^2}^2\Big) + (\lambda_0+2\mu_0)\|\Lambda\Psi_k^\perp\|_{L^2}^2\\ =&(p_k|\dot{\Delta}_k F_1 + \mathcal{R}_k^3) + \frac{1}{2}(|p_k|^2|\nabla\cdot u) + (\Psi_k^\perp|\dot{\Delta}_k\mathcal{P}^\perp F_2)\\ &+\frac{\alpha}{2}(|\Omega_k^\perp|^2|\nabla\cdot u) + \alpha(\Omega_k^\perp|\Lambda^{-1}\mathcal{P}^\perp\dot{\Delta}_k\nabla\cdot F_3 + \mathcal{R}_k^4 \triangleq \tilde{F}(t).\end{aligned} \tag{3.24}$$

Now, we multiply a small constant $\nu_1 > 0$ to (3.22) and add the resulting equation to (3.24). Choosing $\nu_1 > 0$ suitably small such that the coefficient of $\|p_k\|_{L^2}^2$ is positive. Consequently, we have

$$\frac{d}{dt}\Big(\|p_k\|_{L^2}^2 + \|u_k\|_{L^2}^2 + \|\tau_k\|_{L^2}^2\Big) \lesssim |\tilde{F}(t)| + |\hat{F}(t)|. \tag{3.25}$$

Furthermore, bounding the right-hand side of (3.25) by Cauchy-Schwarz inequality leads to the following inequality owing to $2^k \le R_0$,

$$\begin{aligned}&\frac{d}{dt}\big(\|p_k\|_{L^2}+\|u_k\|_{L^2}+\|\tau_k\|_{L^2}\big)\\&\lesssim \|\dot{\Delta}_k(F_1,\Lambda^{-1}\tilde{F}_1,F_3,\Lambda^{-1}\tilde{F}_3,F_2)\|_{L^2}\\&\qquad+\|\mathcal{R}_k^i,p_k\nabla\cdot u,\tau_k\nabla\cdot u,u\cdot\nabla\Lambda u_k,\nabla\cdot u\Lambda u_k\|_{L^2}.\end{aligned}$$

Noticing that (3.6)-(3.7), we deduce that

$$\begin{aligned}&\|(p,\tau;u)\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{2}-1}_{2,1}}\\&\lesssim \|(a,\tau;u)(0)\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}+\|(F_1,\Lambda^{-1}\tilde{F}_1,F_2,\Lambda^{-1}\tilde{F}_3,F_3)\|^{\ell}_{\tilde{L}^1_T\dot{B}^{\frac{n}{2}-1}_{2,1}}.\end{aligned} \tag{3.26}$$

Step 2: High-frequency estimates ($2^k > R_0$)
Applying $\dot{\Delta}_k$ to the first and third equations of (1.8) gives

$$\begin{cases}\partial_t p_k+u\cdot\nabla p_k+\nabla\cdot u_k=\dot{\Delta}_kF_1+\mathcal{R}_k^6,\\ \partial_t\tau_k+u\cdot\nabla\tau_k+\nabla\cdot u_k\mathrm{Id}-2D(u_k)=\dot{\Delta}_kF_3+\mathcal{R}_k^7,\end{cases} \tag{3.27}$$

where $\mathcal{R}_k^6=[u\cdot\nabla,\dot{\Delta}_k]p$ and $\mathcal{R}_k^7=[u\cdot\nabla,\dot{\Delta}_k]\tau$.

Multiplying the first equation of (3.27) by p_k and the second by τ_k, and then integrating over $\mathbb{R}^n\times[0,t]$, we obtain

$$\begin{aligned}\|(p_k,\tau_k)(t)\|_{L^2}&\lesssim\|(p_k,\tau_k)(0)\|_{L^2}+\int_0^t\|\nabla u_k\|_{L^2}d\tau+\int_0^t\|\nabla u\|_{L^\infty}\|(p_k,\tau_k)\|_{L^2}d\tau\\&\quad+\int_0^t\|(\dot{\Delta}_kF_1,\dot{\Delta}_kF_3,\mathcal{R}_k^6,\mathcal{R}_k^7)\|_{L^2}d\tau.\end{aligned} \tag{3.28}$$

It follows from commutator estimates in [2] that

$$\sum_{k\in\mathbb{Z}}2^{ks}\|(\mathcal{R}_k^6,\mathcal{R}_k^7)\|_{L^2}\lesssim\|\nabla u\|_{\dot{B}^{n/2}_{2,1}}\|(p,\tau)\|_{\dot{B}^s_{2,1}}.$$

Now multiplying (3.28) by $2^{k\frac{n}{2}}$, and then summing over the index k satisfying $2^k > R_0$, we are led to

$$\begin{aligned}\|(p,\tau)\|^h_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}&\lesssim\|(p,\tau)(0)\|^h_{\dot{B}^{n/2}_{2,1}}+\|\nabla u\|^h_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}\\&\quad+\|\nabla u\|_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}\|(p,\tau)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}+\|(F_1,F_3)\|^h_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}.\end{aligned}$$

By using (5.3), we have

$$\begin{aligned}&\|(F_1,F_3)\|^h_{\tilde L^1_T\dot B^{n/2}_{2,1}}\\ &\lesssim \|K(a)\|_{\tilde L^\infty_T\dot B^{n/2}_{2,1}}\|\nabla u\|_{\tilde L^1_T\dot B^{n/2}_{2,1}}+\|\tau\|_{\tilde L^\infty_T\dot B^{n/2}_{2,1}}\|\nabla u\|_{\tilde L^1_T\dot B^{n/2}_{2,1}}\\ &\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}$$

So, we get

$$\begin{aligned}\|(p,\tau)\|^h_{\tilde L^\infty_T\dot B^{n/2}_{2,1}}\lesssim{}&\|(p,\tau)(0)\|^h_{\dot B^{n/2}_{2,1}}+\|\nabla u\|^h_{\tilde L^1_T\dot B^{n/2}_{2,1}}\\ &+(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}\tag{3.29}$$

3.3. Estimate for nonlinear terms

Finally, we devote ourselves to bound those nonlinear terms, which occur in the first two parts. The following interpolation inequality is frequently used in our analysis

$$\|f\|_{\tilde L^2_T\dot B^{n/2}_{2,1}}\lesssim\|f\|^{1/2}_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}\|f\|^{1/2}_{L^1_T\dot B^{n/2+1}_{2,1}}.$$

Step 1: Low-frequency estimates ($2^k > R_0$)
Combining (3.8) and (3.26), we have

$$\begin{aligned}&\|(p,\tau)\|^\ell_{\tilde L^\infty_T\dot B^{\frac n2-1}_{2,1}}+\|(u,\Lambda^{-1}\theta)\|^\ell_{\tilde L^\infty_T\dot B^{\frac n2-1}_{2,1}\cap\tilde L^1_T\dot B^{\frac n2+1}_{2,1}}\lesssim\|(a,\tau;u)(0)\|^\ell_{\dot B^{\frac n2-1}_{2,1}}\\ &+\|(F_1,\Lambda^{-1}\tilde F_1,F_2,\Lambda^{-1}\tilde F_3,F_3)\|^\ell_{\tilde L^1_T\dot B^{\frac n2-1}_{2,1}}+\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}$$

More precisely, we need to deal with the following nonlinear terms

$$K(a)\nabla\cdot u,\ \Lambda^{-1}((\nabla u)^T\nabla p),\ Q(\tau,\nabla u),\ \Lambda^{-1}((\nabla u)^T\cdot\nabla)\cdot\tau)$$

in $\tilde F_1,\ F_1,\tilde F_3,\ F_3$ and

$$I(a)\mathcal{A}u,\ u\cdot\nabla u,\ I(a)\theta,\ \frac{1}{1+a}\mathrm{div}\big(2\tilde\mu(a)D(u)+\tilde\lambda(a)\mathrm{div}u\mathrm{Id}\big)$$

in F_2. Regarding $K(a)\nabla\cdot u$, by taking $r_1=1, r_2=\infty,\ f=\nabla\cdot u,\ g=K(a)$ in (5.3) and using (5.1), we have

$$\begin{aligned}&\sum_{2^k\le R_0}2^{k(n/2-1)}\|\dot\Delta_k(K(a)\nabla\cdot u)\|_{L^1_TL^2}\\ &\lesssim\|\nabla u\|_{\tilde L^1_T\dot B^{n/2}_{2,1}}\|K(a)\|_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}\\ &\lesssim(1+\|p\|_{\tilde L^\infty_T\dot B^{n/2}_{2,1}})^{1+[n/2]}\|p\|_{\tilde L^\infty_T\dot B^{n/2-1}_{2,1}}\|u\|_{\tilde L^1_T\dot B^{n/2+1}_{2,1}}\end{aligned}$$

$$\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T}^2.$$

The terms $Q(\tau,\nabla u)$, $u\cdot\nabla u$ can be treated along the same line as $K(a)\nabla u$ by taking $f=\nabla u$ and $g=\tau, u$ respectively. Also $I(a)\mathcal{A}u$ can be treated by setting $r_1=\infty, r_2=1,\ f=I(a),\ g=\nabla^2 u$ in (5.3) and using (5.1). In order to bound the term $\Lambda^{-1}((\nabla u)^T\nabla p)$, we apply (5.3) by taking $r_1=1, r_2=\infty,\ f=\nabla u,\ g=\nabla p$. Then we get

$$\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\Lambda^{-1}((\nabla u)^T\nabla p)\|_{L^1_TL^2}\\
&=\sum_{2^k\le R_0} 2^{k(n/2-2)}\|((\nabla u)^T\nabla p)\|_{L^1_TL^2}\\
&\lesssim \|p\|_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}}\|u\|_{\tilde{L}^1_T\dot{B}^{n/2+1}_{2,1}}\lesssim \|(p,\tau;u,\theta)\|_{\mathcal{E}_T}^2.
\end{aligned}$$

The term $\Lambda^{-1}((\nabla u\cdot\nabla)\cdot\tau)$ can be treated along the same line as $\Lambda^{-1}((\nabla u)^T\nabla p)$.

For $I(a)\theta$, we take $r_1=1,\ r_2=\infty,\ f=\theta,\ g=I(a)$ in (5.12) and using (5.1). Then we have

$$\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot{\Delta}_k(I(a)\theta)\|_{L^1_TL^2}\\
&\lesssim \|\theta\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2,n/2-1}_{2,1}}\|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/2}_{2,1}}\\
&\lesssim (1+\|p\|_{L^\infty_TL^\infty})^{n/2+1}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T}^2\\
&\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T}^2.
\end{aligned}$$

Next we bound nonlinear terms in F_2. Denote

$$\begin{aligned}
I&=\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(u)\big)\\
&=\frac{1}{1+a}\tilde{\mu}(a)\nabla^2u+\frac{1}{1+a}\nabla\tilde{\mu}(a)\nabla u\\
&=\tilde{\mu}(a)\nabla^2u-I(a)\tilde{\mu}(a)\nabla^2u+\nabla\tilde{\mu}(a)\nabla u-I(a)\nabla\tilde{\mu}(a)\nabla u\\
&\triangleq I_1+I_2+I_3+I_4.
\end{aligned}$$

The term I_1 can be treated along the same line as $I(a)\mathcal{A}u$ and I_3 can be dealt with by applying (5.3) with $f=\nabla u,\ g=\nabla\tilde{\mu}(a)$ and $r_1=1, r_2=\infty$. To bound I_2, we have

$$\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot{\Delta}_k(I(a)\tilde{\mu}(a)\nabla^2 v)\|_{L^1_TL^2}\\
&\lesssim \|I(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\|\tilde{\mu}(a)\nabla^2u\|_{\tilde{L}^1_T\dot{B}^{n/2-1}_{2,1}}\\
&\lesssim \|I(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\|\tilde{\mu}(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\|\nabla^2u\|_{\tilde{L}^1_T\dot{B}^{n/2-1}_{2,1}}\\
&\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n+3}\|(p,\tau;u,\theta)\|_{\mathcal{E}_T}^2.
\end{aligned}$$

Regarding I_4, it is easy to show that

$$\begin{aligned}&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot{\Delta}_k(I(a)\nabla\tilde{\mu}(a)\nabla u)\|_{L^1_T L^2}\\ &\lesssim \|I(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\|\nabla\tilde{\mu}(a)\nabla u\|_{\tilde{L}^1_T\dot{B}^{n/2-1}_{2,1}}\\ &\lesssim \|I(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\|\nabla u\|_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}\|\nabla\tilde{\mu}(a)\|_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}}\\ &\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n+3}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}$$

Since bounding $\frac{1}{1+a}\mathrm{div}\big(\tilde{\lambda}(a)\mathrm{div}u\mathrm{Id}\big)$ is the same as I, we feel free to omit those details. Summing up above all estimates, we conclude that

$$\begin{aligned}&\|(p,\tau)\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|(u,\Lambda^{-1}\theta)\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{2}-1}_{2,1}\cap\tilde{L}^1_T\dot{B}^{\frac{n}{2}+1}_{2,1}}\\ &\lesssim \|(p,\tau;u)(0)\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n+3}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}\tag{3.30}$$

Step 2: High-frequency estimates ($2^k > R_0$)

Multiply (3.29) by a small constant ν_2 and then add the resulting equation to (3.20). Note that the term $\|\nabla u\|^h_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}$ on the right-hand side of (3.29) can be absorbed by the dissipative term on left-hand side of (3.20). Consequently, we obtain

$$\begin{aligned}&\|(p,\tau)\|^h_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}+\|u\|^h_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}\cap L^1_T\dot{B}^{n/2+1}_{2,1}}+\|\theta\|^h_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}\cap L^1_T\dot{B}^{n/2-1}_{2,1}}\\ &\lesssim \|(p,\tau)(0)\|^h_{\dot{B}^{n/2}_{2,1}}+\|u(0)\|^h_{\dot{B}^{n/2-1}_{2,1}}\\ &\quad+(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}+\|(\tilde{F}_1,F_2,\tilde{F}_3)\|^h_{L^1_T\dot{B}^{n/2-1}_{2,1}}.\end{aligned}$$

Likely, we bound those nonlinear terms arising in $\tilde{F}_1,\ F_2,\ \tilde{F}_3$, see following:

$$\begin{aligned}&\nabla(K(a)\nabla\cdot u),\ (\nabla u)^T\nabla p\quad\text{in }\tilde{F}_1,\\ &\nabla\cdot Q,\ (\nabla u\cdot\nabla)\cdot\tau\quad\text{in }\tilde{F}_3,\end{aligned}$$

and

$$I(a)\mathcal{A}u,\ u\cdot\nabla u,\ I(a)\theta,\ \frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(u)+\tilde{\lambda}(a)\mathrm{div}u\mathrm{Id}\big)\quad\text{in }F_2.$$

In order to bound $(\nabla u)^T\nabla p$, by (5.3), we have

$$\begin{aligned}&\sum_{2^k> R_0} 2^{k(n/2-1)}\|\dot{\Delta}_k\big((\nabla u)^T\nabla p\big)\|_{L^1_T L^2}\\ &\lesssim \|\nabla u\|_{\tilde{L}^1_T\dot{B}^{n/2}_{2,1}}\|\nabla p\|_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}}\lesssim \|u\|_{\tilde{L}^1_T\dot{B}^{n/2+1}_{2,1}}\|p\|_{\tilde{L}^\infty_T\dot{B}^{n/2}_{2,1}}\\ &\lesssim \|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.\end{aligned}$$

Regarding $\nabla(K(a)\nabla\cdot u)$, we write $\nabla(K(a)\nabla\cdot u)=K(a)\nabla\nabla\cdot u+\nabla\cdot u\nabla K(a)$. The estimate for $\nabla\cdot u\nabla K(a)$ can be handled with at the same way as $(\nabla u)^T\nabla p$. For $K(a)\nabla\nabla\cdot u$, we arrive at

$$
\begin{aligned}
&\sum_{2^k>R_0} 2^{k(n/2-1)}\|\dot{\Delta}_k\big(K(a)\nabla\nabla\cdot u\big)\|_{L^1_T L^2}\\
&\lesssim \|K(a)\|_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}}\|\nabla^2 u\|_{\tilde{L}^1_T \dot{B}^{n/2-1}_{2,1}}\\
&\lesssim (1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n/2+1}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.
\end{aligned}
$$

Bounding $\tilde{F}_3$ can be treated along the same line as $\tilde{F}_1$. The high frequency of F_2 can be dealt with at the similar way as the low frequency, which is left to the interested reader. Consequently, we deduce that

$$
\begin{aligned}
&\|\theta\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2-1}_{2,1}}+\|u\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2+1}_{2,1}}\\
&\lesssim \|(p,\tau)(0)\|^h_{\dot{B}^{n/2}_{2,1}}+\|u(0)\|^h_{\dot{B}^{n/2-1}_{2,1}}\\
&\quad+(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n+3}\|(p,\tau;u)\|^2_{\mathcal{E}_T}.
\end{aligned}
\tag{3.31}
$$

At last, combining (3.29) and (3.31), we achieve the high-frequency estimate

$$
\begin{aligned}
&\|(p,\tau)\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}}+\|u\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2+1}_{2,1}}+\|\theta\|^h_{\tilde{L}^\infty_T \dot{B}^{n/2-1}_{2,1}\cap L^1_T \dot{B}^{n/2-1}_{2,1}}\\
&\lesssim \|(p_0,\tau_0)\|^h_{\dot{B}^{n/2}_{2,1}}+\|u_0\|^h_{\dot{B}^{n/2-1}_{2,1}}\\
&\quad+(1+\|(p,\tau;u,\theta)\|_{\mathcal{E}_T})^{n+3}\|(p,\tau;u,\theta)\|^2_{\mathcal{E}_T}.
\end{aligned}
\tag{3.32}
$$

The inequality (3.1) is followed by (3.30) and (3.32). Hence, the proof of Proposition 3.1 is complete. □

4. Proof of Theorem 1.1

Let us recall a local-in-time existence result of (1.2)-(1.3) which has been achieved by [41].

Proposition 4.1. *Assume* $(\rho_0-1,F_0-I)\in\dot{B}^{n/2}_{2,1}$ *and* $u_0\in\dot{B}^{n/2-1}_{2,1}$ *with* ρ_0 *bounded away from* 0. *There exists a time* $T>0$ *such that* (1.2)-(1.3) *has a unique solution* $(\rho,F;u)$ *with* ρ *bounded away from zero and*

$$
(\rho-1,F-I)\in C([0,T);\dot{B}^{n/2}_{2,1}),\quad u\in C([0,T);\dot{B}^{n/2-1}_{2,1})\cap L^1([0,T);\dot{B}^{n/2+1}_{2,1}).
$$

Based on Proposition 4.1, the proof of Theorem 1.1 can be finished by the standard continuity argument. Indeed, Proposition 4.1 indicates that there exists a maximal time $T>0$ such that system (1.1) admits a unique solution. Clearly, the system (1.9) also has a solution $(p,\tau;u)$ which locally exits on $[0,T)$. It follows from the assumption of Theorem 1.1 and Lemma 5.2 that

$$\|(p_0, \tau_0; u_0)\|_{\mathcal{E}_0} \leq C_0\eta,$$

for some positive constant C_0. Fixed a constant $M > 0$ (to be determined later), we define

$$T^* \triangleq \sup\{t \in [0, T)\Big| \|(p, \tau; u, \theta)\|_{\mathcal{E}_t} \leq M\eta\}.$$

Claim that

$$T^* = T.$$

According to the continuity argument, it suffices to show

$$\|(p, \tau; u, \theta)\|_{\mathcal{E}_T} \leq \frac{1}{2}M\eta. \tag{4.1}$$

Indeed, noting that

$$\|a\|_{L^\infty([0,T)\times\mathbb{R}^n)} = \|h(p)\|_{L^\infty([0,T)\times\mathbb{R}^n)} \leq C_1\|p\|_{L^\infty_T\dot{\mathcal{B}}^{n/2-1,n/2}}.$$

We can choose η sufficiently small such that

$$M\eta \leq \frac{1}{\eta_0 C_1},$$

so

$$\|a\|_{L^\infty([0,T)\times\mathbb{R}^n)} \leq \eta_0.$$

By applying Proposition 3.1, we obtain

$$\|(p, \tau; u, \theta)\|_{\mathcal{E}_T} \leq C\{C_0\eta + (M\eta)^2(1 + M\eta)^{n+3}\}. \tag{4.2}$$

By choosing $M = 3CC_0$ and η sufficient small enough such that

$$C(M\eta)(1 + M\eta)^{n+3} \leq \frac{1}{6},$$

so (4.1) is followed by (4.2) directly. Actually the above argument implies

$$\|(p, \tau; u, \theta)\|_{\mathcal{E}_T} \leq C\|(p, \tau; u)\|_{\mathcal{E}_0}. \tag{4.3}$$

Consequently, the continuity argument ensures that $T = +\infty$. It follows from the third equation of (1.2) that

$$\partial_t(F - I) + u \cdot \nabla(F - I) = \nabla u + \nabla u(F - I).$$

By using Lemma 5.4, we have

$$
\begin{aligned}
&\|F-I\|_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}}\\
&\leq \exp\left\{C\int_0^\infty \|\nabla u\|_{\dot{B}^{n/2}_{2,1}} d\tau\right\}\times\\
&\quad\Big(\|F_0-I\|_{\dot{B}^{n/2}_{2,1}}+\int_0^\infty (\|\nabla u\|_{\dot{B}^{n/2}_{2,1}}+\|\nabla u(F-I)\|_{\dot{B}^{n/2}_{2,1}})d\tau\Big)\\
&\leq C\Big(\|F_0-I\|_{\dot{B}^{n/2}_{2,1}}+\|F-I\|_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}}\|\nabla u\|_{\tilde{L}^1_T \dot{B}^{n/2}_{2,1}}+M\eta\Big)\\
&\leq C\|F_0-I\|_{\dot{B}^{n/2}_{2,1}}+CM\eta\|F-I\|_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}}+CM\eta.
\end{aligned}
$$

Furthermore, we chose η small enough such that $CM\eta \leq 1/2$ and thus obtain

$$
\|F-I\|_{\tilde{L}^\infty_T \dot{B}^{n/2}_{2,1}} \leq C\|F_0-I\|_{\dot{B}^{n/2}_{2,1}}+CM\eta. \tag{4.4}
$$

The continuity argument and (4.3)-(4.4) enable us to finish the proof of Theorem 1.1 eventually.

5. Appendix

To make the manuscript self-contained as soon as possible, we would like to collect nonlinear estimates in the last section. See [2] for more details.

Lemma 5.1. *For the Besov space, we have the following properties:*

- $\dot{\mathcal{B}}^{s_2,\sigma} \subseteq \dot{\mathcal{B}}^{s_1,\sigma}$ *for* $s_1 \geq s_2$ *and* $\dot{\mathcal{B}}^{s,\sigma_2} \subseteq \dot{\mathcal{B}}^{s,\sigma_1}$ *for* $\sigma_1 \leq \sigma_2$.
- *Interpolation: For* $s_1, s_2, \sigma_1, \sigma_2 \in \mathbb{R}$ *and* $\theta \in [0,1]$, *we have*

$$
\|f\|_{\dot{\mathcal{B}}^{\theta s_1+(1-\theta)s_2,\theta\sigma_1+(1-\theta)\sigma_2}} \leq \|f\|^{\theta}_{\dot{\mathcal{B}}^{s_1,\sigma_1}}\|f\|^{(1-\theta)}_{\dot{\mathcal{B}}^{s_2,\sigma_2}}.
$$

System (1.3) involves in compositions of functions and they are bounded according to the following lemma.

Lemma 5.2. *Let* $F:\mathbb{R}\to\mathbb{R}$ *be smooth with* $F(0)=0$. *For all* $1\leq p,r\leq\infty$ *and* $s>0$, *we have*

$$
\|F(f)\|_{\tilde{L}^r_T(\dot{B}^s_{p,1})} \leq C\|f\|_{\tilde{L}^r_T(\dot{B}^s_{p,1})}, \tag{5.1}
$$

where C *depending only on* $\|f\|_{L_T(L^\infty)}$, F' *(and higher derivatives),* s, p *and* n.

For the heat equation, one has the following optimal regularity estimate.

Lemma 5.3. *Let* $p,r\in[1,\infty]$, $s\in\mathbb{R}$, *and* $1\leq\rho_2\leq\rho_1\leq\infty$. *Assume that* $u_0\in\dot{B}^{s-1}_{p,r}$, $f\in \tilde{L}^{\rho_2}_T\dot{B}^{s-3+\frac{2}{\rho_2}}_{p,r}$. *Let* u *be a solution of the equation*

$$
\partial_t u-\mu\Delta u=f,\quad u|_{t=0}=u_0.
$$

Then for $t \in [0, T]$, there holds

$$\mu^{\frac{1}{\rho_1}} \|u\|_{\tilde{L}_T^{\rho_1} \dot{B}_{p,r}^{s-1+2/\rho_1}} \leq C\big(\|u_0\|_{\dot{B}_{p,r}^{s-1}} + \mu^{1/\rho_2 - 1} \|f\|_{\tilde{L}_T^{\rho_2} \dot{B}_{p,r}^{s-3+\frac{2}{\rho_2}}}\big). \tag{5.2}$$

In order to obtain the L^∞ estimate of the original variable F with respect to time t, we need the estimate for the transport equation.

Lemma 5.4. *Let $s \in (-n \min(1/p, 1/p'), 1 + n/p)$ and $1 \leq p, q \leq \infty$. Let v be a vector field such that $\nabla v \in L_T^1 \dot{B}_{p,1}^{n/p}$. Assume that $f_0 \in \dot{B}_{p,q}^s$, $g \in L_T^1 \dot{B}_{p,q}^s$, and f is a solution of the transport equation*

$$\partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0.$$

Then for $t \in [0, T]$, it holds that

$$\|f\|_{\tilde{L}_t \dot{B}_{p,q}^s} \leq \exp\big(C \int_0^t \|\nabla v(\tau)\|_{\dot{B}_{p,1}^{n/p}} d\tau\big)\big(\|f_0\|_{\dot{B}_{p,q}^s} + \int_0^t \|g(\tau)\|_{\dot{B}_{p,q}^s} d\tau\big).$$

The standard product estimate is also used in our analysis.

Proposition 5.1 *([7]). Let $1 \leq r, r_1, r_2 \leq \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $s, t \leq n/2$, $s + t \geq 0$. Then we have*

$$\|fg\|_{\tilde{L}_t^r \dot{B}_{2,1}^{s+t-n/2}} \lesssim \|f\|_{\tilde{L}_t^{r_1} \dot{B}_{2,1}^s} \|g\|_{\tilde{L}_t^{r_2} \dot{B}_{2,1}^t}. \tag{5.3}$$

In addition, we develop a product estimate in the framework of hybrid Besov spaces by using Bony's decompositions. Let us by denote $\chi\{\cdot\}$ the characteristic function in $\mathbb{Z}$ and $\{c(j)\}_{j \in \mathbb{Z}}$ be some sequence on ℓ^1 satisfying $\|\{c(j)\}\|_{\ell^1} = 1$.

Lemma 5.5. *Let $s, t, \sigma, \tau \in \mathbb{R}$. Then we have the following:*

(i) For $2^j \leq R_0$, if $s \leq n/2$, then

$$\|\dot{\Delta}_j (T_f g)\|_{L^2} \leq C c(j) 2^{j(n/2-s-t)} \|f\|_{\dot{B}_{2,1}^s}^\ell \|g\|_{\dot{B}_{2,1}^t}^\ell. \tag{5.4}$$

(ii) For $2^j > R_0$, if $s, \sigma \leq n/2$, then

$$\begin{aligned} &\|\dot{\Delta}_j (T_f g)\|_{L^2} \\ \leq& C c(j)\big(2^{j(n/2-s-t)} \|f\|_{\dot{B}_{2,1}^s}^\ell \|g\|_{\dot{B}_{2,1}^t}^h + 2^{j(n/2-\sigma-\tau)} \|f\|_{\dot{B}_{2,1}^\sigma}^h \|g\|_{\dot{B}_{2,1}^\tau}^h\big). \end{aligned} \tag{5.5}$$

Proof. With our choice of φ in Introduction, it is easy to see that

$$\begin{aligned} &\dot{\Delta}_j \dot{\Delta}_k f = 0 \quad \text{if } |j-k| \geq 2, \\ &\dot{\Delta}_j (\dot{S}_{k-1} f \dot{\Delta}_k f) = 0 \quad \text{if } |j-k| \geq 5. \end{aligned} \tag{5.6}$$

Thanks to (5.6), we have

$$\begin{aligned}\dot\Delta_j(T_f g) &= \sum_{|k-j|\le 4} \dot\Delta_j(\dot S_{k-1} f \dot\Delta_k g)\\ &= \sum_{|k-j|\le 4}\sum_{k'\le k-2} \dot\Delta_j(\dot\Delta_{k'} f \dot\Delta_k g).\end{aligned}$$

Denote $J := \{(k,k') : |k-j| \le 4, k' \le k-2\}$, then for $2^j \le R_0$,

$$\begin{aligned}\|\dot\Delta_j(T_f g)\|_{L^2} &\le \sum_J \|\dot\Delta_j(\dot\Delta_{k'} f \dot\Delta_k g)\|_{L^2}\\ &\lesssim \sum_J 2^{k's}\|\dot\Delta_{k'} f\|_{L^2} 2^{k'(n/2-s)} 2^{kt}\|\dot\Delta_k g\|_{L^2} 2^{-kt}\\ &\lesssim c(j) 2^{j(n/2-s-t)}\|f\|^{\ell}_{\dot B^s_{2,1}}\|g\|^{\ell}_{\dot B^t_{2,1}}.\end{aligned}$$

Next we turn to prove (5.5). We write $J = J_1 + J_2$, where

$$J_1 = J \cap \{2^{k'} \le R_0\}, \quad J_2 = J \cap \{2^{k'} > R_0\}.$$

For $2^j > R_0$ and $s, \sigma \le n/2$, one has

$$\begin{aligned}&\|\dot\Delta_j(T_f g)\|_{L^2}\\ &\le \sum_J \|\dot\Delta_j(\dot\Delta_{k'} f \dot\Delta_k g)\|_{L^2}\\ &\lesssim \sum_J \|\dot\Delta_{k'} f\|_{L^\infty}\|\dot\Delta_k g\|_{L^2}\\ &\lesssim \sum_{J_1} \|\dot\Delta_{k'} f\|_{L^\infty}\|\dot\Delta_k g\|_{L^2} + \sum_{J_2} \|\dot\Delta_{k'} f\|_{L^\infty}\|\dot\Delta_k g\|_{L^2}\\ &\lesssim \sum_{J_1} 2^{k's}\|\dot\Delta_{k'} f\|_{L^2} 2^{k'(n/2-s)} 2^{kt}\|\dot\Delta_k g\|_{L^2} 2^{-kt}\\ &\quad + \sum_{J_1} 2^{k'\sigma}\|\dot\Delta_{k'} f\|_{L^2} 2^{k'(n/2-\sigma)} 2^{k\tau}\|\dot\Delta_k g\|_{L^2} 2^{-k\tau}\\ &\lesssim c(j) 2^{j(n/2-s-t)}\|f\|^{\ell}_{\dot B^s_{2,1}}\|g\|^{h}_{\dot B^t_{2,1}} + c(j) 2^{j(n/2-\sigma-\tau)}\|f\|^{h}_{\dot B^\sigma_{2,1}}\|g\|^{h}_{\dot B^\tau_{2,1}},\end{aligned}$$

which is just (5.5). □

Lemma 5.6. *Let $s, t, \sigma, \tau \in \mathbb{R}$. Assume that $s + t \ge 0, \sigma + \tau \ge 0$. It holds that*

$$\begin{aligned}&\|\dot\Delta_j R(f,g)\|_{L^2}\\ &\le Cc(j)\big(2^{j(n/2-s-t)}\|f\|^{\ell}_{\dot B^s_{2,1}}\|g\|^{\ell}_{\dot B^t_{2,1}} + 2^{j(n/2-\sigma-\tau)}\|f\|^{h}_{\dot B^\sigma_{2,1}}\|g\|^{h}_{\dot B^\tau_{2,1}}\big).\end{aligned} \tag{5.7}$$

Proof. Thanks to (5.6), we have

$$\dot{\Delta}_j R(f,g) = \sum_{k\ge j-3}\sum_{|k-k'|\le 1} \dot{\Delta}_j(\dot{\Delta}_k f\,\dot{\Delta}_{k'} g).$$

Denote $J := \{(k,k') : k \ge j-3, |k-k'| \le 1\}$ and

$$J_1 = J \cap \{2^{k'} \le R_0\}, \quad J_2 = J \cap \{2^{k'} > R_0\}.$$

Then when $s+\tau \ge 0$, we have

$$\begin{aligned}
&\|\dot{\Delta}_j R(f,g)\|_{L^2}\\
&\lesssim 2^{jn/2}\sum_{(k,k')\in J}\|\dot{\Delta}_k f\,\dot{\Delta}_{k'} g\|_{L^1}\\
&= 2^{jn/2}\sum_{(k,k')\in J_1}\|\dot{\Delta}_k f\,\dot{\Delta}_{k'} g\|_{L^1} + 2^{jn/2}\sum_{(k,k')\in J_2}\|\dot{\Delta}_k f\,\dot{\Delta}_{k'} g\|_{L^1}\\
&\lesssim 2^{jn/2}\sum_{(k,k')\in J_1}\|\dot{\Delta}_k f\|_{L^2}\|\dot{\Delta}_{k'} g\|_{L^2} + 2^{jn/2}\sum_{(k,k')\in J_2}\|\dot{\Delta}_k f\|_{L^2}\|\dot{\Delta}_{k'} g\|_{L^2}\\
&\lesssim 2^{jn/2}\sum_{(k,k')\in J_1}2^{ks}\|\dot{\Delta}_k f\|_{L^2}2^{-ks}2^{k't}\|\dot{\Delta}_{k'} g\|_{L^2}2^{-k't}\\
&\quad +2^{jn/2}\sum_{(k,k')\in J_2}2^{k\sigma}\|\dot{\Delta}_k f\|_{L^2}2^{-k\sigma}2^{k'\tau}\|\dot{\Delta}_{k'} g\|_{L^2}2^{-k'\tau}\\
&\lesssim c(j)2^{j(n/2-s-t)}\|f\|^{\ell}_{\dot{B}^s_{2,1}}\|g\|^{\ell}_{\tilde{L}^{r_2}_t\dot{B}^t_{2,1}} + c(j)2^{j(n/2-\sigma-\tau)}\|f\|_{\dot{B}^\sigma_{2,1}}\|g\|_{\tilde{L}^{r_2}_t\dot{B}^\tau_{2,1}}.
\end{aligned}$$

This finishes the proof of Lemma 5.6. □

Having above estimates of para-product operator and remaining operator, one can get the key estimate for product.

Proposition 5.2. *It holds that*

$$\|fg\|_{\dot{B}^{n/2-1}_{2,1}} \lesssim \|f\|_{\dot{\mathcal{B}}^{n/2,n/2-1}}\|g\|_{\dot{\mathcal{B}}^{n/2-1,n/2}}. \tag{5.8}$$

Proof. By Bony's decomposition, we write $fg = T_f g + T_g f + R(f,g)$. At low frequencies, we take $s = n/2, t = n/2-1$ in (5.4) for $T_f g$ and $s = n/2-1, t = n/2$ in (5.4) for $T_g f$. Then we get

$$\sum_{2^j\le R_0} 2^{j(n/2-1)}\big(\|\dot{\Delta}_j(T_f g)\|_{L^2} + \|\dot{\Delta}_j(T_g f)\|_{L^2}\big) \lesssim \|f\|^{\ell}_{\dot{B}^{n/2}_{2,1}}\|g\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}}. \tag{5.9}$$

For the high frequency, we choose $s = n/2, t = n/2-1, \sigma = n/2-1, \tau = n/2$ in (5.5) for $T_f g$; $s = n/2-1, t = n/2, \sigma = n/2, \tau = n/2-1$ in (5.5) for $T_g f$, which lead to

$$\sum_{2^j\geq R_0} 2^{j(n/2-1)}\big(\|\dot{\Delta}_j(T_f g)\|_{L^2}+\|\dot{\Delta}_j(T_g f)\|_{L^2}\big)\lesssim \|f\|_{\dot{\mathcal{B}}^{n/2,n/2-1}}\|g\|_{\dot{\mathcal{B}}^{n/2-1,n/2}}. \tag{5.10}$$

Finally, we choose $s=n/2, t=n/2-1, \sigma=n/2-1, \tau=n/2$ in (5.7) and get

$$\sum_{j\in\mathbb{Z}} 2^{j(n/2-1)}\|\dot{\Delta}_j(R(f,g)\|_{L^2}\lesssim \|f\|^{\ell}_{\dot{B}^{n/2}_{2,1}}\|g\|^{\ell}_{\dot{\mathcal{B}}^{n/2-1}}+\|f\|^{h}_{\dot{B}^{n/2-1}_{2,1}}\|g\|^{h}_{\dot{B}^{n/2}_{2,1}}. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), we arrive at (5.8). Therefore, the proof of Proposition 5.2 is completed. □

Finally, let us point out the new product estimate remains valid in Chemin-Lerner's spaces whereas the time exponent r behaves according to the Hölder inequality.

Remark 5.1. The inequality

$$\|fg\|_{\tilde{L}^r_t\dot{B}^{n/2-1}_{2,1}}\lesssim \|f\|_{\tilde{L}^{r_1}_t\dot{\mathcal{B}}^{n/2,n/2-1}}\|g\|_{\tilde{L}^{r_2}_t\dot{\mathcal{B}}^{n/2-1,n/2}} \tag{5.12}$$

holds whenever $1\leq r, r_1, r_2\leq\infty$ and $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$.

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