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ONE COMPONENT REGULARITY CRITERIA FOR THE AXIALLY SYMMETRIC MHD-BOUSSINESQ SYSTEM

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Abstract. In this paper, we consider regularity criteria of a class of 3D axially symmetric MHD-Boussinesq systems without magnetic resistivity or thermal diffusivity. Under some Prodi-Serrin type critical assumptions on the horizontal angular component of the velocity, we will prove that strong solutions of the axially symmetric MHD-Boussinesq system can be smoothly extended beyond the possible blow-up time T_* if the magnetic field contains only the horizontal swirl component. No a priori assumption on the magnetic field or the temperature fluctuation is imposed.

1. **Introduction.**

In this paper, we study the 3D MHD-Boussinesq system without magnetic resistivity and thermal diffusivity:

$$\begin{cases}\partial_t u + u\cdot\nabla u + \nabla p - \mu\Delta u = h\cdot\nabla h + \rho e_3,\\ \partial_t h + u\cdot\nabla h - h\cdot\nabla u = 0,\\ \partial_t \rho + u\cdot\nabla\rho = 0,\\ \nabla\cdot u = \nabla\cdot h = 0.\end{cases} \tag{1.1}$$

Here $u\in\mathbb{R}^3$ is the velocity and $h\in\mathbb{R}^3$ is the magnetic field, while $p\in\mathbb{R}$ and $\rho\in\mathbb{R}$ represent the pressure and the temperature fluctuation, respectively. $e_3=(0,0,1)^T$ is the unit vector in the vertical direction, and $\mu>0$ stands for the viscosity constant, which is assumed to be one without loss of generality in the following.

Physically, equation $(1.1)_1$ describes the conservation law of the momentum with the influence of buoyant effect ρe_3, while $(1.1)_2$ is the non-resistive Maxwell-Faraday

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equation which describes the Faraday's law of induction. The third line of (1.1) represents the ideal temperature fluctuation, while the fourth line describes the incompressibility of the fluid and Gauss's law for magnetism. The MHD-Boussinesq system, which models the convection of an incompressible conductive flow driven by the Lorenz force and buoyant effect of a thermal field, plays a vital role in atmospheric science and geophysical applications. It is closely related to a type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with a presence of a magnetic field. For detailed physical background, we refer readers to [28, 24, 23, 26].

Our main result and its proof will be presented in the cylindrical coordinates $(r,\, \theta,\, z)$. For $x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$, we denote

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3,$$

and

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

We say a solution of (1.1) is axially symmetric, if and only if

$$\begin{cases} u = u^r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u^z(t, r, z)e_z, \\ h = h^r(t, r, z)e_r + h^\theta(t, r, z)e_\theta + h^z(t, r, z)e_z, \\ \rho = \rho(t, r, z), \end{cases}$$

satisfy the system (1.1). By the local existence and uniqueness results, it is clear that one only needs to assume $h_0^r = h_0^z \equiv 0$, then vanishing of h^r and h^z holds for all time (see [18]). In this case, (1.1) can be rewritten as

$$\begin{cases} \partial_t u^r + (u^r \partial_r + u^z \partial_z) u^r - \dfrac{(u^\theta)^2}{r} + \partial_r P = -\dfrac{(h^\theta)^2}{r} + \left(\Delta - \dfrac{1}{r^2}\right) u^r, \\ \partial_t u^\theta + (u^r \partial_r + u^z \partial_z) u^\theta + \dfrac{u^\theta u^r}{r} = \left(\Delta - \dfrac{1}{r^2}\right) u^\theta, \\ \partial_t u^z + (u^r \partial_r + u^z \partial_z) u^z + \partial_z P = \Delta u^z + \rho, \\ \partial_t h^\theta + (u^r \partial_r + u^z \partial_z) h^\theta - \dfrac{h^\theta u^r}{r} = 0, \\ \partial_t \rho + (u^r \partial_r + u^z \partial_z) \rho = 0, \\ \nabla \cdot u = \partial_r u^r + \dfrac{u^r}{r} + \partial_z u^z = 0, \end{cases} \tag{1.2}$$

where $P := p + \frac{1}{2}|h^\theta|^2$. To state the regularity theorem of the initial value problem of the axially symmetric solution of (1.1), we present here a Prodi-Serrin type condition on the horizontal swirl component of the velocity:

Condition 1.1. For any $s \geq 0$,

$$\int_0^{T_*} \left\| \frac{u^\theta}{r^s}(t, \cdot) \right\|_{L^p}^q dt < \infty, \quad \text{where} \quad \frac{3}{p} + \frac{2}{q} \leq 1 + s \quad \text{for} \quad \frac{3}{1+s} < p \leq \infty. \tag{1.3}$$

Meanwhile, for the borderline case $p = \frac{3}{1+s}$, we assume that:

Condition 1.2. For any $s \geq 0$,

$$\sup_{0\leq t\leq T_*}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}} < \varepsilon_0, \tag{1.4}$$

where $\varepsilon_0 = \varepsilon_0\left(s,\, ru_0^\theta\right) << 1$ will be decided in the proof of Theorem 1.3.

Now we are ready for the main result:

Theorem 1.3. *Let (u, h, ρ) be the classical axially symmetric solution of (1.1) whose initial data $(u_0, h_0, \rho_0) \in H^m(\mathbb{R}^3)$, for $m \geq 3$, and $\nabla \cdot u_0 = h_0^r = h_0^z = 0$. Then (u, h, ρ) can be smoothly extended beyond T_* if and only if Condition 1.1 or Condition 1.2 holds.*

Critical regularity criteria of incompressible fluid dynamic systems date back to pioneer works of G. Prodi [29] and J. Serrin [31, 32] around the 1960s, where the famous Prodi-Serrin criterion for 3D Naiver-Stokes equations was given. Readers can see [10, 11, 33, 34, 2] for more regularity results on the Navier-Stokes equations.

If the fluid (say e.g., plasma) is affected by the Lorentz force, then the Navier-Stokes system is generalized to the magnetohydrodynamics system. Many fruitful studies and researches on the MHD system has been achieved in recent years. Some partial regularity and blow-up criteria could be found in [8, 12] and references therein. Lin-Xu-Zhang [21] proved the global well-posedness of classical solutions to the 2D non-resistive MHD system with smooth initial data which is close to a non-trivial steady state. See also [30] for similar results. Lei [19] proved the global regularity of classical solutions to a 3D MHD system with a family of large axisymmetric data. Moreover, if the temperature influences the fluid, then the fluid equations can be modeled by the classical Boussinesq system. Hou-Li [14] and Chae [5] independently proved the global regularity of solutions to the 2D Boussinesq system. Abidi et al. [1] and Hmidi-Rousset [13] proved the global well-posedness of the Cauchy problem for the 3D axisymmetric Boussinesq system without swirl. To overcome difficulties coming from the temperature fluctuation ρ, authors in [13] firstly proved the boundedness of operators $\mathcal{L} = \left(\Delta + \frac{2}{r}\partial_r\right)^{-1}\frac{\partial_r}{r}$ and $\tilde{\mathcal{L}} = \left(\Delta + \frac{2}{r}\partial_r\right)^{-1}\frac{\partial_z}{r}$ in L^p spaces ($p \geq 2$). This strategy will also be applied in the proof of our main theorem later in this paper. For more regularity results on the Boussinesq system, we refer readers to [17, 4] and references therein.

Recently there are more and more studies concerning the full 3D MHD-Boussinesq system. We refer readers to papers such as [18, 22, 3, 27] for the regularity criteria, local and global well-posedness of weak and strong solutions of the MHD-Boussinesq system. The local well-posedness results were proved in Larios-Pei [18]. If a nonlinear damping term is added in the momentum equations, Liu-Bian-Pu [22] proved the global well-posedness of strong solutions. Recently, Bian-Pu [3] proved the global regularity of axially symmetric large solutions to the MHDB system (1.2) without the horizontal swirl component u^θ of the velocity under the assumption that the support of the initial thermal fluctuation is away from the z-axis and its projection on to the z-axis is compact. Later, this result was improved in Pan [27] by removing the "support set" assumption on the initial data of the thermal fluctuation.

Throughout the paper, $C_{a,b,c,\ldots}$ denotes a positive constant depending on a, b, c, ... which may be different from line to line. We also apply $A \lesssim B$ to denote $A \leq CB$. Meanwhile, $A \simeq B$ means both $A \lesssim B$ and $B \lesssim A$. $[\mathcal{A},\, \mathcal{B}] := \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ denotes the communicator of the operator $\mathcal{A}$ and the operator $\mathcal{B}$. For a domain D, and $1 \le p \le \infty$, we denote $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. When $D = \mathbb{R}^3$, we simply write $\|\cdot\|_{L^p(D)} = \|\cdot\|_{L^p}$. For any $m \in \mathbb{N}$, $H^m = H^m(\mathbb{R}^3)$ denotes the L^2-based Sobolev space $W^{m,2}(\mathbb{R}^3)$. Given a Banach space X, we say $v : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ belongs to the Bochner space $L^p(0,T;X)$, if

$$\|v(t,\cdot)\|_X \in L^p(0,T),$$

and we usually use $L^p_T X$ for short notation of $L^p(0,T;X)$.

Our proof of the main result in this paper consists of the following steps: First, we investigate a reformulated system (3.8) which is motivated by [13, 7] and derive a closed $L^\infty_{T_*} L^2 \cap L^2_{T_*} H^1$ estimate of (3.8) under the condition (1.3) or (1.4). Based on this estimate, we further derive the $L^\infty_{T_*} L^2 \cap L^2_{T_*} H^1$ estimate of ∇u (3.25). Using the maximal regularity result of the heat flow, we arrive the $L^1_{T_*} L^\infty$ estimate of ∇u (3.31), then the $L^1_{T_*} L^\infty$ estimates of $\nabla \times h$ and $\nabla \rho$, (3.40) and (3.41), follow. Finally, using these $L^1_{T_*} L^\infty$ estimates, the estimates of higher-order norms of the solution follow from a classical communicator estimate by Kato-Ponce [15].

The remaining of this paper is organized as follows. In Section 2, we provide some useful Lemmas concerning interpolation inequalities, some L^p boundedness of singular operators related to the problem, a Hardy type inequality, the maximal regularity for the heat flow, and logarithmic imbedding inequalities. Finally, in Section 3, we provide the proof of regularity criteria in Theorem 1.3.

2. Preliminaries.

At the beginning, let us introducte the well-known *Gagliardo−Nirenberg* interpolation inequality. We list here without proof.

Lemma 2.1 (Gagliardo-Nirenberg)**.** *Fix $q, r \in [1,\infty]$ and $j, m \in \mathbb{N} \cup \{0\}$ with $j \le m$. Suppose that $f \in L^q(\mathbb{R}^d) \cap \dot{W}^{m,r}(\mathbb{R}^d)$ and there exists a real number $\alpha \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{d} + \alpha\left(\frac{1}{r} - \frac{m}{d}\right) + \frac{1-\alpha}{q}.$$

Then $f \in \dot{W}^{j,p}(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that

$$\|\nabla^j f\|_{L^p(\mathbb{R}^d)} \le C\|\nabla^m f\|^{\alpha}_{L^r(\mathbb{R}^d)} \|f\|^{1-\alpha}_{L^q(\mathbb{R}^d)},$$

except the following two cases:

(i) $j = 0$, $mr < d$ and $q = \infty$; (In this case it is necessary to assume also that either $|u| \to 0$ at infinity, or $u \in L^s(\mathbb{R}^d)$ for some $s < \infty$.)

(ii) $1 < r < \infty$ and $m - j - d/r \in \mathbb{N}$. (In this case it is necessary to assume also that $\alpha < 1$.)

In the following we state a useful space-time interpolation which is frequently used in the research of Navier-Stokes equations:

Lemma 2.2. *If $u \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1)$, then*

$$u \in L^q\left(0,T;L^p\right), \tag{2.1}$$

where

$$\frac{2}{q} + \frac{3}{p} \ge \frac{3}{2}, \quad 2 \le p \le 6.$$

Proof. The Sobolev inequality implies $u \in L^2\left(0,T;L^6\right)$. Then we interpolate the L^s norm between L^2 and L^6 to derive

$$\|u(t,\cdot)\|_{L^p} \le \|u(t,\cdot)\|_{L^2}^{(6-p)/2p}\|u(t,\cdot)\|_{L^6}^{(3p-6)/2p}.$$

This indicates

$$\int_0^T \|u(t,\cdot)\|_{L^p}^q dt \le \int_0^T \|u(t,\cdot)\|_{L^2}^{(6-p)q/2p}\|u(t,\cdot)\|_{L^6}^{(3p-6)q/2p} dt.$$

Since $u \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;L^6\right)$, the integral on the right-hand side of the above inequality is bounded when $(3p-6)q/2p \le 2$, which corresponds to

$$\frac{2}{q} + \frac{3}{p} \ge \frac{3}{2}.$$

□

Next, we focus on the following estimates of a triple product form with commutator:

Lemma 2.3. *Let $m \in \mathbb{N}$, $m \ge 2$, and $f,g,k \in C_0^\infty(\mathbb{R}^3)$. Then the following estimate holds:*

$$\left|\int_{\mathbb{R}^3} [\nabla^m,\, f\cdot\nabla] g \nabla^m k dx\right| \le C\, \|\nabla^m (f,g,k)\|_{L^2}^2\, \|\nabla (f,\, g)\|_{L^\infty}. \tag{2.2}$$

Proof. Applying Hölder's inequality, one derives

$$\left|\int_{\mathbb{R}^3} [\nabla^m,\, f\cdot\nabla] g \nabla^m k dx\right| \le \|[\nabla^m,\, f\cdot\nabla] g\|_{L^2}\|\nabla^m k\|_{L^2}. \tag{2.3}$$

Due to the commutator estimate by Kato-Ponce [15], it follows that

$$\|[\nabla^m,\, f\cdot\nabla] g\|_{L^2} \le C\left(\|\nabla f\|_{L^\infty}\|\nabla^m g\|_{L^2} + \|\nabla g\|_{L^\infty}\|\nabla^m f\|_{L^2}\right). \tag{2.4}$$

Then (2.2) follows from plugging (2.4) into (2.3). □

The following lemma was introduced by Hmidi-Rousset [13] where the authors derived regularity of the axisymmetric Boussinesq system without swirl. It states the L^p-boundedness of two operators related to axially symmetric vector fields.

Lemma 2.4. *Denote $\mathcal{L} = \left(\Delta + \frac{2}{r}\partial_r\right)^{-1}\frac{\partial_r}{r}$ and $\tilde{\mathcal{L}} = \left(\Delta + \frac{2}{r}\partial_r\right)^{-1}\frac{\partial_z}{r}$. Suppose $\rho \in H^2\left(\mathbb{R}^3\right)$ be axisymmetric, then for every $p \in [2,+\infty)$, there exists an absolute constant $C_p > 0$ such that*

$$\|\mathcal{L}\rho\|_{L^p} \le C_p\|\rho\|_{L^p}, \quad \|\tilde{\mathcal{L}}\rho\|_{L^p} \le C_p\|\rho\|_{L^p}.$$

Moreover, for any smooth axisymmetric function f, we have the identity

$$\mathcal{L}\partial_r f = \frac{f}{r} - \mathcal{L}\left(\frac{f}{r}\right) - \partial_z\tilde{\mathcal{L}} f.$$

Proof. The detailed proof can be found in Proposition 3.1, 3.2 and Lemma 3.3 in [13]. We omit the details here. □

The following famous estimate will be applied later in our proof.

Lemma 2.5. *Let $u = u^r e_r + u^\theta e_\theta + u^z e_z$ be an axially symmetric vector field on $\mathbb{R}^3$, $w = \nabla\times u = w^r e_r + w^\theta e_\theta + w^z e_z$. Define $\Omega := \frac{w^\theta}{r}$. For $1 < p < +\infty$, there exists an absolute constant $C_p > 0$ such that*

$$\left\|\nabla\frac{u^r}{r}(t,\cdot)\right\|_{L^p} \le C_p\|\Omega(t,\cdot)\|_{L^p}.$$

The proof of this lemma can be founded in many literatures, such as [19] (equation (A.5)) and [25] (Proposition 2.5).

Next we give a Sobolev-Hardy inequality. We omit the detailed proof since it could be found in the Lemma 2.4 of [7].

Lemma 2.6. *Set $\mathbb{R}^d=\mathbb{R}^k\times\mathbb{R}^{d-k}$ with $2\le k\le d$, and write $x=(x',z)\in\mathbb{R}^k\times\mathbb{R}^{d-k}$. For $1<q<d, 0\le\theta\le q$ and $\theta<k$, let $q_*\in\left[q,\frac{q(d-\theta)}{d-q}\right]$. Then there exists a positive constant $C=C(\theta,q,d,k)$ such that for all $f\in C_0^\infty\left(\mathbb{R}^d\right)$,*

$$\left(\int_{\mathbb{R}^d}\frac{|f|^{q_*}}{|x'|^\theta}dx\right)^{\frac{1}{q_*}}\le C\|f\|_{L^q}^{\frac{d-\theta}{q_*}-\frac{d}{q}+1}\|\nabla f\|_{L^q}^{\frac{d}{q}-\frac{d-\theta}{q_*}}.$$

In particular, we pick $d=3,k=2,q=2,q_*\in[2,2(3-\theta)]$, and assume $0\le\theta<2, r=\sqrt{x_1^2+x_2^2}$. Then there exists a positive constant $C=C\left(q_*,\theta\right)$ such that for all $f\in C_0^\infty\left(\mathbb{R}^d\right)$

$$\left\|\frac{f}{r^{\frac{\theta}{q_*}}}\right\|_{L^{q_*}}\le C\|f\|_{L^2}^{\frac{3-\theta}{q_*}-\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{3}{2}-\frac{3-\theta}{q_*}}. \tag{2.5}$$

Using the Biot-Savart law and the L^p boundedness of Calderon-Zygmund singular integral operators, we have the following lemma whose detailed proof can be found for example in [6, 9].

Lemma 2.7. *Let $u=u^re_r+u^\theta e_\theta+u^ze_z$ be an axially symmetric vector field on $\mathbb{R}^3$, $w=\nabla\times u=w^re_r+w^\theta e_\theta+w^ze_z$ and $b=u^re_r+u^ze_z$. Then we have*

$$\|\nabla u\|_{L^p}\le C_p\|w\|_{L^p},\quad \|\nabla^2u\|_{L^p}\le C_p\|\nabla w\|_{L^p}, \tag{2.6}$$

and

$$\|\nabla b\|_{L^p}\le C_p\|w^\theta\|_{L^p},\quad \|\nabla^2 b\|_{L^p}\le C_p\left(\|\nabla w^\theta\|_{L^p}+\left\|\frac{w^\theta}{r}\right\|_{L^p}\right), \tag{2.7}$$

for all $1<p<\infty$.

Now we recall the standard maximal regularity of heat flows in $L^r_TL^p$-type spaces. Detailed proof could be found in [20, Theorem 7.3] for instance.

Lemma 2.8 (Maximal $L^r_TL^p$ regularity for the heat flow)**.** *Let us define the operator $\mathcal{A}$ by the formula*

$$\mathcal{A}:\quad f\longmapsto\int_0^t\nabla^2e^{(t-s)\Delta}f(s,\cdot)ds.$$

Then $\mathcal{A}$ is bounded from $L^r\left(0,T;L^p(\mathbb{R}^d)\right)$ to $L^r\left(0,T;L^p(\mathbb{R}^d)\right)$ for every $T\in(0,\infty]$ and $1<p,r<\infty$. Moreover, there holds:

$$\|\mathcal{A}f\|_{L^r_TL^p}\le C\|f\|_{L^r_TL^p}. \tag{2.8}$$

Finally, we recall the following logarithmic imbedding inequality which is proved in [16].

Lemma 2.9. *Let $1<p<\infty$ and $s>d/p$. There exists a constant $C=C_{d,p,s}$ such that the estimate*

$$\|f\|_{L^\infty(\mathbb{R}^d)}\le C\left(1+\|f\|_{BMO(\mathbb{R}^d)}\log(e+\|f\|_{W^{s,p}(\mathbb{R}^d)})\right) \tag{2.9}$$

holds for all $f\in W^{s,p}(\mathbb{R}^d)$.

In this paper, the following corollary of Lemma 2.9 is more convenient for us. That is:

Corollary 2.10. *For any divergence free vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$ such that $g \in H^3(\mathbb{R}^3)$, the following estimate holds:*

$$\|\nabla g\|_{L^\infty(\mathbb{R}^3)} \lesssim 1 + \|\nabla \times g\|_{BMO(\mathbb{R}^3)} \log\left(e + \|g\|_{H^3(\mathbb{R}^3)}\right). \tag{2.10}$$

Proof. Using Fourier transform and noting that

$$\xi \otimes \hat{g} = -\frac{\xi}{|\xi|} \otimes \left(\frac{\xi}{|\xi|} \times (\xi \times \hat{g})\right)$$

provided $\xi \cdot \hat{g} \equiv 0$, (2.10) is proven by combining the estimate (2.9) and the fact that the Riesz operator is bounded in the BMO space. □

3. **Proof of Theorem 1.3.** In this section, we focus on the proof of Theorem 1.3. Denoting $\Gamma := ru^\theta$ and $H := \frac{h^\theta}{r}$, by $(1.2)_2$ and $(1.2)_4$, one derives that

$$\partial_t \Gamma + (u^r \partial_r + u^z \partial_z)\Gamma = \left(\Delta - \frac{2}{r}\partial_r\right)\Gamma.$$

$$\partial_t H + (u^r \partial_r + u^z \partial_z) H = 0.$$

At the beginning, the following Lemma states fundamental estimates of the system (1.2):

Lemma 3.1 (Fundamental Energy Estimates)**.** *Let (u, h, ρ) be a smooth solution of (1.2), then we have*

(i) For $p \in [1, \infty]$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} \|\Gamma(t,\cdot)\|_{L^p} &\le \|\Gamma_0\|_{L^p}; \\ \|H(t,\cdot)\|_{L^p} &\le \|H_0\|_{L^p}; \\ \|\rho(t,\cdot)\|_{L^p} &\le \|\rho_0\|_{L^p}. \end{aligned} \tag{3.1}$$

(ii) For $u_0, h_0, \rho_0 \in L^2$ and $t \in \mathbb{R}_+$,

$$\|(u,h)(t,\cdot)\|_{L^2}^2 + \int_0^t \|\nabla u(s,\cdot)\|_{L^2}^2 ds \le C_0(1+t)^2, \tag{3.2}$$

where C_0 depends only on $\|(u_0, h_0, \rho_0)\|_{L^2}$.

Proof. The estimate in (3.1) is classical for the heat equation when $p < \infty$ and follows from the maximum principle when $p = \infty$. Meanwhile, (3.2) follows from the standard L^2 estimate of the system (1.1), together with the result in (3.1). See also [27, Proposition 2.1]. We omit the details here. □

3.1. **$L^\infty_{T_*} L^2 \cap L^2_{T_*} H^1$ estimate of a reformulated system.** First we see the vorticity w of the axially symmetric velocity u is defined by

$$w = \nabla \times u = w^r(t,r,z)e_r + w^\theta(t,r,z)e_\theta + w^z(t,r,z)e_z,$$

where

$$w^r = -\partial_z u^\theta, \quad w^\theta = \partial_z u^r - \partial_r u^z, \quad w^z = \partial_r u^\theta + \frac{u^\theta}{r}.$$

By the first three equations of (1.2), (w^r, w^θ, w^z) satisfies

$$\begin{cases} \partial_t w^r + (u^r\partial_r + u^z\partial_z)w^r = \left(\Delta - \frac{1}{r^2}\right) w^r + (w^r\partial_r + w^z\partial_z)u^r, \\ \\ \partial_t w^\theta + (u^r\partial_r + u^z\partial_z)w^\theta = \left(\Delta - \frac{1}{r^2}\right) w^\theta + \frac{u^r}{r}w^\theta + \frac{1}{r}\partial_z(u^\theta)^2 - \frac{1}{r}\partial_z(h^\theta)^2 - \partial_r\rho, \\ \\ \partial_t w^z + (u^r\partial_r + u^z\partial_z)w^z = \Delta w^z + (w^r\partial_r + w^z\partial_z)u^z. \end{cases} \tag{3.3}$$

Applying $\mathcal{L} = \left(\Delta + \frac{2}{r}\partial_r\right)^{-1}\frac{\partial_r}{r}$ to the equation of ρ, one derives

$$\partial_t \mathcal{L}\rho + u\cdot\nabla\mathcal{L}\rho = -[\mathcal{L}, u\cdot\nabla]\rho. \tag{3.4}$$

Meanwhile, $(3.3)_2$ indicates $\Omega := \frac{w^\theta}{r}$ satisfies

$$\partial_t\Omega + u\cdot\nabla\Omega = \left(\Delta + \frac{2}{r}\partial_r\right)\Omega - \partial_z H^2 - \frac{\partial_r\rho}{r} - \frac{2u^\theta w^r}{r^2}. \tag{3.5}$$

Now we denote $L := \Omega - \mathcal{L}\rho$. Subtracting (3.4) from (3.5) and noting the axially symmetric condition, we have

$$\partial_t L + (u^r\partial_r + u^z\partial_z)L = \left(\Delta + \frac{2}{r}\partial_r\right) L - \partial_z H^2 + [\mathcal{L}, u\cdot\nabla]\rho - 2\frac{u^\theta w^r}{r^2}. \tag{3.6}$$

On the other hand, by denoting $J := \frac{w^r}{r}$, we can get the following equation from $(3.3)_1$:

$$\partial_t J + (u^r\partial_r + u^z\partial_z)J = \left(\Delta + \frac{2}{r}\partial_r\right) J + (w^r\partial_r + w^z\partial_z)\frac{u^r}{r}. \tag{3.7}$$

Therefore, we have the following reformulated system by combining (3.6) and (3.7):

$$\begin{cases} \partial_t L + (u^r\partial_r + u^z\partial_z)L = \left(\Delta + \frac{2}{r}\partial_r\right) L - \partial_z H^2 + [\mathcal{L}, u\cdot\nabla]\rho - 2\frac{u^\theta}{r}J, \\ \partial_t J + (u^r\partial_r + u^z\partial_z)J = \left(\Delta + \frac{2}{r}\partial_r\right) J + (w^r\partial_r + w^z\partial_z)\frac{u^r}{r}. \end{cases} \tag{3.8}$$

Now we are ready for an a prior $L^\infty_{T_*}L^2 \cap L^2_{T_*}H^1$ estimate for the above reformulated system. We have the following Lemma.

Lemma 3.2. *Under the same conditions as Theorem 1.3, the following a priori estimate of* (L, J) *holds:*

$$\sup_{0\le t\le T_*}\|(L,J)(t,\cdot)\|^2_{L^2} + \int_0^{T_*}\|\nabla(L,J)(t,\cdot)\|^2_{L^2}dt < \infty. \tag{3.9}$$

Proof. Performing the L^2 inner product of $(3.8)_1$, using integration by parts and divergence-free condition, one finds

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|L(t,\cdot)\|^2_{L^2} + \|\nabla L(t,\cdot)\|^2_{L^2} \\ \le& \int_{\mathbb{R}^3}\mathcal{L}(u\cdot\nabla\rho)Ldx - \int_{\mathbb{R}^3}u\cdot\nabla(\mathcal{L}\rho)Ldx - \int_{\mathbb{R}^3}\partial_z H^2 Ldx - 2\int_{\mathbb{R}^3}\frac{u^\theta}{r}JLdx \\ =& \int_{\mathbb{R}^3}\mathcal{L}(u\cdot\nabla\rho)Ldx + \int_{\mathbb{R}^3}(\mathcal{L}\rho)u\cdot\nabla Ldx + \int_{\mathbb{R}^3}H^2\partial_z Ldx - 2\int_{\mathbb{R}^3}\frac{u^\theta}{r}JLdx \\ :=& I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.10}$$

Using the method in the proof of Proposition 2.2 of [27], the first 3 terms above can be estimated by

$$\begin{aligned}\sum_{j=1}^{3} I_j \leq & C_{h_0,\rho_0}\left(1+\|\nabla u(t,\cdot)\|_{L^2}^2\right)+C(1+t)^2 \\ & + C_{\rho_0}\left(1+\|L(t,\cdot)\|_{L^2}^2\right)+\frac{1}{4}\|\nabla L(t,\cdot)\|_{L^2}^2.\end{aligned} \tag{3.11}$$

Meanwhile, using the Cauchy-Schwartz inequality, I_4 can be estimated by

$$\begin{aligned}I_4 \leq & \frac{1}{2}\int_{\mathbb{R}^3}\frac{|u^\theta|}{r}|L|^2 dx+\frac{1}{2}\int_{\mathbb{R}^3}\frac{|u^\theta|}{r}|J|^2 dx \\ := & I_{41}+I_{42}.\end{aligned} \tag{3.12}$$

For any $p \in [\frac{3}{1+s}, \infty]$, we estimate I_{41} and I_{42} in the following 2 cases:

Case I: $0 \leq s \leq 1$.

We use Hölder inequality to derive

$$I_{41} = \int_{\mathbb{R}^3}\frac{u^\theta}{r^s}\frac{|L|^2}{r^{1-s}}dx \leq \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}\left(\int_{\mathbb{R}^3}\left|\frac{L^{2p'}}{r^{(1-s)p'}}\right|dx\right)^{1/p'}, \tag{3.13}$$

where $p' = \frac{p}{p-1}$ is the conjugate number of p. By choosing $\theta = (1-s)p'$, $q_* = 2p'$ in (2.5) of Lemma 2.6, one finds that

$$\left(\int_{\mathbb{R}^3}\left|\frac{L^{2p'}}{r^{(1-s)p'}}\right|dx\right)^{1/p'} \leq C_{s,p}\|L(t,\cdot)\|_{L^2}^{1+s-\frac{3}{p}}\|\nabla L(t,\cdot)\|_{L^2}^{1-s+\frac{3}{p}}. \tag{3.14}$$

Substituting (3.14) in (3.13) and using Young inequality, one derives that

$$I_{41} \leq \begin{cases} C_{s,p}\left\|\dfrac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}}\|L(t,\cdot)\|_{L^2}^2+\dfrac{1}{4}\|\nabla L(t,\cdot)\|_{L^2}^2, & \text{for } p > \dfrac{3}{1+s}; \\ C_s\left\|\dfrac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}\|\nabla L(t,\cdot)\|_{L^2}^2, & \text{for } p = \dfrac{3}{1+s}. \end{cases} \tag{3.15}$$

Similarly, I_{42} satisfies

$$I_{42} \leq \begin{cases} C_{s,p}\left\|\dfrac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}}\|J(t,\cdot)\|_{L^2}^2+\dfrac{1}{4}\|\nabla J(t,\cdot)\|_{L^2}^2, & \text{for } p > \dfrac{3}{1+s}; \\ C_s\left\|\dfrac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}\|\nabla J(t,\cdot)\|_{L^2}^2, & \text{for } p = \dfrac{3}{1+s}. \end{cases} \tag{3.16}$$

Remark 3.3. *Actually the above estimate in Case I is also feasible for $-1 < s < 0$. However we do not pursue it because the following $L^\infty_{T_*}L^2 \cap L^2_{T_*}H^1$ estimate of J fails in this situation.*

Case II: $s > 1$.

Using Hölder inequality and (3.1) in Lemma 3.1, one finds

$$I_{41} = \int_{\mathbb{R}^3} |ru^\theta|^{\frac{s-1}{s+1}} \left|\frac{u^\theta}{r^s}\right|^{\frac{2}{1+s}} |L|^2 dx \leq \|\Gamma_0\|_{L^\infty}^{\frac{s-1}{s+1}} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2}{1+s}} \|L(t,\cdot)\|^2_{L^{\frac{2p(1+s)}{p(1+s)-2}}}. \tag{3.17}$$

Noting that $\frac{2p(1+s)}{p(1+s)-2} \in [2,6]$ when $p \geq \frac{3}{1+s}$ and applying Lemma 2.1, one has

$$\|L(t,\cdot)\|_{L^{\frac{2p(1+s)}{p(1+s)-2}}} \leq C_{s,p}\|L(t,\cdot)\|_{L^2}^{1-\frac{3}{p(1+s)}} \|\nabla L(t,\cdot)\|_{L^2}^{\frac{3}{p(1+s)}}. \tag{3.18}$$

Thus by inserting (3.18) into (3.17) and using Hölder inequality, the estimate (3.15) is still valid for $s > 1$ with the constant C depending on s, p and $\|\Gamma_0\|_{L^\infty}$. The proof of (3.16) when $s > 1$ is similar. This finishes the estimate of I_4 in (3.10). Plugging (3.11), (3.15) and (3.16) into (3.10), we have the following estimate of L when $p > \frac{3}{1+s}$:

$$\begin{aligned}\frac{d}{dt}\|L(t,\cdot)\|_{L^2}^2 + \|\nabla L(t,\cdot)\|_{L^2}^2 \leq& C_{h_0,\rho_0}\left(\|\nabla u(t,\cdot)\|_{L^2}^2 + (1+t)^2 + \|L(t,\cdot)\|_{L^2}^2\right)\\ &+ C_{s,p,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}} \left(\|L(t,\cdot)\|_{L^2}^2 + \|J(t,\cdot)\|_{L^2}^2\right)\\ &+ \frac{1}{4}\|\nabla L(t,\cdot)\|_{L^2}^2 + \frac{1}{4}\|\nabla J(t,\cdot)\|_{L^2}^2,\end{aligned} \tag{3.19}$$

and the following estimate when $p = \frac{3}{1+s}$:

$$\begin{aligned}\frac{d}{dt}\|L(t,\cdot)\|_{L^2}^2 + \|\nabla L(t,\cdot)\|_{L^2}^2 \leq& C_{h_0,\rho_0}\left(\|\nabla u(t,\cdot)\|_{L^2}^2 + (1+t)^2 + \|L(t,\cdot)\|_{L^2}^2\right)\\ &+ C_{s,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p} \left(\|\nabla L(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2\right).\end{aligned} \tag{3.20}$$

Next we work on the equation of J in (3.8). Taking L^2 inner product of $(3.8)_2$, we arrive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 &= \int_{\mathbb{R}^3} \left(\nabla \times (u^\theta e_\theta)\right)\cdot\left(J\nabla\frac{u^r}{r}\right)dx\\ &= \int_{\mathbb{R}^3} u^\theta e_\theta \cdot \left(\nabla J \times \nabla\frac{u^r}{r}\right)dx\\ &= \int_{\mathbb{R}^3} u^\theta\left(\partial_r\frac{u^r}{r}\partial_z J - \partial_z\frac{u^r}{r}\partial_r J\right)dx\\ &\leq \frac{1}{2}\int_{\mathbb{R}^3}|u^\theta|^2\left|\nabla\frac{u^r}{r}\right|^2 dx + \frac{1}{2}\|\nabla J(t,\cdot)\|_{L^2}^2,\end{aligned}$$

which follows that

$$\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \lesssim \int_{\mathbb{R}^3}|u^\theta|^2\left|\nabla\frac{u^r}{r}\right|^2 dx. \tag{3.21}$$

Different from (3.12), here we should be very careful to avoid the appearance of second-order gradients of $\frac{u^r}{r}$. Even though the following estimate

$$\left\|\nabla^2\frac{u^r}{r}(t,\cdot)\right\|_{L^2} \leq C\|\partial_z\Omega(t,\cdot)\|_{L^2}$$

holds (see [19], equation (A.6)), we still have no idea to bound $\|\partial_z \Omega\|_{L^2}$ due to the appearance of $\nabla(\mathcal{L}\rho)$ at the moment. Therefore $\nabla^2 \frac{u^r}{r}$ term cannot be eliminated by $\|\nabla L(t,\cdot)\|_{L^2}^2$ on the left hand side of (3.10). This is, in the authors' opinion, a key difference from Navier-Stokes and MHD systems in which $\rho \equiv 0$.

Nevertheless, noting that $\Gamma = ru^\theta$ is uniformly bounded according to Lemma 3.1, for a fixed $s \in [0,\infty)$, (3.21) indicates that

$$\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \le C\|\Gamma_0\|_{L^\infty}^{\frac{2s}{1+s}} \int_{\mathbb{R}^3} \left|\frac{u^\theta}{r^s}\right|^{\frac{2}{s+1}} \left|\nabla \frac{u^r}{r}\right|^2 dx.$$

For any $p \ge \frac{3}{1+s}$, using Hölder inequality, one derives that

$$\begin{aligned}\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \le C\|\Gamma_0\|_{L^\infty}^{\frac{2s}{1+s}} &\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2}{1+s}} \\ &\times \left\|\nabla \frac{u^r}{r}(t,\cdot)\right\|_{L^2}^{2-\frac{6}{(1+s)p}} \left\|\nabla \frac{u^r}{r}(t,\cdot)\right\|_{L^6}^{\frac{6}{(1+s)p}}.\end{aligned}$$

By Lemma 2.5 and the definition of L, one notes that

$$\begin{aligned}&\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \\ \le& C_{s,p}\|\Gamma_0\|_{L^\infty}^{\frac{2s}{1+s}} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2}{1+s}} \|\Omega(t,\cdot)\|_{L^2}^{2-\frac{6}{(1+s)p}} \|\Omega(t,\cdot)\|_{L^6}^{\frac{6}{(1+s)p}} \\ \le& C_{s,p}\|\Gamma_0\|_{L^\infty}^{\frac{2s}{1+s}} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2}{1+s}} \left(\|L(t,\cdot)\|_{L^2} + \|\mathcal{L}\rho(t,\cdot)\|_{L^2}\right)^{2-\frac{6}{(1+s)p}} \\ &\times \left(\|L(t,\cdot)\|_{L^6} + \|\mathcal{L}\rho(t,\cdot)\|_{L^6}\right)^{\frac{6}{(1+s)p}}.\end{aligned}$$

Applying the boundedness of the operator $\mathcal{L}$ in Lemma 2.4, together with the time-uniform estimate of ρ in Lemma 3.1, one arrives that when $p > \frac{3}{1+s}$:

$$\begin{aligned}&\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \\ \le& C_{s,p}\|\Gamma_0\|_{L^\infty}^{\frac{2s}{1+s}} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2}{1+s}} \left(\|L(t,\cdot)\|_{L^2} + \|\rho_0\|_{L^2}\right)^{2-\frac{6}{(1+s)p}} \left(\|\nabla L(t,\cdot)\|_{L^2} + \|\rho_0\|_{L^6}\right)^{\frac{6}{(1+s)p}} \\ \le& C_{s,p,\Gamma_0} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}} \left(\|L(t,\cdot)\|_{L^2} + \|\rho_0\|_{L^2}\right)^2 + \frac{1}{4}\left(\|\nabla L(t,\cdot)\|_{L^2} + \|\rho_0\|_{L^6}\right)^2 \\ \le& C_{s,p,\Gamma_0} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}} \|L(t,\cdot)\|_{L^2}^2 + C_{s,p,\rho_0,\Gamma_0}\left(\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}} + 1\right) \\ &+ \frac{1}{4}\|\nabla L(t,\cdot)\|_{L^2}^2.\end{aligned} \tag{3.22}$$

Similarly when $p = \frac{3}{1+s}$, one derives

$$\begin{aligned}\frac{d}{dt}\|J(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2 \le& C_{s,\Gamma_0} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}}^{\frac{2}{1+s}} \|\nabla L(t,\cdot)\|_{L^2}^2 \\ &+ C_{s,\rho_0,\Gamma_0} \left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}}^{\frac{2}{1+s}}.\end{aligned} \tag{3.23}$$

Therefore, when $p > \frac{3}{s+1}$, (3.19) and (3.22) imply that

$$\begin{aligned}
&\frac{d}{dt}\left(\|L(t,\cdot)\|_{L^2}^2 + \|J(t,\cdot)\|_{L^2}^2\right) + \left(\|\nabla L(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2\right)\\
\le& C_{s,p,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}}\left(\|L(t,\cdot)\|_{L^2}^2 + \|J(t,\cdot)\|_{L^2}^2\right)\\
&+ C_{s,p,\rho_0,\Gamma_0}\left(\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^p}^{\frac{2p}{(1+s)p-3}} + 1\right)\\
&+ C_{h_0,\rho_0}\left(\|\nabla u(t,\cdot)\|_{L^2}^2 + (1+t)^2 + \|L(t,\cdot)\|_{L^2}^2\right).
\end{aligned}$$

Thus the condition (1.3) and Gronwall inequality indicates (3.9). Finally when $p = \frac{3}{1+s}$, (3.20) and (3.23) lead to

$$\begin{aligned}
&\frac{d}{dt}\left(\|L(t,\cdot)\|_{L^2}^2 + \|J(t,\cdot)\|_{L^2}^2\right) + \left(\|\nabla L(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2\right)\\
\le& C_{s,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}}\left(\|\nabla L(t,\cdot)\|_{L^2}^2 + \|\nabla J(t,\cdot)\|_{L^2}^2\right)\\
&+ C_{s,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}}^{\frac{2}{1+s}}\|\nabla L(t,\cdot)\|_{L^2}^2 + C_{s,\rho_0,\Gamma_0}\left\|\frac{u^\theta}{r^s}(t,\cdot)\right\|_{L^{\frac{3}{1+s}}}^{\frac{2}{1+s}}\\
&+ C_{h_0,\rho_0}\left(\|\nabla u(t,\cdot)\|_{L^2}^2 + (1+t)^2 + \|L(t,\cdot)\|_{L^2}^2\right).
\end{aligned} \tag{3.24}$$

Choosing $\varepsilon_0 = (4C_{s,\Gamma_0})^{-\max\left\{1,\frac{1+s}{2}\right\}}$, we find the first and second terms on the right hand of (3.24) can be absorbed by the left hand providing

$$\left\|\frac{u^\theta}{r^s}\right\|_{L^\infty(0,T_*;L^{\frac{3}{1+s}})} < \varepsilon_0.$$

Using Gronwall inequality, (3.9) also holds when $p = \frac{3}{1+s}$. $\square$

Corollary 3.4. *Under the same conditions as Theorem 1.3, we have*

$$\sup_{0\le t\le T_*}\|\Omega(t,\cdot)\|_{L^2}^2 < \infty.$$

Proof. By Lemma 3.1 and Lemma 2.4, $\mathcal{L}\rho$ satisfies:

$$\|\mathcal{L}\rho(t,\cdot)\|_{L^2}^2 \le C\|\rho(t,\cdot)\|_{L^2}^2 \le C\|\rho_0\|_{L^2}^2 < \infty, \quad \forall t\in[0,T_*].$$

Thus the corollary is proved by noting the $L^\infty_{T_*}L^2$ boundedness of $L = \Omega - \mathcal{L}\rho$ in (3.9). $\square$

3.2. $L^\infty_{T_*}L^2\cap L^2_{T_*}H^1$ **estimate of** ∇u**.** This part is devoted to the $L^\infty_{T_*}L^2\cap L^2_{T_*}H^1$ estimate of ∇u, that is:

Lemma 3.5. *Under the same conditions as Theorem 1.3, the following a priori estimate of the gradient of the velocity holds:*

$$\sup_{0\le t\le T_*}\|\nabla u(t,\cdot)\|_{L^2}^2 + \int_0^{T_*}\|\nabla^2 u(t,\cdot)\|_{L^2}^2 dt < \infty. \tag{3.25}$$

Proof. To do this, we first estimate the horizontal angular component of the vorticity.

3.2.1. *Estimate of w^θ.* For $(3.3)_2$, we perform the standard L^2 inner product to derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^\theta(t,\cdot)\|_{L^2}^2+\|\nabla w^\theta(t,\cdot)\|_{L^2}^2+\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}^2\\
=&\int_{\mathbb{R}^3}\frac{u^r}{r}(w^\theta)^2dx+\int_{\mathbb{R}^3}\partial_z\frac{(u^\theta)^2}{r}w^\theta dx-\int_{\mathbb{R}^3}\partial_r\rho w^\theta dx-\int_{\mathbb{R}^3}\partial_z\frac{(h^\theta)^2}{r}w^\theta dx\\
:=&I_1+I_2+I_3+I_4.
\end{aligned}$$

Now we estimate I_i, $i=1,2,3,4$ separately. By Hölder inequality, Young inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
I_1\leq&\|u^r(t,\cdot)\|_{L^3}\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}\|w^\theta(t,\cdot)\|_{L^6}\\
\leq&C\|u^r(t,\cdot)\|_{L^3}\|\Omega(t,\cdot)\|_{L^2}\|\nabla w^\theta(t,\cdot)\|_{L^2}\\
\leq&C\|u^r(t,\cdot)\|_{L^3}^2\|\Omega(t,\cdot)\|_{L^2}^2+\frac{1}{8}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2\\
\leq&C\|u^r(t,\cdot)\|_{L^2}\|\nabla u^r(t,\cdot)\|_{L^2}\|\Omega(t,\cdot)\|_{L^2}^2+\frac{1}{8}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2=&\,2\int_{\mathbb{R}^3}\frac{\partial_z u^\theta}{r}u^\theta w^\theta dx\\
\leq&\,2\|J(t,\cdot)\|_{L^2}\|w^\theta(t,\cdot)\|_{L^6}\|u^\theta(t,\cdot)\|_{L^3}\\
\leq&\,C\|J(t,\cdot)\|_{L^2}\|\nabla w^\theta(t,\cdot)\|_{L^2}\|u^\theta(t,\cdot)\|_{L^3}\\
\leq&\,C\|u^\theta(t,\cdot)\|_{L^3}^2\|J(t,\cdot)\|_{L^2}^2+\frac{1}{8}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2\\
\leq&\,C\|u^\theta(t,\cdot)\|_{L^2}\|\nabla u^\theta(t,\cdot)\|_{L^2}\|J(t,\cdot)\|_{L^2}^2+\frac{1}{8}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2.
\end{aligned}$$

Meanwhile, one derives the following for I_3:

$$\begin{aligned}
I_3=&-2\pi\int_{-\infty}^{\infty}\int_0^\infty\partial_r\rho w^\theta rdrdz\\
=&2\pi\int_{-\infty}^{\infty}\int_0^\infty\rho\partial_r(w^\theta r)drdz\\
=&2\pi\int_{-\infty}^{\infty}\int_0^\infty\rho\partial_r w^\theta rdrdz+2\pi\int_{-\infty}^{\infty}\int_0^\infty\rho\frac{w^\theta}{r}rdrdz\\
\leq&\|\rho(t,\cdot)\|_{L^2}\|\nabla w^\theta(t,\cdot)\|_{L^2}+\|\rho(t,\cdot)\|_{L^2}\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}\\
\leq&C\|\rho(t,\cdot)\|_{L^2}^2+\frac{1}{8}\left(\|\nabla w^\theta(t,\cdot)\|_{L^2}^2+\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}^2\right),
\end{aligned}$$

also similarly for I_4:

$$\begin{aligned}
I_4=&\int_{\mathbb{R}^3}\frac{(h^\theta)^2}{r}\partial_z w^\theta dx\\
\leq&\|H(t,\cdot)\|_{L^\infty}\|h^\theta(t,\cdot)\|_{L^2}\|\nabla w^\theta(t,\cdot)\|_{L^2}\\
\leq&C\|H(t,\cdot)\|_{L^\infty}^2\|h^\theta(t,\cdot)\|_{L^2}^2+\frac{1}{8}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2.
\end{aligned}$$

The above estimates for I_i, $i = 1, 2, 3, 4$ along with Lemma 3.1 indicate that

$$
\begin{aligned}
&\frac{d}{dt}\|w^\theta(t,\cdot)\|_{L^2}^2 + \|\nabla w^\theta(t,\cdot)\|_{L^2}^2 + \left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}^2 \\
\le& C\left(\|u^r(t,\cdot)\|_{L^2}\|\nabla u^r(t,\cdot)\|_{L^2}\|\Omega(t,\cdot)\|_{L^2}^2 + \|u^\theta(t,\cdot)\|_{L^2}\|\nabla u^\theta(t,\cdot)\|_{L^2}\|J(t,\cdot)\|_{L^2}^2\right.\\
&\left.+\|\rho(t,\cdot)\|_{L^2}^2 + \|H(t,\cdot)\|_{L^\infty}^2\|h^\theta(t,\cdot)\|_{L^2}^2\right).
\end{aligned}
$$

Integrating with t on $[0, T_*]$, the following final inequality follows from the $L^\infty_{T_*}L^2$ estimates of u, h and ρ, together with $L^\infty_{T_*}L^\infty$ estimate of H in Lemma 3.1, and $L^\infty_{T_*}L^2$ estimate of (Ω, J) in Lemma 3.2 and Corollary 3.4. That is:

$$
\begin{aligned}
&\sup_{0\le t\le T_*}\|w^\theta(t,\cdot)\|_{L^2}^2 + \int_0^{T_*}\|\nabla w^\theta(t,\cdot)\|_{L^2}^2dt + \int_0^{T_*}\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}^2dt \\
\lesssim& \sup_{0\le t\le T_*}\|u(t,\cdot)\|_{L^2}\sup_{0\le t\le T_*}\left(\|\Omega(t,\cdot)\|_{L^2}^2 + \|J(t,\cdot)\|_{L^2}^2\right)\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^2}dt + T_*\|\rho_0\|_{L^2}^2\\
&+\|H_0\|_{L^\infty}^2T_*\sup_{0\le t\le T_*}\|h^\theta(t,\cdot)\|_{L^2}^2\\
<&\infty.
\end{aligned}
\tag{3.26}
$$

3.2.2. *Estimate of w^r and w^z.* We multiply $(3.3)_1$ by w^r and integrate over $\mathbb{R}^3$ to derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^r(t,\cdot)\|_{L^2}^2 + \|\nabla w^r(t,\cdot)\|_{L^2}^2 + \left\|\frac{w^r}{r}(t,\cdot)\right\|_{L^2}^2 \\
=&\int_{\mathbb{R}^3} w^r(w^r\partial_r + w^z\partial_z)u^r dx\\
=&-\int_{\mathbb{R}^3} u^r(w^r\partial_r + w^z\partial_z)w^r dx\\
\le&\|u^r(t,\cdot)\|_{L^\infty}\left(\|w^r(t,\cdot)\|_{L^2}\|\partial_r w^r(t,\cdot)\|_{L^2} + \|w^z(t,\cdot)\|_{L^2}\|\partial_z w^r(t,\cdot)\|_{L^2}\right)\\
\le&\frac{1}{4}\|\nabla w^r(t,\cdot)\|_{L^2}^2 + C\|u^r(t,\cdot)\|_{L^\infty}^2\left(\|w^r(t,\cdot)\|_{L^2}^2 + \|w^z(t,\cdot)\|_{L^2}^2\right).
\end{aligned}
\tag{3.27}
$$

Here the last three lines follow from the integration by parts, Hölder inequality and Young inequality. Meanwhile, by a similar performance on $(3.3)_3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^z(t,\cdot)\|_{L^2}^2 + \|\nabla w^z(t,\cdot)\|_{L^2}^2\\
=&\int_{\mathbb{R}^3} w^z(w^r\partial_r + w^z\partial_z)u^z dx\\
=&-\int_{\mathbb{R}^3} u^z(w^r\partial_r + w^z\partial_z)w^z dx\\
\le&\|u^z(t,\cdot)\|_{L^\infty}\left(\|w^r(t,\cdot)\|_{L^2}\|\partial_r w^z(t,\cdot)\|_{L^2} + \|w^z(t,\cdot)\|_{L^2}\|\partial_z w^z(t,\cdot)\|_{L^2}\right)\\
\le&\frac{1}{4}\|\nabla w^z(t,\cdot)\|_{L^2}^2 + C\|u^z(t,\cdot)\|_{L^\infty}^2\left(\|w^r(t,\cdot)\|_{L^2}^2 + \|w^z(t,\cdot)\|_{L^2}^2\right).
\end{aligned}
\tag{3.28}
$$

Summing up (3.27) and (3.28) and applying Gronwall inequality, one derives

$$\sup_{0\le t\le T_*}\|(w^r,w^z)(t,\cdot)\|_{L^2}^2+\int_0^{T_*}\left(\|(\nabla w^r(t,\cdot),\nabla w^z(t,\cdot))\|+\left\|\frac{w^r}{r}(t,\cdot)\right\|_{L^2}\right)dt$$
$$\le\|(w_0^r,w_0^z)\|_{L^2}^2\exp\left(C\int_0^{T_*}\|b(t,\cdot)\|_{L^\infty}^2dt\right). \tag{3.29}$$

Finally, it remains to estimate the part inside the exponential function on the right-hand-side of (3.29). Using Gagliardo-Nirenberg interpolation inequality, (2.7) and Hölder inequality, together with estimates (3.2) and (3.26), one has

$$\begin{aligned}&\int_0^{T_*}\|b(t,\cdot)\|_{L^\infty}^2dt\\&\le C\int_0^{T_*}\|\nabla b(t,\cdot)\|_{L^2}\|\nabla^2 b(t,\cdot)\|_{L^2}dt\\&\le C\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^2}\left(\|\nabla w^\theta(t,\cdot)\|_{L^2}+\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}\right)dt\\&\le C\left(\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^2}^2ds\right)^{1/2}\left(\int_0^{T_*}\left(\|\nabla w^\theta(t,\cdot)\|_{L^2}^2+\left\|\frac{w^\theta}{r}(t,\cdot)\right\|_{L^2}^2\right)dt\right)^{1/2}<\infty.\end{aligned}$$

Inserting the above estimate in (3.29), we have

$$\sup_{0\le t\le T_*}\|(w^r,w^z)(t,\cdot)\|_{L^2}^2+\int_0^{T_*}\left(\|\nabla(w^r,w^z)(t,\cdot)\|_{L^2}^2+\left\|\frac{w^r}{r}(t,\cdot)\right\|_{L^2}^2\right)dt<\infty. \tag{3.30}$$

Combining (3.26) and (3.30), we have the $L^\infty_{T_*}L^2\cap L^2_{T_*}H^1$ estimate for the vorticity. Then using (2.6), (3.25) follows. □

3.3. $L^1_{T_*}L^\infty$ **estimate of** ∇u**.** Recall the equation for the vorticity:

$$\begin{cases}\partial_t w-\Delta w=\nabla\times(u\cdot\nabla u)-\nabla\times(h\cdot\nabla h)+\nabla\times(\rho e_3);\\ w(0,x)=\nabla\times u_0(x).\end{cases}$$

For the further convenience, we split w into three parts:

$$w:=w_0+w_1+w_2,$$

where w_0 solves the linear parabolic equation with the initial value $\nabla\times u_0(x)$:

$$\begin{cases}\partial_t w_0-\Delta w_0=0;\\ w(0,x)=\nabla\times u_0(x).\end{cases}$$

Clearly, when $t>0$, w_0 is regular enough for our argument in this paper, so we only need to consider the rest parts. Meanwhile, w_1 and w_2, which have homogeneous initial data, satisfy

$$\partial_t w_1-\Delta w_1=-\nabla\times(h\cdot\nabla h)$$

and

$$\partial_t w_2-\Delta w_2=\nabla\times(u\cdot\nabla u)+\nabla\times(\rho e_3),$$

respectively.

Now we **claim that**

$$\boldsymbol{\nabla u \in L^1\left(0, T_*; L^\infty\right)}. \tag{3.31}$$

To prove it, we first observe that

$$h \cdot \nabla h = -\frac{(h^\theta)^2}{r} e_r = -H h^\theta e_r,$$

since $h = h^\theta(t, r, z) e_\theta$. Noting that

$$H \in L^\infty\left(0, T_*; L^\infty\right) \tag{3.32}$$

follows from (3.1) in Lemma 3.1, the following estimate of h^θ holds by performing L^4 inner product of h^θ:

$$\begin{aligned}
\frac{d}{dt}\left\|h^\theta(t,\cdot)\right\|_{L^4}^4 &\le 4\left|\int_{\mathbb{R}^3} \frac{u^r}{r}\left(h^\theta\right)^4 dx\right| \\
&\le 4\|H\|_{L^\infty}\int_{\mathbb{R}^3}|u^r|\,|h^\theta|^3 dx \\
&\le 4\left\|H_0\right\|_{L^\infty}\left\|u^r(t,\cdot)\right\|_{L^4}\left\|h^\theta(t,\cdot)\right\|_{L^4}^3 \\
&\le C\left\|H_0\right\|_{L^\infty}\left\|\nabla u^r(t,\cdot)\right\|_{L^2}^{3/4}\left\|u^r(t,\cdot)\right\|_{L^2}^{1/4}\left\|h^\theta(t,\cdot)\right\|_{L^4}^3.
\end{aligned}$$

Integration from 0 to t on time for $t \in (0, T_*]$, one derives

$$\begin{aligned}
&\sup_{0\le t\le T_*}\|h^\theta(t,\cdot)\|_{L^4} \\
&\le \|h_0^\theta\|_{L^4} + C\|H_0\|_{L^\infty}\sup_{0\le t\le T_*}\|u^r(t,\cdot)\|_{L^2}^{1/4}\int_0^{T_*}\|\nabla u^r(t,\cdot)\|_{L^2}^{3/4}\,dt \\
&\le \|h_0^\theta\|_{L^4} + C\|H_0\|_{L^\infty}\sup_{0\le t\le T_*}\|u^r(t,\cdot)\|_{L^2}^{1/4}\left(\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^2}^2\,dt\right)^{3/8} {T_*}^{5/8} \\
&< \infty.
\end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33), we find

$$h \cdot \nabla h \in L^\infty\left(0, T_*; L^4\right) \subset L^{4/3}\left(0, T_*; L^4\right).$$

Then ∇w_1 satisfies

$$\nabla w_1 \in L^{4/3}\left(0, T_*; L^4\right) \tag{3.34}$$

by applying (2.8), the maximal regularity for the heat flow in Lemma 2.8. To treat w_2, by interpolating $L^2_{T_*}H^1$ and $L^\infty_{T_*}L^2$ as shown in (2.1) of Lemma (2.2), we arrive

$$\nabla u \in L^{8/3}\left(0, T_*; L^4\right). \tag{3.35}$$

Also we have the following interpolation inequality by Lemma 2.1:

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \|\nabla u(t,\cdot)\|_{L^4}^{6/7}\|u(t,\cdot)\|_{L^2}^{1/7}.$$

Then considering the fundamental energy estimate (3.2), one deduces that

$$\begin{aligned}
\int_0^{T_*}\|u(t,\cdot)\|_{L^\infty}^{8/3}dt &\lesssim \|u\|_{L^\infty(0,T_*;L^2)}^{8/21}\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^4}^{16/7}dt \\
&\lesssim \|u\|_{L^\infty(0,T_*;L^2)}^{8/21}\left(\int_0^{T_*}\|\nabla u(t,\cdot)\|_{L^4}^{8/3}dt\right)^{6/7} T_*^{1/7} < \infty.
\end{aligned} \tag{3.36}$$

Then (3.35) and (3.36) assert that

$$u \cdot \nabla u \in L^{4/3}\left(0, T_*; L^4\right).$$

Meanwhile, by (3.1), it is clear that

$$\rho \in L^\infty(0, T_*; L^4) \subset L^{4/3}(0, T_*; L^4).$$

Following from (2.8) in Lemma 2.8, it is clear that

$$\nabla w_2 \in L^{4/3}\left(0, T_*; L^4\right). \tag{3.37}$$

Then (3.37), together with (3.34), imply that

$$\nabla w \in L^{4/3}\left(0, T_*; L^4\right). \tag{3.38}$$

Now the interpolation inequality in Lemma 2.1, together with the lower order estimate of w in (3.9) and (2.6), assert that

$$\begin{aligned}
\|\nabla u(t,\cdot)\|_{L^\infty} &\lesssim \|\nabla u(t,\cdot)\|_{L^2}^{1/7} \|\nabla^2 u(t,\cdot)\|_{L^4}^{6/7} \\
&\lesssim \|w(t,\cdot)\|_{L^2}^{1/7} \|\nabla w(t,\cdot)\|_{L^4}^{6/7}.
\end{aligned}$$

Then using (3.38), we find the claim is proved since

$$\begin{aligned}
\int_0^{T_*} \|\nabla u(t,\cdot)\|_{L^\infty} dt &\lesssim \|w\|_{L^\infty(0,T_*;L^2)}^{1/7} \int_0^{T_*} \|\nabla w(t,\cdot)\|_{L^4}^{6/7} dt \\
&\le \|w\|_{L^\infty(0,T_*;L^2)}^{1/7} \left(\int_0^{T_*} \|\nabla w(t,\cdot)\|_{L^4}^{4/3} dt \right)^{14/9} T_*^{5/14} < \infty.
\end{aligned}$$

3.4. $L^1_{T_*} L^\infty$ **estimate of** $\nabla \times h$**.** Let $j := \nabla \times h$. By $h = h^\theta(t,r,z) e_\theta$, it follows that

$$j = j^r(t,r,z) e_r + j^z(t,r,z) e_z$$

is an axially symmetric swirl-free vector field with

$$j^r = -\partial_z h^\theta, \quad j^z = \partial_r h^\theta + \frac{h^\theta}{r}.$$

Taking derivative of $(1.2)_4$, noting the divergence-free condition of u, one obtains

$$\begin{cases}
\partial_t j^r + (u^r \partial_r + u^z \partial_z) j^r = -\left(\partial_r u^r + 2\partial_z u^z\right) j^r + \partial_z u^r j^z - 2\partial_z u^r H; \\
\\
\partial_t j^z + (u^r \partial_r + u^z \partial_z) j^z = \partial_r u^z j^r - \left(2\partial_r u^r + \partial_z u^z\right) j^z + \left(4\partial_r u^r + 2\partial_z u^z\right) H.
\end{cases} \tag{3.39}$$

Before we perform the $L^1_{T_*} L^\infty$-estimate of $\nabla \times h$, we denote $X(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ the particle trajectory mapping of the velocity b, which solves the initial value problem:

$$\frac{\partial X(t,\zeta)}{\partial t} = b(t, X(t,\zeta)), \quad X(0,\zeta) = \zeta.$$

Integrating (3.39) along the particle trajectory mapping, we have

$$\begin{aligned}
j^r(t, X(t,\zeta)) =& j_0^r(\zeta) + \int_0^t \left[-\left(\partial_r u^r + 2\partial_z u^z\right) j^r + \partial_z u^r j^z - 2\partial_z u^r H\right](s, X(s,\zeta)) ds; \\
j^z(t, X(t,\zeta)) =& j_0^z(\zeta) \\
&+ \int_0^t \left[\partial_r u^z j^r - \left(2\partial_r u^r + \partial_z u^z\right) j^z + \left(4\partial_r u^r + 2\partial_z u^z\right) H\right](s, X(s,\zeta)) ds.
\end{aligned}$$

Taking the L^∞ norm over $\zeta \in \mathbb{R}^3$, noting the estimate of H in (3.1), one derives that

$$\begin{aligned}
\|(j^r, j^z)(t,\cdot)\|_{L^\infty} \le& \|(j_0^r\,, j_0^z)\|_{L^\infty} \\
&+ C\int_0^t \|\nabla b(s,\cdot)\|_{L^\infty} \left(\|(j^r, j^z)(s,\cdot)\|_{L^\infty} + \|H(s,\cdot)\|_{L^\infty}\right) ds \\
\le& \|(j_0^r\,, j_0^z)\|_{L^\infty} + C\|H_0\|_{L^\infty} \int_0^t \|\nabla b(s,\cdot)\|_{L^\infty} ds \\
&+ C\int_0^t \|\nabla b(s,\cdot)\|_{L^\infty} \|(j^r, j^z)(s,\cdot)\|_{L^\infty} ds.
\end{aligned}$$

Applying (3.31) and Gronwall inequality, one arrives that

$$\begin{aligned}
&\|(j^r, j^z)(t)\|_{L^\infty} \\
\le& \left(\|(j_0^r\,, j_0^z)\|_{L^\infty} + C\|H_0\|_{L^\infty} \int_0^t \|\nabla b(s,\cdot)\|_{L^\infty} ds\right) \exp\left(C\int_0^t \|\nabla b(s,\cdot)\|_{L^\infty} ds\right) \\
\le& \left(\|(j_0^r\,, j_0^z)\|_{L^\infty} + C\|H_0\|_{L^\infty} \int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} ds\right) \exp\left(C\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} ds\right)
\end{aligned}$$

holds for any $t \in (0, T_*]$. This implies

$$\int_0^{T_*} \|\nabla \times h(t,\cdot)\|_{L^\infty} dt = \int_0^{T_*} \|(j^r, j^z)(t,\cdot)\|_{L^\infty} dt < \infty, \tag{3.40}$$

which finishes the proof of the desired estimate.

3.5. $L^1_{T_*} L^\infty$ estimate of $\nabla\rho$.

Now it remains to esitmate $\nabla\rho$. Taking ∇ to $(1.1)_3$, we know that

$$\partial_t \nabla\rho + u \cdot \nabla\nabla\rho = -\nabla u \cdot \nabla\rho.$$

The routine L^∞ estimate follows that

$$\|\nabla\rho(t,\cdot)\|_{L^\infty} \le \|\nabla\rho_0\|_{L^\infty} + \int_0^t \|\nabla u(s,\cdot)\|_{L^\infty} \|\nabla\rho(s,\cdot)\|_{L^\infty} ds.$$

By Gronwall inequality and using (3.31), we arrive

$$\sup_{0\le t\le T_*} \|\nabla\rho(t,\cdot)\|_{L^\infty} \le \|\nabla\rho_0\|_{L^\infty} \exp\left(\int_0^{T_*} \|\nabla u(s,\cdot)\|_{L^\infty} ds\right) < \infty. \tag{3.41}$$

3.6. Estimates of higher order norms & proof of Theorem 1.3. Combining (3.31), (3.40) and (3.41), we have

$$\int_0^{T_*} \|(\nabla \times u, \nabla \times h)(t,\cdot)\|_{L^\infty} dt + \int_0^{T_*} \|\nabla\rho(t,\cdot)\|_{L^\infty} dt < \infty. \tag{3.42}$$

We now show H^m $(m \ge 3)$ regularity of the solution by using the above inequality. We note that the proof below is still valid for the case that the viscous coefficient $\mu = 0$.

Apply ∇^m $(m \in \mathbb{N},\ m \geq 3)$ to $(1.1)_{1,2,3}$ to derive that

$$\begin{cases} \partial_t \nabla^m u + u \cdot \nabla \nabla^m u + \nabla \nabla^m p - \mu \Delta \nabla^m u \\ \qquad\qquad = h \cdot \nabla \nabla^m h + \nabla^m (\rho e_3) - [\nabla^m, u \cdot \nabla] u + [\nabla^m, h \cdot \nabla] h, \\ \partial_t \nabla^m h + u \cdot \nabla \nabla^m h - h \cdot \nabla \nabla^m u = -[\nabla^m, u \cdot \nabla] h + [\nabla^m, h \cdot \nabla] u, \\ \partial_t \nabla^m \rho + u \cdot \nabla \nabla^m \rho = -[\nabla^m, u \cdot \nabla] \rho. \end{cases} \tag{3.43}$$

Performing the L^2 energy estimate of (3.43), noting that

$$\int_{\mathbb{R}^3} h \cdot \nabla \nabla^m h \cdot \nabla^m u dx + \int_{\mathbb{R}^3} h \cdot \nabla \nabla^m u \cdot \nabla^m h dx = 0,$$

we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla^m (u, h, \rho)(t, \cdot)\|_{L^2}^2 + \mu \left\|\nabla^{m+1} u(t, \cdot)\right\|_{L^2}^2 \\ &\quad = - \int_{\mathbb{R}^3} [\nabla^m, u \cdot \nabla] u \nabla^m u dx + \int_{\mathbb{R}^3} [\nabla^m, h \cdot \nabla] h \nabla^m u dx - \int_{\mathbb{R}^3} [\nabla^m, u \cdot \nabla] h \nabla^m h dx \\ &\qquad + \int_{\mathbb{R}^3} [\nabla^m, h \cdot \nabla] u \nabla^m h dx - \int_{\mathbb{R}^3} [\nabla^m, u \cdot \nabla] \rho \nabla^m \rho dx + \int_{\mathbb{R}^3} \nabla^m (\rho e_3) \nabla^m u dx. \end{aligned}$$

By Lemma 2.3, the above equation implies

$$\begin{aligned} &\frac{d}{dt} \|\nabla^m (u, h, \rho)(t, \cdot)\|_{L^2}^2 + \mu \left\|\nabla^{m+1} u(t, \cdot)\right\|_{L^2}^2 \\ \lesssim &\|\nabla^m (u, h, \rho)(t, \cdot)\|_{L^2}^2 \left(\|\nabla (u, h, \rho)(t, \cdot)\|_{L^\infty} + 1\right). \end{aligned} \tag{3.44}$$

By denoting

$$E_m(t) := \sup_{0 \leq s \leq t} \|\nabla^m (u, h, \rho)(s, \cdot)\|_{L^2}^2, \quad 0 \leq t < T_*, \quad m \geq 3,$$

(3.44), together with (2.10) in Corollary 2.10, $(3.1)_3$ and (3.2) in Lemma 3.1, indicate that

$$\begin{aligned} &\frac{d}{dt} \|\nabla^m (u, h, \rho)(t, \cdot)\|_{L^2}^2 + \mu \|\nabla^{m+1} u(t, \cdot)\|_{L^2}^2 \\ \lesssim &\left(1 + \|(\nabla \times u, \nabla \times h, \nabla \rho)(t, \cdot)\|_{BMO} \log(e + E_m(t))\right) (e + E_m(t)) \\ \lesssim &\left(1 + \|(\nabla \times u, \nabla \times h, \nabla \rho)(t, \cdot)\|_{L^\infty} \log(e + E_m(t))\right) (e + E_m(t)). \end{aligned}$$

Integrating the above inequality over $(0,\, t)$, where $t \in [0,\, T_*)$, one has

$$\begin{aligned} &e + \|\nabla^m (u,\, h,\, \rho)(t)\|_{L^2}^2 \\ \lesssim &e + \|\nabla^m (u_0,\, h_0,\, \rho_0)\|_{L^2}^2 \\ &+ \int_0^t \Big\{1 + \|(\nabla \times u, \nabla \times h, \nabla \rho)(s, \cdot)\|_{L^\infty} \log\left(e + E_m(s)\right)\Big\} \left(e + E_m(s)\right) ds, \end{aligned}$$

which implies

$$\begin{aligned} e + E_m(t) \lesssim &e + \|\nabla^m (u_0,\, h_0,\, \rho_0)\|_{L^2}^2 \\ &+ \int_0^t \Big\{1 + \|(\nabla \times u, \nabla \times h, \nabla \rho)(s, \cdot)\|_{L^\infty} \log\left(e + E(s)\right)\Big\} \left(e + E(s)\right) ds. \end{aligned}$$

Using Gronwall inequality twice, one deduces that

$$e + E_m(t) \leq C_{0,T_*} \left(e + \|\nabla^m(u_0,\, h_0,\, \rho_0)\|_{L^2}^2\right)^{\exp\left(C_{0,T_*} \int_0^t (1+\|(\nabla\times u,\nabla\times h,\nabla\rho)(s,\cdot)\|_{L^\infty})ds\right)}, \quad \forall t \in [0, T_*),$$

where $C_{0,T_*} > 0$ is a constant depends on initial data and T_*. Hence (u, h, ρ) can be regularly extended beyond T_* under the condition (3.42). This completes the proof of Theorem 1.3.

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