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Stability of smooth solutions for the compressible Euler equations with time-dependent damping and one-side physical vacuum

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Abstract

In this paper, the one-side physical vacuum problem for the one dimensional compressible Euler equations with time-dependent damping is considered. Near the physical vacuum boundary, the sound speed is $C^{1/2}$-Hölder continuous. The coefficient of the time-dependent damping is given by $\frac{\mu}{(1+t)^\lambda}$, $(0 < \lambda, 0 < \mu)$ which decays by order $-\lambda$ in time. First we give an one-side physical vacuum background solution whose density and velocity have a growing order with respect to time. Then the main purpose of this paper is to prove the stability of this background solution under the assumption that $0 < \lambda < 1, 0 < \mu$ or $\lambda = 1, 2 < \mu$. The pointwise convergence rate of the density, velocity and the expanding rate of the physical vacuum boundary are also given. The proof is based on the space-time weighted energy estimates, elliptic estimates and the Hardy inequality in the Lagrangian coordinates.

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Keywords: One-side physical vacuum; Compressible Euler equations; Time-dependent damping

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1. Introduction

In this paper, we investigate the stability of the one-side physical vacuum solution for the following 1-d compressible Euler equations with time-dependent damping.

$$\begin{cases} \rho_t + (\rho u)_x = 0 \quad \text{in } I(t) := \{(x,t) | x_b(t) < x < +\infty, t > 0\}, \\ (\rho u)_t + (p(\rho) + \rho u^2)_x = -\dfrac{\mu}{(1+t)^\lambda} \rho u \quad \text{in } I(t), \\ \rho > 0 \quad \text{in } I(t), \quad \rho = 0 \quad \text{on } x_b(t), \\ (\rho, u) = (\rho_0, u_0) \quad \text{on } I(0) := \{x | x_b(0) < x < +\infty\}, \end{cases} \tag{1.1}$$

where the boundary $x_b(t)$ satisfies

$$\dot{x}_b(t) = u(x_b(t), t).$$

Here $(x,t) \in \mathbb{R} \times [0, \infty)$, ρ, u, and p denote the space and time variable, density, velocity, and pressure, respectively. I(t), $x_b(t)$, and $\dot{x}_b(t)$ represent the changing domain occupied by the gas, the moving vacuum boundary and the velocity of $x_b(t)$, respectively. $-\frac{\mu}{(1+t)^\lambda}\rho u$, appearing on the right-hand side of $(1.1)_2$ describes the frictional damping which will decay by order $-\lambda$ in time. We assume the gas is the isentropic flow and the pressure satisfies the γ law:

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma \quad \text{for } \gamma > 1.$$

(Here the adiabatic constant is set to be $\frac{1}{\gamma}$.) Let $c = \sqrt{p'(\rho)}$ be the sound speed. A vacuum boundary is called physical if

$$0 < \left| \frac{\partial c^2}{\partial x} \right| < +\infty$$

in a small neighborhood of the boundary. In order to capture this physical singularity, the initial density is supposed to satisfy

$$\rho_0(x) > 0 \quad \text{for } x_b(0) < x < +\infty,$$
$$\rho_0(x_b(0)) = 0, \quad \text{and} \quad 0 < \left| \left(\rho_0^{\gamma-1}\right)_x (x_b(0)) \right| < \infty.$$

The size of damping can affect the asymptotic behavior of solutions of the Euler equations. When the damping vanishes (the damping coefficient is zero), shock will form. For the mathematical analysis of finite-time formation of singularities, readers can see Alinhac [1], Chemin [2], Courant-Friedrichs [3], Christodoulou [4], Rammaha [36] as well as Sideris [38] and references therein for more details. While for the Euler equations with non-zero constant-coefficient damping, global existence and stability of smooth solutions away from or near physical vacuum can be founded in [16,31,30] and references therein. It is natural to ask whether there are some global or blow-up results of solutions of the Euler equations with variant-coefficient damping, especially for the damping which decays in time. Actually for the Euler equations with the damping coefficient given by $\frac{\mu}{(1+t)^\lambda}$, now there are numerous works concerning about the global existence, finite-time blow up, and asymptotic behaviors of smooth solutions. As far as the author knows, the pioneer works come from Hou-Witt-Yin [19,20] considering the multi-dimensional Euler equations and Pan [32–34] considering the one-dimensional case, where a critical couple of numbers (λ, μ), depending on the space dimension, are given to separate the global existence and finite-time blow up of smooth solutions when the initial data is a small perturbation of the equilibrium $(\rho, u) = (1, 0)$. In one dimensional case, Pan [32–34] show that the critical couple numbers are $(1, 2)$, which means smooth solutions exist globally in time if $0 < \lambda < 1, 0 < \mu$ and $\lambda = 1, 2 < \mu$, and blow up in finite time if $\lambda = 1, \mu < 2$ and $1 < \lambda, 0 < \mu$. Later, various results are shown in this aspect. Sugiyama [35] studies the blow up mechanism of smooth solutions with $\lambda = 1, \mu < 2$ and $1 < \lambda,\ 0 < \mu$. Li *et al.* [27] and Cui *et al.* [9] proved the time global asymptotic profile of the solution when $\lambda < 1,\ 0 < \mu$ and (ρ, u) approach to different constants at space infinity $+\infty$ and $-\infty$. See also some recent works in [5,13,22,23] and references therein.

Unfortunately, as far as the author knows, there isn't any global result concerning about the Euler equations with time-dependent damping and physical vacuum by now. Our main purpose of this paper is to prove the global existence and stability of smooth solutions of system (1.1) which contains time-decayed damping and physical vacuum.

A solution of system (1.1) with one-side physical vacuum boundary is given by

$$\bar{u}(t) = \begin{cases} -\int_0^t e^{\frac{\mu}{1-\lambda}\left[(1+s)^{1-\lambda} - (1+t)^{1-\lambda}\right]} ds, & \text{for } \lambda \neq 1, \\ -\frac{1}{\mu+1}\left[(1+t) - (1+t)^{-\mu}\right], & \text{for } \lambda = 1, \end{cases} \tag{1.2}$$

and

$$\bar{\rho}(x,t) = (\gamma - 1)^{\frac{1}{\gamma-1}} \left(x - \int_0^t \bar{u}(\tau) d\tau \right)^{\frac{1}{\gamma-1}}, \tag{1.3}$$

in $I(t)$. Here

$$I(t) := \{(x,t) | \int_0^t \bar{u}(\tau) d\tau =: \bar{x}_b(t) < x < +\infty, t \geq 0\}.$$

We will give a short explanation in Appendix A to show how we get (1.2) and (1.3). It is not hard to derive that there exists a constant $M_{\lambda,\mu}$, depending on λ and μ, such that when $t \geq M_{\lambda,\mu}$, we have

$$\bar{u}(t) \approx \begin{cases} -(1+t) & \text{for } \lambda \geq 1, \\ -(1+t)^\lambda & \text{for } \lambda < 1. \end{cases}$$

$$\bar{\rho}^{\gamma-1}(x,t) \approx \begin{cases} x + (1+t)^2 & \text{for } \lambda \geq 1, \\ x + (1+t)^{\lambda+1} & \text{for } \lambda < 1. \end{cases}$$

It is easy to see that this solution has one-side physical vacuum boundary since

$$\left|\frac{\partial \bar{c}^2}{\partial x}\right| = \left|\frac{\partial \bar{\rho}^{\gamma-1}}{\partial x}\right| = \gamma - 1,$$

at the boundary $x = \bar{x}_b(t)$.

We see that such a solution has growing order in time as t goes to infinity. The sound speed grows at order $\frac{\lambda+1}{2}$ for $\lambda < 1$ and order 1 for $\lambda \geq 1$, while the velocity grows order at order λ for $\lambda < 1$ and order 1 for $\lambda \geq 1$.

Our main purpose is to show the stability of this one-side physical vacuum solution (1.2) and (1.3) when the initial data of system (1.1) is a small perturbation of it. In particular, the pointwise convergence rate of the density, the velocity and the expanding rate of the vacuum boundary in time are obtained.

The physical vacuum problem of the compressible Euler equations in which the sound speed is $C^{1/2}$-Hölder continuous across the vacuum boundary is a challenging and interesting problem in the study of free boundary problems for compressible fluids. Even the local-in-time existence theory is hard to prove since standard methods of symmetric hyperbolic systems do not apply.

The phenomena of a physical vacuum arises naturally in several important physical situations such as the equilibrium and dynamics of boundaries of gaseous stars (cf [21,28]). The local-in-time well-posedness for the one and three dimensional compressible Euler equations with physical vacuum has been achieved by Coutand *et al.* [6–8] and Jang-Masmoudi [24,25]. However, due to the strong degeneracy and singular behaviors near the vacuum boundary, it is a great challenge to extend the local-in-time existence theory to the global one of smooth solutions. In analysis, it is hard to establish the uniform-in-time higher-order a prior energy estimates to obtain the global-in-time regularity of solutions near vacuum boundaries. Huang-Marcati-Pan [17], Huang-Pan-Wang [18] and Geng-Huang [10] proved the L^p convergence of L^∞-weak solutions for the Cauchy problem of the one-dimensional compressible Euler equations with constant-coefficient damping to Barenblatt solutions of the porous media equations. They use entropy-type estimates for the solution itself without deriving estimates for derivatives. However, the interfaces separating gases and vacuum cannot be traced in the framework of L^∞-weak solutions. In order to understand the behavior and long-time dynamics of physical vacuum boundaries, study on the global-in-time regularity of solutions is essential. To the best of our knowledge, the first global-in-time result of smooth solutions in Euler equations with constant-coefficient damping comes from Luo-Zeng [30], where the authors proved the global existence of smooth solutions and convergence to Barenblatt solutions for the physical vacuum free boundary problem.

From [6–8,24,25], a powerful tool in the study of physical vacuum free boundary problems of the Euler equations is the weighted energy estimate. By introducing the spatial weight to overcome the singularity at the vacuum boundary, the authors there establish the local-in-time well-posedness theory. Yet weighted estimates only involving spatial weights seem to be limited to proving local existence results. Later, Luo-Zeng [30,41] introduce time weights to quantify the large-time behavior of solutions for the Euler equation with constant-coefficient damping in one dimension and three dimension with spherically symmetric data. The choice of time weights are suggested by looking at the linearized problem to get hints on how the solution decays.

The a prior estimates for the time weighted energy in the paper of Luo-Zeng [30] rely heavily on the constant-coefficient damping term $-\rho u$. Here we consider the Euler equations with time-dependent damping with physical vacuum boundary. The damping coefficient decays by order

$-\lambda$ in time as t go to infinity, which makes the problem more challenging since now we not only have degenerate vacuum boundary but also have degenerate damping.

Inspired by their space-time weighted higher energy, we can construct a similar weighted energy to study the stability of smooth solutions (1.2) and (1.3) of the Euler equations with time-dependent damping (1.1). Although the time-dependent damping is degenerate in our paper, the time weight for our space and time mixed energy is stronger compared with that in [30,41]. See $(2.9)_2$ below and (2.16) in [30]. The reason is that in [30], the first order space derivative of the corrected Barenblatt flow $\tilde{\eta}_x$ in Lagrangian coordinates has a growing order $(1+t)^{\frac{1}{\gamma+1}}$. In their linearized equation, there is a factor $\tilde{\eta}_x^{-\gamma-1} \approx (1+t)^{-1}$ in front of the space derivative of the solution, which results in less decay rate compared with ours. Also, in our linearized equation, when $\lambda = 1$, we need $2 < \mu$ to ensure the closure of our time weighted energy despite in the lower or higher derivative estimates. This seems to be essential to prove the global stability of system (1.1) since in our previous papers [32,33], we have showed that $\mu = 2$ is the threshold to separate the global existence and finite time blow up of smooth solutions to system (1.1) when the initial data is a small perturbation of equilibrium $(\rho, u) = (1, 0)$.

The strategy of our proof will share the similar idea as that in [30,41]. First to simplify the energy estimates, we will use elliptic estimates to show that the weighted space-derivative energy can be controlled by the time-derivative energy. In this process, we need to use the Hardy inequality repeatedly. Then we perform the time-derivative energy estimates in L^2 norms by a prior assumption. To close the energy, the weighted L^∞ norms of the solutions is needed which can be achieved by Sobolev embedding and Hardy inequality. The advantages of this approach can prove the global existence and large-time convergence of solutions with the detailed convergence rates simultaneously.

Before ending this introduction, we review some prior results on vacuum free boundary problems for the compressible Euler equations and related modes besides the results mentioned above. Liu-Yang [29] proved the local existence theory when the singularity near the vacuum is mild in the sense that c^α $(0 < \alpha \leq 1)$ (c denote the sound speed) is smooth across the vacuum boundary for the one-dimensional Euler equations with constant-coefficient damping. Their method is based on the theory of symmetric hyperbolic systems which is not applicable to physical vacuum boundary problems since only c^2, instead of c^α is required to be smooth across the gas-vacuum interface (further development of this type of theory can be found in [39]). A nice review of singular behavior of solutions near vacuum boundaries for compressible fluids can be found in [40]. An instability theory of stationary solutions to the physical vacuum free boundary problem for the spherically symmetric compressible Euler-Poisson equations of gaseous stars for $6/5 < \gamma < 4/3$ was established in Jang [21]. The local-in-time well-posedness of the physical vacuum free boundary problem for the one-dimensional and three dimensional Euler-Poisson equations was investigated in [11] and [12], respectively. The stabilizing mechanism of the expanding background solutions and global existence for 3 dimensional Euler and Euler-Poisson equations with physical vacuum have been shown in [14,15,37] and references therein. See also [42,43] for recent progress on the compressible Euler equations with constant-coefficient damping.

Throughout the rest of paper, C will denote a positive constant that only depends on the parameters of the problem $\lambda,\ \mu,\ \gamma$ and $C_{a,b,c,...}$ denotes a positive constant depending on a, b, c, ... which may be different from line to line. We will employ the notation $a \lesssim b$ to denote $a \leq Cb$ and $a \sim b$ to denote $C^{-1}b \leq a \leq Cb$.

2. Reformulation of the problem and main results

2.1. Fix the domain and Lagrangian variables

We make the initial interval of the background solution (1.2) and (1.3), $(\bar{x}_b(0), +\infty) = (0, +\infty)$, as the reference interval and define a diffeomorphism

$$\eta_0 : (0, +\infty) \to (x_b(0), +\infty)$$

by

$$\int_{x_b(0)}^{\eta_0(x)} \rho_0(y)dy = \int_0^x \bar{\rho}_0(y)dy \quad \text{for } x \in (0, +\infty),$$

where $\bar{\rho}_0(y) := \bar{\rho}(y, 0)$ is the initial density of the solution (1.3). Differentiating the above equality by x indicates

$$\rho_0(\eta_0(x))\, \eta_0'(x) = \bar{\rho}_0(x) \quad \text{for } x \in (0, +\infty). \tag{2.1}$$

To simplify the presentation, set

$$\mathcal{I} := (0, +\infty).$$

To fix the boundary, we transform system (1.1) into Lagrangian variables. For $x \in \mathcal{I}$, we define the Lagrangian variable $\eta(x, t)$ by

$$\begin{cases} \eta_t(x,t) = u(\eta(x,t), t) & \text{for } t > 0, \\ \eta(x, 0) = \eta_0(x), \end{cases}$$

and set the Lagrangian density and velocity by

$$f(x,t) = \rho(\eta(x,t), t) \quad \text{and} \quad v(x,t) = u(\eta(x,t), t).$$

Then the Lagrangian version of system (1.1) can be written on the reference domain $\mathcal{I}$ as

$$\begin{cases} f_t + f v_x/\eta_x = 0 & \text{in } \mathcal{I} \times (0, \infty), \\ f v_t + \frac{1}{\gamma}(f^\gamma)_x/\eta_x = -\frac{\mu}{(1+t)^\lambda} f v & \text{in } \mathcal{I} \times (0, \infty), \\ f > 0 \text{ in } \mathcal{I} \times (0, \infty), & f = 0 \text{ on } \partial\mathcal{I} \times (0, \infty), \\ (f, v) = (\rho_0(\eta_0), u_0(\eta_0)) & \text{on } \mathcal{I} \times \{t = 0\}. \end{cases} \tag{2.2}$$

The map $\eta(\cdot, t)$ defined above can be extended to $\overline{\mathcal{I}} = [0, +\infty)$. In the setting, the vacuum free boundary for problem (1.1) is given by

$$x_b(t) = \eta(\bar{x}_b(0), t) = \eta(0, t) \quad \text{for } t \geq 0. \tag{2.3}$$

It follows from solving $(2.2)_1$ and using (2.1) that

$$f(x,t)\eta_x(x,t)=\rho_0(\eta_0(x))\,\eta_0'(x)=\bar\rho_0(x),\quad x\in\mathcal{I}. \tag{2.4}$$

It should be noticed that we need $\eta_x(x,t)>0$ for $x\in\mathcal{I}$ and $t\geq 0$ to make the Lagrangian transformation sensible, which will be verified later. So, the initial density of the background solution, $\bar\rho_0$, can be regarded as a parameter, and system (2.2) can be rewritten as

$$\begin{cases}\bar\rho_0\eta_{tt}+\frac{\mu}{(1+t)^\lambda}\bar\rho_0\eta_t+\frac{1}{\gamma}\left(\bar\rho_0^\gamma/\eta_x^\gamma\right)_x=0 & \text{in } \mathcal{I}\times(0,\infty),\\ (\eta,\eta_t)=(\eta_0,u_0(\eta_0)) & \text{on } \mathcal{I}\times\{t=0\}.\end{cases} \tag{2.5}$$

2.2. Main results

Our main purpose is to study the stability of the background solution (1.2) and (1.3). Set $\bar\eta(x,t):=x+\int_0^t\bar u(\tau)d\tau$. It is not hard to check that

$$\bar\rho_0\bar\eta_{tt}+\frac{\mu}{(1+t)^\lambda}\bar\rho_0\bar\eta_t+\frac{1}{\gamma}\left(\bar\rho_0^\gamma/\bar\eta_x^\gamma\right)_x=0\quad\text{in } \mathcal{I}\times(0,\infty). \tag{2.6}$$

Let

$$w(x,t)=\eta(x,t)-\bar\eta(x,t). \tag{2.7}$$

Then subtract (2.6) from $(2.5)_1$, we see that w satisfy

$$\begin{cases}\bar\rho_0 w_{tt}+\dfrac{\mu}{(1+t)^\lambda}\bar\rho_0 w_t+\dfrac{1}{\gamma}\left[\bar\rho_0^\gamma\left((1+w_x)^{-\gamma}-1\right)\right]_x=0 & \text{in } \mathcal{I}\times(0,\infty),\\ (w,w_t)=(\eta_0-x,u_0(\eta_0)) \quad \text{on } \mathcal{I}\times\{t=0\}.\end{cases} \tag{2.8}$$

In the rest of the paper, we will use the notation

$$\int=:\int_{\mathcal{I}},\qquad \|\cdot\|=:\|\cdot\|_{L^2(\mathcal{I})},\qquad\text{and}\qquad \|\cdot\|_{L^\infty}=:\|\cdot\|_{L^\infty(\mathcal{I})}.$$

Denote $\alpha=\frac{1}{\gamma-1}$ and $m=3+[\frac{1}{\gamma-1}]$. Let $\delta\in(0,\lambda+1)$. For $j=0,\ldots,m$ and $i=0,\ldots,m-j$, we set

$$\begin{aligned}\mathcal{E}_j(t):=(1+t)^{2j-\delta\mathbf{1}_{\lambda<1}}\int_{\mathcal{I}}\Big[&\bar\rho_0(\partial_t^j w)^2\\ &+(1+t)^{\lambda+1}\left(\bar\rho_0(\partial_t^{j+1}w)^2+\bar\rho_0^\gamma(\partial_t^j w_x)^2\right)\Big](x,t)dx,\\ \mathcal{E}_{j,i}(t):=(1+t)^{2j+\lambda+1-\delta\mathbf{1}_{\lambda<1}}\int_{\mathcal{I}}\Big[&\bar\rho_0^{1+(i+1)(\gamma-1)}(\partial_t^j\partial_x^{i+1}w)^2\\ &+\bar\rho_0^{1+(i-1)(\gamma-1)}(\partial_t^j\partial_x^i w)^2\Big](x,t)dx,\end{aligned}$$

where $\mathbf{1}_{\lambda<1}$ is the characteristic function on $\{\lambda < 1\}$, which means

$$\mathbf{1}_{\lambda<1} = \begin{cases} 1, & \text{if } \lambda < 1, \\ 0, & \text{if } 1 \le \lambda. \end{cases}$$

If we set

$$\sigma(x) := \bar{\rho}_0^{\gamma-1}(x) = (\gamma - 1)x \approx x, \quad x \in \mathcal{I},$$

then $\mathcal{E}_j$ and $\mathcal{E}_{j,i}$ can be rewritten as

$$\begin{aligned} \mathcal{E}_j(t) &= (1+t)^{2j-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[\sigma^\alpha \left(\partial_t^j w\right)^2 \\ &\quad + (1+t)^{\lambda+1} \left(\sigma^\alpha (\partial_t^{j+1} w)^2 + \sigma^{\alpha+1} (\partial_t^j w_x)^2\right)\Big](x,t)dx, \\ \mathcal{E}_{j,i}(t) &= (1+t)^{2j+\lambda+1-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \Big[\sigma^{\alpha+i+1} \left(\partial_t^j \partial_x^{i+1} w\right)^2 \\ &\quad + \sigma^{\alpha+i-1} \left(\partial_t^j \partial_x^i w\right)^2\Big](x,t)dx. \end{aligned} \tag{2.9}$$

Remark 2.1. By using (3.6) below, we see that actually

$$\mathcal{E}_{j,i}(t) \approx (1+t)^{2j+1+\lambda-\delta\mathbf{1}_{\lambda<1}} \int_{\mathcal{I}} \sigma^{\alpha+i+1} \left(\partial_t^j \partial_x^{i+1} w\right)^2 dx. \tag{2.10}$$

The total energy is defined by

$$\mathcal{E}(t) := \sum_{j=0}^{m} \left(\mathcal{E}_j(t) + \sum_{i=0}^{m-j} \mathcal{E}_{j,i}(t)\right).$$

The bound of $\mathcal{E}(t)$ gives the uniform bound and decay of w and its derivatives. See (2.11) below. Now we are ready to state the main result.

Theorem 2.2. *Suppose that $\lambda = 1, 2 < \mu$ or $0 < \lambda < 1, 0 < \mu$. There exists a constant ϵ_0 such that if $\mathcal{E}(0) \le \epsilon_0$, then the problem (2.8) admits a global unique smooth solution in $\mathcal{I} \times [0, \infty)$ satisfying for all $t \ge 0$*

$$\mathcal{E}(t) \le C\mathcal{E}(0),$$

and

$$
\begin{aligned}
&\sum_{j=0}^{m}(1+t)^{2j-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{j-3,0\}}{2}}\partial_t^j w(\cdot,t)\right\|_{L^\infty}^2 \\
&+\sum_{\substack{i+j\le m\\ i\ge 1}}(1+t)^{2j+\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{2i+j-3,0\}}{2}}\partial_t^j\partial_x^i w(\cdot,t)\right\|_{L^\infty}^2\le C\mathcal{E}(t),
\end{aligned}
\tag{2.11}
$$

where C is a positive constant independent of t.

As a corollary of Theorem 2.2, we have the following theorem for solutions to the original vacuum free boundary problem (1.1).

Theorem 2.3. *Suppose that $\lambda=1, 2<\mu$ or $0<\lambda<1, 0<\mu$. There exists a constant $\epsilon_0>0$ such that if $\mathcal{E}(0)\le\epsilon_0$, then the problem (1.1) admits a global unique smooth solution $(\rho, u, \mathrm{I}(t))$ for $t\in[0,\infty)$ satisfying*

$$|\rho(\eta(x,t),t)-\bar\rho(\bar\eta(x,t),t)|\le Cx^{\frac{1}{\gamma-1}}(1+t)^{-\frac{\lambda+1}{2}+\frac{\delta}{2}\mathbf{1}_{\lambda<1}}, \tag{2.12}$$

$$|u(\eta(x,t),t)-\bar u(\bar\eta(x,t),t)|\le C(1+t)^{-1+\frac{\delta}{2}\mathbf{1}_{\lambda<1}}, \tag{2.13}$$

$$x_b(t)\approx-(1+t)^{\lambda+1}, \tag{2.14}$$

$$\left|\frac{d^k x_b(t)}{dt^k}\right|\le C(1+t)^{\lambda+1-k},\quad k=1,2, \tag{2.15}$$

for all $x\in\mathcal{I}$ and $t\ge 0$. Here C is a positive constant independent of t.

The pointwise behavior of the density and the convergence of the velocity for the vacuum free boundary problem (1.1) to (1.2) and (1.3) is given by (2.12) and (2.13), respectively. (2.14) gives the precise expanding rate of the vacuum boundaries, which is the same as $\bar x_b(t)$. It is also shown in (2.12) that the difference of the density of problem (1.1) and (1.3) decays at the rate of $(1+t)^{-\left(\frac{\lambda+1}{2}\right)^-}$ when $\lambda<1$ and $(1+t)^{-2}$ for $\lambda=1$ in L^∞, where a^- denotes a constant which is smaller than but can be arbitrarily close to a.

Due to the finite-time blow up of smooth solutions for (2.8) with $\lambda=1, 0<\mu\le 2$ or $1<\lambda, 0<\mu$ in [32,33] under the assumption that the density and the velocity is a small perturbation of $(\bar\rho,\bar u)=(1,0)$, we give the following conjecture, which will be considered in our further work.

Conjecture 2.4. *Suppose that $\lambda=1, 0<\mu\le 2$ or $1<\lambda, 0<\mu$. The smooth solution of (2.8) will blow up in finite time for a family of smooth initial data $(w,\partial_t w)|_{t=0}$ even if $(w,\partial_t w)|_{t=0}$ are sufficiently small.*

3. Proof of Theorem 2.2

At the beginning, we give a weighted Sobolev L^∞ embedding Lemma for later use.

Lemma 3.1. *Suppose that $\mathcal{E}(t)$ is finite, then it holds that*

$$\underbrace{\sum_{j=0}^{m}(1+t)^{2j-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{j-3,0\}}{2}}\partial_t^j w(\cdot,t)\right\|_{L^\infty}^2}_{E_1}$$
$$+\underbrace{\sum_{\substack{i+j\le m\\ i\ge 1}}(1+t)^{2j+\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{2i+j-3,0\}}{2}}\partial_t^j\partial_x^i w(\cdot,t)\right\|_{L^\infty}^2}_{E_2}\lesssim\mathcal{E}(t).$$

The idea of proving Lemma 3.1 comes from Lemma 3.7 of [30]. But there are some differences since now the integration interval, $\mathcal{I}:=(0,+\infty)$ is infinite while in Lemma 3.7 of [30], the reference interval is finite. We give the proof in Appendix B.

The proof of Theorem 2.2 is based on the local existence of smooth solutions (cf. [7,24]) and continuation arguments. The uniqueness of the smooth solutions can be obtained as in section 11 of [28]. In order to prove the global existence of smooth solutions, we need to obtain the uniform-in-time a priori estimates on any given time interval $[0,T]$ satisfying $\sup_{t\in[0,T]}\mathcal{E}(t)<\infty$. To this end, we use a bootstrap argument by making the following a priori assumption: there exists a suitably small fixed positive number $\epsilon_0\in(0,1)$ independent of t such that

$$\sup_{0\le t\le T}\mathcal{E}(t)\le M\epsilon_0, \tag{3.1}$$

for some constant M, independent of ϵ_0, to be determined later. Under this a priori assumption, and by using Lemma 3.1, we see that

$$\begin{aligned}&\sum_{j=0}^{m}(1+t)^{2j-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{j-3,0\}}{2}}\partial_t^j w(\cdot,t)\right\|_{L^\infty}^2\\&+\sum_{\substack{i+j\le m\\ i\ge 1}}(1+t)^{2j+\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\max\{2i+j-3,0\}}{2}}\partial_t^j\partial_x^i w(\cdot,t)\right\|_{L^\infty}^2\\&\le\mathcal{E}(t)\le CM\epsilon_0,\quad t\in[0,T].\end{aligned} \tag{3.2}$$

Here we can assume that $M\epsilon_0$ is sufficiently small such that $M\epsilon_0\ll 1$. Then we show in subsection 3.2 the following elliptic estimates:

$$\mathcal{E}_{j,i}(t)\le C\sum_{\ell=0}^{i+j}\mathcal{E}_\ell(t)\quad\text{when } i,\ j\ge 0,\ i+j\le m, \tag{3.3}$$

where C is a positive constant independent of t.

With (3.2) and elliptic estimates (3.3), we show in subsection 3.3 the following nonlinear weighted energy estimate: for some positive constant C independent of t

$$\mathcal{E}_j(t) \le C \sum_{\ell=0}^{j} \mathcal{E}_\ell(0), \quad j = 0, 1, \dots, m. \tag{3.4}$$

Combining (3.3) and (3.4), we see that

$$\mathcal{E}(t) \le C_* \mathcal{E}(0). \tag{3.5}$$

By choosing $M = 2C_*$, we see that

$$\mathcal{E}(t) \le \frac{1}{2} M \epsilon_0,$$

which closes the energy estimate.

3.1. Preliminaries

In this subsection, we present some embedding estimates for weighted Sobolev spaces that will be used later.

Lemma 3.2. *If $f \in C^1([0, +\infty))$ and decays sufficiently fast as $x \to +\infty$, then we have, for $\theta > 1$ and $\theta \in \mathbb{R}$,*

$$\int_0^\infty x^{\theta-2} f^2 dx \le C_\theta \int_0^\infty x^\theta f_x^2 dx. \tag{3.6}$$

If $f \in C^1([0, 1])$, then we have, for $\theta > 1$ and $\theta \in \mathbb{R}$,

$$\int_0^1 x^{\theta-2} f^2 dx \le C_\theta \int_0^1 x^\theta (f^2 + f_x^2) dx. \tag{3.7}$$

Proof.

$$\begin{aligned}
\int_0^\infty x^{\theta-2} f^2 dx &= \int_0^\infty f^2 d\frac{x^{\theta-1}}{\theta-1} \\
&= -\frac{1}{\theta-1} \int_0^\infty x^{\theta-1} (f^2)_x dx \\
&\le \nu \int_0^\infty x^{\theta-2} f^2 dx + C_\nu \int_0^\infty x^\theta f^2 dx.
\end{aligned}$$

By choosing small ν to let the first term of the righthand of the above inequality be absorbed by the left, we can get (3.6).

For (3.7), we have

$$
\begin{aligned}
\int_0^1 x^{\theta-2} f^2 dx &= \int_0^1 f^2 d\frac{x^{\theta-1}}{\theta-1} \\
&= \frac{f^2(1)}{\theta-1} - \frac{1}{\theta-1}\int_0^1 x^{\theta-1}(f^2)_x dx \\
&= \frac{1}{\theta-1}\int_0^1 (x^{\theta+1} f^2)_x dx - \frac{1}{\theta-1}\int_0^1 x^{\theta-1}(f^2)_x dx \\
&= \frac{\theta+1}{\theta-1}\int_0^1 x^{\theta} f^2 dx + \frac{1}{\theta-1}\int_0^1 (x^{\theta+1} - x^{\theta-1})(f^2)_x dx \\
&\le C_\theta \int_0^1 x^{\theta} f^2 dx + C_\theta \int_0^1 (x^{\theta-1}) f f_x dx \\
&\le \nu \int_0^\infty x^{\theta-2} f^2 dx + C_{\theta,\nu}\int_0^\infty x^{\theta}(f^2 + f_x^2) dx.
\end{aligned}
$$

Also by choosing small ν to let the first term of the righthand of the above inequality be absorbed by the left, we can get (3.7). □

For any $a > 0$ and nonnegative integer b, the weighted Sobolev space $H^{a,b}([0,1])$ is given by

$$
H^{a,b}([0,1]) := \sum_{k=0}^{b} \int_0^1 x^a |\partial_x^k F|^2 dx.
$$

Then for $b \ge a/2$, we have the following embedding of weighted Sobolev spaces (cf [26])

$$
H^{a,b}([0,1]) \hookrightarrow H^{b-a/2}([0,1])
$$

with the estimate

$$
\|F\|_{H^{b-a/2}([0,1])} \le C_{a,b} \|F\|_{H^{a,b}([0,1])}. \tag{3.8}
$$

3.2. Elliptic estimates

We prove the following elliptic estimates in this subsection.

Proposition 3.3. *Under the assumption of* (3.1) *for suitably small positive number* $\epsilon_0 \in (0,1)$*, then for* $0 \le t \le T$*, we have*

$$\mathcal{E}_{j,i}(t) \lesssim \sum_{\ell=0}^{i+j} \mathcal{E}_\ell(t) \quad \text{when } i,\ j \ge 0,\ i+j \le m.$$

The proof of this proposition consists of Lemma 3.4 and Lemma 3.5 below.

Lower-Order Elliptic Estimates

Equation $(2.8)_1$ can be rewritten as

$$\left(\bar\rho_0^\gamma w_x\right)_x = \bar\rho_0 w_{tt} + \frac{\mu}{(1+t)^\lambda}\bar\rho_0 w_t + \frac{1}{\gamma}\left[\bar\rho_0^\gamma\left((1+w_x)^{-\gamma} - 1 + \gamma w_x\right)\right]_x.$$

Divide the above equation above by $\bar\rho_0$ and expand the resulting equation to obtain

$$\begin{aligned}\sigma w_{xx} + \gamma w_x = {} & w_{tt} + \frac{\mu}{(1+t)^\lambda} w_t - \sigma\left[(1+w_x)^{-\gamma-1} - 1\right] w_{xx} \\ & + \left[(1+w_x)^{-\gamma} - 1 + \gamma w_x\right].\end{aligned} \tag{3.9}$$

Lemma 3.4. *Under the assumption of* (3.1) *for suitably small positive number* $\epsilon_0 \in (0,1)$*, then*

$$\mathcal{E}_{0,0}(t) \lesssim \mathcal{E}_0(t), \quad \mathcal{E}_{1,0}(t) + \mathcal{E}_{0,1}(t) \lesssim \mathcal{E}_1(t),\ 0 \le t \le T.$$

Proof. When $i = 0$, we using (3.6) to see that

$$\begin{aligned}\mathcal{E}_{j,0}(t) &:= (1+t)^{2j+1+\lambda-\delta\mathbf{1}_{\lambda<1}} \int \left[\sigma^{\alpha+1}(\partial_t^j w_x)^2 + \sigma^{\alpha-1}(\partial_t^j w)^2\right](x,t)dx \\ &\lesssim (1+t)^{2j+1+\lambda-\delta\mathbf{1}_{\lambda<1}} \int \sigma^{\alpha+1}(\partial_t^j w_x)^2 dx \\ &\lesssim \mathcal{E}_j(t),\end{aligned}$$

which implies that $\mathcal{E}_{0,0}(t) \lesssim \mathcal{E}_0(t)$ and $\mathcal{E}_{1,0}(t) \lesssim \mathcal{E}_1(t)$. We mainly focus on the proof of $\mathcal{E}_{0,1}(t) \lesssim \mathcal{E}_1(t)$.

Multiply equation (3.9) by $\sigma^{\alpha/2}$ and perform the spatial L^2-norm to obtain

$$\begin{aligned}&(1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx} + \gamma\sigma^{\frac{\alpha}{2}} w_x\right\|^2 \\ \le & C\left((1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha}{2}} w_{tt}\right\|^2 + (1+t)^{1-\lambda-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha}{2}} w_t\right\|^2\right) \\ & + C\left((1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{1+\frac{\alpha}{2}} w_x w_{xx}\right\|^2 + (1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha}{2}} w_x^2\right\|^2\right) \\ \le & C\mathcal{E}_1 + C\|w_x\|_{L^\infty}^2 (1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left(\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 + \left\|\sigma^{\frac{\alpha}{2}} w_x\right\|^2\right) \\ \le & C\mathcal{E}_1 + C\epsilon_0\left(\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 + \left\|\sigma^{\frac{\alpha}{2}} w_x\right\|^2\right),\end{aligned} \tag{3.10}$$

where we have used the Taylor expansion, the smallness of w_x (which is the consequence of (3.2)) to derive the first inequality and the definition of $\mathcal{E}_1$ to the second. Note that the left-hand side of (3.10) can be expanded as

$$\begin{aligned}
&\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx} + \gamma\sigma^{\frac{\alpha}{2}} w_x\right\|^2 \\
=&\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 + \gamma^2\left\|\sigma^{\frac{\alpha}{2}} w_x\right\|^2 + \gamma\int \sigma^{1+\alpha}\left(w_x^2\right)_x dx \\
=&\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 + \gamma^2\left\|\sigma^{\frac{\alpha}{2}} w_x\right\|^2 - \gamma(1+\alpha)\int \sigma^{\alpha}\sigma_x w_x^2 dx \\
=&\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 + \gamma^2\left\|\sigma^{\frac{\alpha}{2}} w_x\right\|^2 - \gamma(1+\alpha)(\gamma-1)\int \sigma^{\alpha} w_x^2 dx \\
=&\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 .
\end{aligned} \tag{3.11}$$

At the last line of the above inequality, we use the fact that $(1+\alpha)(\gamma-1)=\gamma$.

By combining (3.10) and (3.11) and using (3.6), we get

$$\begin{aligned}
&(1+t)^{\lambda+1-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 \\
\leq& C\mathcal{E}_1 + C\epsilon_0\left\|\sigma^{1+\frac{\alpha}{2}} w_{xx}\right\|^2 .
\end{aligned}$$

Remembering (2.10) and smallness of ϵ_0, the above inequality indicates that

$$\mathcal{E}_{0,1}(t) \leq C\mathcal{E}_1(t). \quad \square$$

Higher-Order Elliptic Estimates

For $i \geq 1$ and $j \geq 0$, applying $\partial_t^j\partial_x^{i-1}$ to (3.9) yields that

$$\begin{aligned}
&\sigma\partial_t^j\partial_x^{i+1} w + ((\gamma-1)i+1)\,\partial_t^j\partial_x^i w \\
=&\partial_t^{j+2}\partial_x^{i-1} w + \frac{\mu}{(1+t)^{\lambda}}\partial_t^{j+1}\partial_x^{i-1} w + Q_1 + Q_2,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
Q_1 :=& -\partial_t^j\partial_x^{i-1}\left\{\sigma\left[(1+w_x)^{-\gamma-1}-1\right]w_{xx}\right\} \\
&+ \partial_t^j\partial_x^{i-1}\left[(1+w_x)^{-\gamma}-1+\gamma w_x\right], \\
Q_2 :=& \mu\sum_{\ell=1}^{j} C_j^{\ell}\partial_t^{\ell}(1+t)^{-\lambda}\partial_t^{j+1-\ell}\partial_x^{i-1} w.
\end{aligned} \tag{3.13}$$

Multiply equation (3.12) by $\sigma^{\frac{\alpha+i-1}{2}}$ and perform the spatial L^2-norm to obtain

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_x^{i+1}w+((\gamma-1)i+1)\sigma^{\frac{\alpha+i-1}{2}}\partial_t^j\partial_x^i w\right\|^2\\
&\lesssim\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_x^{i-1}w\right\|^2+(1+t)^{-2\lambda}\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_x^{i-1}w\right\|^2\\
&\quad+\left\|\sigma^{\frac{\alpha+i-1}{2}}(Q_1,Q_2)\right\|^2.
\end{aligned}$$

Similar to the derivation of (3.11), we can get

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_x^{i+1}w\right\|^2\\
&\lesssim\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_x^{i-1}w\right\|^2+(1+t)^{-2\lambda}\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_x^{i-1}w\right\|^2\\
&\quad+\left\|\sigma^{\frac{\alpha+i-1}{2}}(Q_1,Q_2)\right\|^2.
\end{aligned}\tag{3.14}$$

We will use this estimate to prove the following lemma by induction.

Lemma 3.5. *Under the assumption of* (3.1) *for suitably small positive number* $\epsilon_0\in(0,1)$. *Then for* $j\ge 0, i\ge 1$, *and* $0\le i+j\le m$

$$\mathcal{E}_{j,i}(t)\lesssim\sum_{\ell=0}^{i+j}\mathcal{E}_\ell(t),\quad t\in[0,T].\tag{3.15}$$

Proof. We use induction on $i+j$ to prove this lemma. As shown in Lemma 3.4, we know that (3.15) holds for $i+j=1$. For $1\le k\le m-1$, we make the induction hypothesis that (3.15) holds for all $i, j\ge 0$, and $i+j\le k$, that is,

$$\mathcal{E}_{j,i}(t)\lesssim\sum_{\ell=0}^{i+j}\mathcal{E}_\ell(t),\quad i\ge 1,\ j\ge 0,\ i+j\le k,\tag{3.16}$$

it then suffices to prove (3.15) for $i\ge 1,\ \ j\ge 0$, and $i+j=k+1$. We will bound $\mathcal{E}_{k+1-\ell,\ell}$ from $\ell=1$ to $k+1$ step by step.

The main difficulty is to control the term $\left\|\sigma^{\frac{\alpha+i-1}{2}}(Q_1,Q_2)\right\|^2$ in (3.14).

We estimate Q_2 given by (3.13) as follows. For Q_2, it is easy to see that

$$|Q_2|\lesssim\sum_{\ell=1}^{j}(1+t)^{-\lambda-\ell}\left|\partial_t^{j+1-\ell}\partial_x^{i-1}w\right|.$$

So that by using the definition of $\mathcal{E}_{j,i}$ and $\mathcal{E}_j$, we have

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i-1}{2}} Q_2\right\|^2 \\
\lesssim&\sum_{\ell=1}^{j}(1+t)^{-2\lambda-2\ell}\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1-\ell}\partial_x^{i-1}w\right\|^2 \\
\lesssim&(1+t)^{-2j-2(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\sum_{\ell=1}^{j}\left(\mathcal{E}_{j+1-\ell}\mathbf{1}_{i=1}+\mathcal{E}_{j+1-\ell,i-2}\mathbf{1}_{i\geq 2}\right).
\end{aligned}$$

For Q_1, it follows from (3.13) and (3.2) that

$$\begin{aligned}
|Q_1|\lesssim&\sum_{n=0}^{j}\sum_{\ell=0}^{i-1}K_{n\ell}\left(\left|\partial_t^{j-n}\partial_x^{i-1-\ell}(\sigma w_{xx})\right|+\left|\partial_t^{j-n}\partial_x^{i-1-\ell}w_x\right|\right) \\
\lesssim&\sum_{n=0}^{j}\sum_{\ell=0}^{i-1}K_{n\ell}\left(\left|\sigma\partial_t^{j-n}\partial_x^{i-\ell+1}w\right|+\left|\partial_t^{j-n}\partial_x^{i-\ell}w\right|\right) \\
=:&\sum_{n=0}^{j}\sum_{\ell=0}^{i-1}Q_{2n\ell}.
\end{aligned}$$

Here the main term of $K_{n\ell}$ is $\partial_t^n\partial_x^\ell w_x$. We only go to estimate $K_{n\ell}$ for $n+\ell\leq 2$ since we can use the same method to estimate $Q_{2n\ell}$ for $n+\ell\geq 3$ as that for $n+\ell\leq 2$.

First, for $n=\ell=0$, using (3.2), we have

$$\begin{aligned}
\left\|\sigma^{\frac{\alpha+i-1}{2}}Q_{100}\right\|^2\lesssim&\|w_x\|_{L^\infty}^2\left(\left\|\sigma^{\frac{\alpha+i+1}{2}}\partial_t^j\partial_x^{i+1}w\right\|^2+\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^j\partial_x^i w\right\|^2\right) \\
\lesssim&\epsilon_0(1+t)^{-2j-2(1+\lambda)+2\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j,i} \\
\lesssim&\epsilon_0(1+t)^{-2j-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j,i}.
\end{aligned}$$

Here we have used the fact that $\delta\in(0,1+\lambda)$. And for $n=0$, $\ell=1$,

$$\begin{aligned}
\left\|\sigma^{\frac{\alpha+i-1}{2}}Q_{101}\right\|^2\lesssim&\|\sigma^{1/2}w_{xx}\|_{L^\infty}^2\left(\left\|\sigma^{\frac{\alpha+i}{2}}\partial_t^j\partial_x^{i}w\right\|^2+\left\|\sigma^{\frac{\alpha+i-2}{2}}\partial_t^j\partial_x^{i-1} w\right\|^2\right) \\
\lesssim&\epsilon_0(1+t)^{-2j-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j,i-1}.
\end{aligned}$$

Also, case $n=1$, $\ell=0$ can be estimated the same by using (3.2)

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i-1}{2}} Q_{110}\right\|^2\\
\lesssim&\|\partial_t w_x\|_{L^\infty}^2 \left(\left\|\sigma^{\frac{\alpha+i+1}{2}} \partial_t^{j-1}\partial_x^{i+1} w\right\|^2 + \left\|\sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1}\partial_x^{i} w\right\|^2\right)\\
\lesssim&\,\epsilon_0(1+t)^{-2-1-\lambda+\delta\mathbf{1}_{\lambda<1}}(1+t)^{-2(j-1)-1-\lambda+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j-1,i}\\
\lesssim&\,\epsilon_0(1+t)^{-2j-1-\lambda+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j-1,i}.
\end{aligned}$$

Here we have $j \geq 1$.

Then for $n = 2$, $\ell = 0$, we have $j \geq 2$ and

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i-1}{2}} Q_{120}\right\|^2\\
\lesssim&\|\sigma^{1/2} w_{xtt}\|_{L^\infty}^2 \left(\left\|\sigma^{\frac{\alpha+i}{2}} \partial_t^{j-2}\partial_x^{i+1} w\right\|^2 + \left\|\sigma^{\frac{\alpha+i-2}{2}} \partial_t^{j-2}\partial_x^{i} w\right\|^2\right)\\
\lesssim&\|\sigma^{1/2} w_{xtt}\|_{L^\infty}^2 \left\|\sigma^{\frac{\alpha+i+2}{2}} \partial_t^{j-2}\partial_x^{i+2} w\right\|^2\\
\lesssim&\epsilon_0(1+t)^{-4-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}(1+t)^{-2(j-2)-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j-2,i+1}\\
\lesssim&\epsilon_0(1+t)^{-2j-1-\lambda+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j-2,i+1}.
\end{aligned}$$

Here we have used the hardy inequality (3.6) to estimate

$$\left\|\sigma^{\frac{\alpha+i-2}{2}} \partial_t^{j-2}\partial_x^{i} w\right\|^2 \lesssim \left\|\sigma^{\frac{\alpha+i}{2}} \partial_t^{j-2}\partial_x^{i+1} w\right\|^2 \lesssim \left\|\sigma^{\frac{\alpha+i+2}{2}} \partial_t^{j-2}\partial_x^{i+2} w\right\|^2,$$

since $\alpha + i - 2 \geq \alpha - 1 > -1$.

In the case $n = \ell = 1$, we have $i \geq 2$.

$$\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i-1}{2}} Q_{111}\right\|^2\\
\lesssim&\|\sigma w_{xxt}\|_{L^\infty}^2 \left(\left\|\sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1}\partial_x^{i} w\right\|^2 + \left\|\sigma^{\frac{\alpha+i-3}{2}} \partial_t^{j-1}\partial_x^{i-1} w\right\|^2\right)\\
\lesssim&\|\sigma w_{xxt}\|_{L^\infty}^2 \left\|\sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1}\partial_x^{i} w\right\|^2\\
\lesssim&\epsilon_0(1+t)^{-2j-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j-1,i}.
\end{aligned}$$

Here due to $i \geq 2$ and (3.6), we have used

$$\left\|\sigma^{\frac{\alpha+i-3}{2}} \partial_t^{j-1}\partial_x^{i-1} w\right\|^2 \lesssim \left\|\sigma^{\frac{\alpha+i-1}{2}} \partial_t^{j-1}\partial_x^{i} w\right\|^2.$$

In the case $n = 0$, $\ell = 2$, we have $i \geq 3$.

$$
\begin{aligned}
&\left\|\sigma^{\frac{\alpha+i-1}{2}} Q_{102}\right\|^2\\
\lesssim&\|\sigma^{3/2} w_{xxx}\|^2_{L^\infty}\left(\left\|\sigma^{\frac{\alpha+i-2}{2}}\partial_t^j\partial_x^{i-1}w\right\|^2+\left\|\sigma^{\frac{\alpha+i-4}{2}}\partial_t^j\partial_x^{i-2}w\right\|^2\right)\\
\lesssim&\|\sigma^{3/2} w_{xxx}\|^2_{L^\infty}\left\|\sigma^{\frac{\alpha+i-2}{2}}\partial_t^j\partial_x^{i-1}w\right\|^2\\
\lesssim&\epsilon_0(1+t)^{-2j-(1+\lambda)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_{j,i-1}.
\end{aligned}
$$

Since the leading term of $Q_{n\ell}$ is

$$
\sum_{\ell}^{n}|\partial_t^n\partial_x^{\ell+1}w|\left\{|\sigma\partial_t^{j-n}\partial_x^{i+1-\ell}w|+|\partial_t^{j-n}\partial_x^{i-\ell}w|\right\},
$$

other terms for $n+\ell\geq 3$ can be handled with the same line.

Now combining the above estimates, we get

$$
\begin{aligned}
&(1+t)^{2j+1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha+i-1}{2}}(Q_1,Q_2)\right\|^2\\
\lesssim&\epsilon_0\mathcal{E}_{j,i}+\sum_{\substack{0\leq\ell\leq j\\ \ell+r\leq i+j-1}}\mathcal{E}_{\ell,r}+\sum_{\ell=0}^{j}\mathcal{E}_\ell.
\end{aligned}
$$

Substituting this into (3.14), we get

$$
\begin{aligned}
\mathcal{E}_{j,i}\lesssim&(1+t)^{2j+1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+2}\partial_x^{i-1}w\right\|^2\\
&+(1+t)^{2j+1-\lambda-\delta\mathbf{1}_{\lambda<1}}\left\|\sigma^{\frac{\alpha+i-1}{2}}\partial_t^{j+1}\partial_x^{i-1}w\right\|^2\\
&+\epsilon_0\mathcal{E}_{j,i}+\sum_{\substack{0\leq\ell\leq j\\ \ell+r\leq i+j-1}}\mathcal{E}_{\ell,r}+\sum_{\ell=0}^{j}\mathcal{E}_\ell.
\end{aligned}
\tag{3.17}
$$

In particularly, when $i\geq 2$, we have

$$
\mathcal{E}_{j,i}\lesssim\mathcal{E}_{j+2,i-2}+\mathcal{E}_{j+1,i-2}+\sum_{\substack{0\leq\ell\leq j\\ \ell+r\leq i+j-1}}\mathcal{E}_{\ell,r}+\sum_{\ell=0}^{j}\mathcal{E}_\ell.
\tag{3.18}
$$

In what follows, we use (3.18) and the induction hypothesis (3.16) to show that (3.15) holds for $i+j=k+1$. First, choosing $j=k$ and $i=1$ in (3.17) gives

$$
\mathcal{E}_{k,1}(t)\lesssim\mathcal{E}_{k+1}(t)+\mathcal{E}_k(t)+\sum_{\substack{0\leq\ell\leq k\\ \ell+r\leq k}}\mathcal{E}_{\ell,r}+\sum_{\ell=0}^{k}\mathcal{E}_\ell,
$$

which, together with (3.16) implies

$$\mathcal{E}_{k,1}(t) \lesssim \sum_{\ell=0}^{k+1} \mathcal{E}_\ell(t).$$

Similarly, using (3.18), we have

$$\mathcal{E}_{k-1,2}(t) \lesssim \mathcal{E}_{k+1}(t) + \mathcal{E}_k(t) + \sum_{\substack{0\le\ell\le k-1\\ \ell+r\le k}} \mathcal{E}_{\ell,r} + \sum_{\ell=0}^{k-1} \mathcal{E}_\ell \lesssim \sum_{\ell=0}^{k+1} \mathcal{E}_\ell(t).$$

For $\mathcal{E}_{k-2,3}$, it follows from (3.18) and (3.16) that

$$\mathcal{E}_{k-2,3}(t) \lesssim \mathcal{E}_{k,1}(t) + \mathcal{E}_{k-1,1}(t) + \sum_{\substack{0\le\ell\le k-2\\ \ell+r\le k}} \mathcal{E}_{\ell,r} + \sum_{\ell=0}^{k-2} \mathcal{E}_\ell \lesssim \sum_{\ell=0}^{k+1} \mathcal{E}_\ell(t).$$

The other cases can be handled similarly. So we have proved (3.16) when $i + j = k + 1$. This finishes the proof of Lemma 3.5. □

3.3. Nonlinear weighted energy estimates

In this subsection, we prove that the weighted energy $\mathcal{E}_j(t)$ can be bounded by the initial data for $t \in [0, T]$.

Proposition 3.6. *Suppose that* (3.1) *holds for a suitably small positive number* $\epsilon_0 \in (0, 1)$. *Then for* $t \in [0, T]$

$$\mathcal{E}_j(t) \lesssim \sum_{\ell=0}^{j} \mathcal{E}_\ell(0), \quad j = 0, 1, \ldots, m.$$

The proof of Proposition 3.6 contains Lemma 3.7 and Lemma 3.8 below.

Basic Energy Estimates

Lemma 3.7. *Suppose that* (3.1) *holds for a suitably small positive number* $\epsilon_0 \in (0, 1)$. *Then*

$$\mathcal{E}_0(t) + \int_0^t \int \left[(1+\tau)^{1-\delta \mathbf{1}_{\lambda<1}} \sigma^\alpha w_\tau^2 + (1+\tau)^{\lambda-\delta \mathbf{1}_{\lambda<1}} \sigma^{\alpha+1} w_x^2\right] dx d\tau \lesssim \mathcal{E}_0(0), \quad t \in [0, T]. \tag{3.19}$$

Proof. In order to simplify the presentation, by using Taylor expansion and smallness of w_x, we rewrite $(2.8)_1$ as follows

$$\sigma^\alpha w_{tt} + \frac{\mu}{(1+t)^\lambda} \sigma^\alpha w_t - [\sigma^{\alpha+1}(1+o(1)) w_x]_x = 0, \tag{3.20}$$

where $o(1)$ means $o(1) \lesssim \sqrt{\epsilon_0}$.

The proof will be divided into two parts. One is for $0 < \lambda < 1, 0 < \mu$ and the other is for $\lambda = 1, 2 < \mu$

Case 1: $0 < \lambda < 1, 0 < \mu$

Multiplying (3.20) by $(K+t)^\lambda w_t$, where $K > 1$ is a suitably large constant, to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\frac{1}{2}\frac{d}{dt}\int \sigma^\alpha (K+t)^\lambda w_t^2 dx - \frac{\lambda}{2}(K+t)^{\lambda-1}\int \sigma^\alpha w_t^2 dx + \mu\left(\frac{K+t}{1+t}\right)^\lambda \int \sigma^\alpha w_t^2 dx$$
$$+ (K+t)^\lambda \int \sigma^{\alpha+1}\left[(1+o(1))w_x\right] w_{xt} dx = 0.$$

We then have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int (K+t)^\lambda \left[\sigma^\alpha w_t^2 + (1+o(1))\sigma^{\alpha+1} w_t^2\right] dx\\&+\left[\mu - \frac{\lambda}{2}(K+t)^{\lambda-1}\right]\int \sigma^\alpha w_t^2 dx\\&-(1+o(1))\frac{\lambda}{2}(K+t)^{\lambda-1}\int \sigma^{\alpha+1} w_x^2 dx \le 0.\end{aligned} \tag{3.21}$$

Now multiplying (3.20) by νw for some small $\nu > 0$, to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\begin{aligned}&\nu\frac{d}{dt}\int \sigma^\alpha w_t w dx - \nu\int \sigma^\alpha w_t^2 dx + \frac{\nu\mu}{2}\partial_t \int \frac{1}{(1+t)^\lambda}\sigma^\alpha w^2 dx\\&+\frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int \sigma^\alpha w^2 dx + \nu\int \sigma^{\alpha+1}(1+o(1))w_x^2 dx = 0.\end{aligned} \tag{3.22}$$

Adding (3.21) and (3.22), we have

$$\begin{aligned}&\frac{d}{dt}\int \tilde{\mathfrak{E}}_0(x,t)dx + \frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int \sigma^\alpha w^2 dx\\&+\left[\mu - \frac{\lambda}{2}(K+t)^{\lambda-1} - \nu\right]\int \sigma^\alpha w_t^2 dx\\&+(1+o(1))\left(\nu - \frac{\lambda}{2}(K+t)^{\lambda-1}\right)\int \sigma^{\alpha+1} w_x^2 dx \le 0.\end{aligned} \tag{3.23}$$

Here

$$\begin{aligned}\tilde{\mathfrak{E}}_0(x,t) :=& \frac{(K+t)^\lambda}{2}\left[\sigma^\alpha w_t^2 + (1+o(1))\sigma^{\alpha+1} w_x^2\right]\\&+\nu\sigma^\alpha w_t w + \frac{\nu\mu}{2(1+t)^\lambda}\sigma^\alpha w^2.\end{aligned}$$

By using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\frac{(K+t)^\lambda}{4}\left[\sigma^\alpha w_t^2+\sigma^{\alpha+1}w_x^2\right]+\left(\frac{\nu\mu}{2}-\nu^2\right)\frac{1}{(1+t)^\lambda}\sigma^\alpha w^2\\
&\le \tilde{\mathfrak{E}}_0 \le\\
&\frac{3(K+t)^\lambda}{4}\left[\sigma^\alpha w_t^2+\sigma^{\alpha+1}w_x^2\right]+\left(\frac{\nu\mu}{2}+\nu^2\right)\frac{1}{(1+t)^\lambda}\sigma^\alpha w^2.
\end{aligned}\tag{3.24}$$

Since $\lambda < 1$, by first choosing small ν and then large K, we can get, from (3.23),

$$\begin{aligned}
&\frac{d}{dt}\int \tilde{\mathfrak{E}}_0(x,t)dx+\frac{\nu\mu\lambda}{2(1+t)^{\lambda+1}}\int \sigma^\alpha w^2 dx\\
&+\frac{\nu}{2}\int \sigma^\alpha w_t^2 dx+\frac{\nu}{2}\int \sigma^{\alpha+1}w_x^2 dx\le 0.
\end{aligned}\tag{3.25}$$

Now multiplying (3.25) by $(K+t)^{\lambda-\delta}$, we can achieve

$$\begin{aligned}
&\frac{d}{dt}\int (K+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(x,t)dx-(\lambda-\delta)(K+t)^{\lambda-1-\delta}\tilde{\mathfrak{E}}_0(x,t)\\
&+\frac{\nu\mu\lambda(K+t)^{\lambda-\delta}}{2(1+t)^{\lambda+1}}\int \sigma^\alpha w^2 dx\\
&+\frac{\nu(K+t)^{\lambda-\delta}}{2}\left\{\int \sigma^\alpha w_t^2 dx+\int \sigma^{\alpha+1}w_x^2 dx\right\}\le 0,
\end{aligned}$$

by using (3.24), we have

$$\begin{aligned}
&\frac{d}{dt}\int (K+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(x,t)dx\\
&+\frac{\nu\mu(K+t)^{\lambda-\delta}}{2(1+t)^{\lambda+1}}\underbrace{\left(\lambda-(\lambda-\delta)(1+\frac{2\nu}{\mu})\right)}_{L_1}\int \sigma^\alpha w^2 dx\\
&+(K+t)^{\lambda-\delta}\underbrace{\left(\frac{\nu}{2}-\frac{3}{4}(K+t)^{\lambda-1}\right)}_{L_2}\left\{\int \sigma^\alpha w_t^2 dx+\int \sigma^{\alpha+1}w_x^2 dx\right\}\le 0.
\end{aligned}$$

Again, by choosing small ν and large K, we can assure that L_1 and L_2 are positive. Then we have for some constant $c_{\lambda,\mu}$

$$\begin{aligned}
&\frac{d}{dt}\int (K+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(x,t)dx\\
&+c_{\lambda,\mu}(K+t)^{\lambda-\delta}\left\{\int \sigma^\alpha w_t^2 dx+\int \sigma^{\alpha+1}w_x^2 dx\right\}\le 0.
\end{aligned}\tag{3.26}$$

Now we multiply (3.21) by $(K+t)^{1-\delta}$ to achieve that

$$\frac{1}{2}\frac{d}{dt}\int (K+t)^{1+\lambda-\delta}\left[\sigma^\alpha w_t^2 + (1+o(1))\sigma^{\alpha+1}w_t^2\right]dx$$
$$+ c_{\lambda,\mu}(K+t)^{1-\delta}\int \sigma^\alpha w_t^2 dx \tag{3.27}$$
$$- c_{\lambda,\mu}(K+t)^{\lambda-\delta}\int \sigma^{\alpha+1}w_x^2 dx \le 0.$$

Multiplying a small number ν_1 to (3.27) and then adding the resulting equation to (3.26), we can get

$$\frac{d}{dt}\int \mathfrak{E}_0(x,t)dx$$
$$+ c_{\lambda,\mu}(1+t)^{1-\delta}\int \sigma^\alpha w_t^2 dx + c_{\lambda,\mu}(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}w_x^2 dx \le 0, \tag{3.28}$$

where

$$\mathfrak{E}_0(x,t) := (K+t)^{\lambda-\delta}\tilde{\mathfrak{E}}_0(x,t)$$
$$+ \nu_1(K+t)^{1+\lambda-\delta}\left[\sigma^\alpha w_t^2 + (1+o(1))\sigma^{\alpha+1}w_t^2\right]$$
$$\approx (1+t)^{1+\lambda-\delta}\left[\sigma^\alpha w_t^2 + \sigma^{\alpha+1}w_x^2\right] + (1+t)^{-\delta}\sigma^\alpha w^2,$$
$$\int \mathfrak{E}_0(x,t)dx \approx \mathcal{E}_0(t).$$

Now integrating (3.28) with respect to time variable from 0 to t, we get (3.19) for $0<\lambda<1, 0<\mu$.

Case 2: $\lambda = 1, 2 < \mu$

Multiplying (3.20) by $(1+t)^2 w_t$ and integrating the product with respect to the spatial variable, then we can get

$$\frac{1}{2}\frac{d}{dt}\int \sigma^\alpha (1+t)^2 w_t^2 dx - (1+t)\int \sigma^\alpha w_t^2 dx + \mu(1+t)\int \sigma^\alpha w_t^2 dx$$
$$+ (1+t)^2\int \sigma^{\alpha+1}\left[(1+o(1))w_x\right]w_{xt}dx = 0.$$

We then have

$$\frac{1}{2}\frac{d}{dt}\int (1+t)^2\sigma^\alpha w_t^2 + (1+t)^2(1+o(1))\sigma^{\alpha+1}w_t^2 dx$$
$$+ (\mu-1)(1+t)\int \sigma^\alpha w_t^2 dx \tag{3.29}$$
$$- (1+o(1))(1+t)\int \sigma^{\alpha+1}w_x^2 dx = 0.$$

Now multiplying (3.20) by $\nu(1+t)w$ for some positive ν to be determined later, and integrating the product with respect to the spatial variable, then we can get

$$\begin{aligned}&\nu\frac{d}{dt}\int\sigma^\alpha(1+t)w_t w dx-\nu(1+t)\int\sigma^\alpha w_t^2dx+\frac{\nu(\mu-1)}{2}\partial_t\int\sigma^\alpha w^2dx\\&+\nu(1+o(1))(1+t)\int\sigma^{\alpha+1}w_x^2dx=0.\end{aligned}\tag{3.30}$$

Adding (3.29) and (3.30), we have

$$\begin{aligned}&\frac{d}{dt}\int\mathfrak{E}_0(t)dx+(\mu-1-\nu)(1+t)\int\sigma^\alpha w_t^2dx\\&+(\nu-1)(1+t)(1+o(1))\int\sigma^{\alpha+1}w_x^2dx=0.\end{aligned}\tag{3.31}$$

Here

$$\begin{aligned}\mathfrak{E}_0(x,t):=&\frac{(1+t)^2}{2}\left[\sigma^\alpha w_t^2+(1+o(1))\sigma^{\alpha+1}w_x^2\right]\\&+\nu(1+t)\sigma^\alpha w_t w+\frac{\nu(\mu-1)}{2}\sigma w^2.\end{aligned}\tag{3.32}$$

Now, since $\mu>2$, we assume $\mu=2+2\kappa$ for some positive κ. Choosing $\nu=1+\kappa$, we can achieve

$$\begin{aligned}\mathfrak{E}_0(x,t):=&\frac{(1+t)^2}{2}\left[\sigma^\alpha w_t^2+(1+o(1))\sigma^{\alpha+1}w_x^2\right]\\&+(1+\kappa)(1+t)\sigma^\alpha w_t w+\frac{(1+\kappa)(1+2\kappa)}{2}\sigma^\alpha w^2.\end{aligned}$$

Then (3.31) becomes

$$\begin{aligned}&\frac{d}{dt}\int\mathfrak{E}_0(t)dx+\kappa(1+t)\int\sigma^\alpha w_t^2dx\\&+\kappa(1+t)(1+o(1))\int\sigma^{\alpha+1}w_x^2dx=0.\end{aligned}\tag{3.33}$$

By using Cauchy-Shwartz inequality to absorb the term involving $w_t w$ in (3.32), it is not hard to deduce that

$$\mathfrak{E}_0(x,t)\approx(1+t)^2\left(\sigma^\alpha w_t^2+\sigma^{\alpha+1}w_x^2\right)+\sigma^\alpha w^2,\qquad\int\mathfrak{E}_0(x,t)dx\approx\mathcal{E}_0(t).$$

Now integrating (3.33) with respect to time variable from 0 to t, we get (3.19) for $\lambda=1,2<\mu$. □

Higher-Order Energy Estimates

For $k \geq 1$, ∂_t^k (2.8)$_1$ yields that

$$\begin{aligned} &\sigma^\alpha \partial_t^{k+2} w + \frac{\mu}{(1+t)^\lambda} \sigma^\alpha \partial_t^{k+1} w + \mu \sigma^\alpha \sum_{\ell=1}^{k} C_k^\ell \partial_t^\ell (1+t)^{-\lambda} \partial_t^{k+1-\ell} w \\ &- \left[\sigma^{\alpha+1} (1+w_x)^{-\gamma-1} \partial_t^k w_x + \sigma^{\alpha+1} J \right]_x = 0, \end{aligned} \tag{3.34}$$

where

$$J := \left\{ \partial_t^{k-1} \left[(1+w_x)^{-\gamma-1} w_{xt} \right] - (1+w_x)^{-\gamma-1} \partial_t^k w_x \right\}.$$

To obtain the leading terms of J, we single out the terms involving $\partial_t^{k-1} w_x$. To this end, we rewrite J as

$$\begin{aligned} J =&(k-1) \left[(1+w_x)^{-\gamma-1} \right]_t \partial_t^{k-1} w_x + w_{xt} \partial_t^{k-1} \left[(1+w_x)^{-\gamma-1} \right] \\ &+ \sum_{\ell=2}^{k-2} C_{k-1}^\ell \left(\partial_t^{k-\ell} w_x \right) \partial_t^\ell \left[(1+w_x)^{-\gamma-1} \right] \\ =&k \left[(1+w_x)^{-\gamma-1} \right]_t \partial_t^{k-1} w_x + \widetilde{J}, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \tilde{J} :=& -(\gamma+1) w_{xt} \sum_{\ell=1}^{k-2} C_{k-2}^\ell \left(\partial_t^{k-1-\ell} w_x \right) \partial_t^\ell \left[(1+w_x)^{-\gamma-2} \right] \\ &+ \sum_{\ell=2}^{k-2} C_{k-1}^\ell \left(\partial_t^{k-\ell} w_x \right) \partial_t^\ell \left[(1+w_x)^{-\gamma-1} \right]. \end{aligned} \tag{3.36}$$

Here summations $\sum_{\ell=1}^{k-2}$ and $\sum_{\ell=2}^{k-2}$ are understood to be 0 when $k = 1, 2$ and $k = 1, 2, 3$, respectively. It should be noted that only the terms of lower-order derivatives, $w_x, \ldots, \partial_t^{k-2} w_x$, are contained in $\tilde{J}$. In particular, $\tilde{J} = 0$ when $k = 1$.

Lemma 3.8. *Suppose that* (3.1) *holds for some small positive number* $\epsilon_0 \in (0, 1)$. *Then for all* $j = 1, \ldots, m$

$$
\begin{aligned}
&\mathcal{E}_j(t)+\int_0^t\int\Big[(1+\tau)^{2j+1-\delta\mathbf{1}_{\lambda<1}}\sigma^\alpha\left(\partial_\tau^{j+1}w\right)^2\\
&\qquad+(1+\tau)^{2j+\lambda-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}\left(\partial_\tau^j w_x\right)^2\Big]dxd\tau \qquad (3.37)\\
&\lesssim\sum_{\ell=0}^{j}\mathcal{E}_\ell(0),\quad t\in[0,T].
\end{aligned}
$$

Proof. We use induction to prove (3.37). As shown in Lemma 3.7 we know that (3.37) holds for $j=0$. For $1\le k\le m$, we make the induction hypothesis that (3.37) holds for all $j=0,1,\dots,k-1$, i.e.,

$$
\begin{aligned}
&\mathcal{E}_j(t)+\int_0^t\int\Big[(1+\tau)^{2j+1-\delta\mathbf{1}_{\lambda<1}}\sigma^\alpha\left(\partial_\tau^{j+1}w\right)^2\\
&\qquad+(1+\tau)^{2j+\lambda-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}\left(\partial_\tau^j w_x\right)^2\Big]dxd\tau \qquad (3.38)\\
&\lesssim\sum_{\ell=0}^{j}\mathcal{E}_\ell(0),\quad t\in[0,T],\quad 0\le j\le k-1.
\end{aligned}
$$

It suffices to prove (3.37) holds for $j=k$ under the induction hypothesis (3.38)

We divide the proof into three steps.

Step one: Setup of the linearized main term

We begin by rewrite (3.34) as follows by using the smallness of w_x,

$$
\begin{aligned}
&\sigma^\alpha\partial_t^{k+2}w-\left[\sigma^{\alpha+1}(1+o(1))\partial_t^k w_x\right]_x+\frac{\mu}{(1+t)^\lambda}\sigma^\alpha\partial_t^{k+1}w\\
=&-\mu\sigma^\alpha\sum_{\ell=1}^{k}C_k^\ell\partial_t^\ell(1+t)^{-\lambda}\partial_t^{k+1-\ell}w+\left[\sigma^{\alpha+1}J\right]_x \qquad (3.39)\\
:=&P(x,t).
\end{aligned}
$$

If we view $\partial_t^k w$ as w in the proof of Lemma 3.7, we can get a similar formula with (3.28) and (3.33) as follows

$$
\begin{aligned}
&\frac{d}{dt}\int\mathfrak{E}_k(t)dx+(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\int\sigma^\alpha(\partial_t^{k+1}w)^2dx\\
&+(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\int\sigma^{\alpha+1}(\partial_t^k w_x)^2dx \qquad (3.40)\\
&\lesssim(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int P(x,t)\partial_t^{k+1}wdx+(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\int P(x,t)\partial_t^k wdx,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{E}_k(x,t) \approx &(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left[\sigma^{\alpha+1}(\partial_t^k w_x)^2+\sigma^{\alpha}(\partial_t^{k+1} w)^2\right] \\
&+(1+t)^{-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}(\partial_t^k w_x)^2, \\
&(1+t)^{2k}\int \mathfrak{E}_k(x,t)dx \approx \mathcal{E}_k(t).
\end{aligned}
$$

Then substituting $P(x,t)$ in (3.39) into (3.40) and using integration by parts, we can get

$$
\begin{aligned}
\frac{d}{dt}\int \mathfrak{E}_k(t)dx &+ \int\left[(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha}(\partial_t^{k+1}w)^2+(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dx \\
\lesssim &-(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}J\partial_t^{k+1}w_x dx \\
&-(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}J\partial_t^{k}w_x dx \\
&+\sum_{\ell=1}^{k}\left|\int (1+t)^{1-\ell-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha}\partial_t^{k+1-\ell}w\,\partial_t^{k+1}w dx\right| \\
&+\sum_{\ell=1}^{k}\left|\int (1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha}\partial_t^{k+1-\ell}w\,\partial_t^{k}w dx\right|.
\end{aligned}
\tag{3.41}
$$

Since the derivative of the term containing $\partial_t^{j+1}w_x$ on the right hand of (3.41) exceeds the highest order derivative on the left side of (3.41), we use (3.35) and integration by parts on time to estimate the first term on the right-hand side of (3.41) as follows:

$$
\begin{aligned}
&-(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha}J\partial_t^{k+1}w_x dx \\
=&-\frac{d}{dt}\int (1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}J\partial_t^{k}w_x dx \\
&+(1+\lambda-\delta\mathbf{1}_{\lambda<1})(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}J\partial_t^{k}w_x dx \\
&+(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}J_t\partial_t^{k}w_x dx \\
=&-\frac{d}{dt}\int \sigma^{\alpha+1}(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}J\partial_t^{k}w_x dx \\
&+(1+\lambda-\delta\mathbf{1}_{\lambda<1})(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}J\partial_t^{k}w_x dx \\
&+k(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}\left[(1+w_x)^{-\gamma-1}\right]_t\left(\partial_t^k w_x\right)^2 dx \\
&+k(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}\left[(1+w_x)^{-\gamma-1}\right]_{tt}\partial_t^{k-1}w_x\,\partial_t^{k}w_x dx \\
&+(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\int \sigma^{\alpha+1}\widetilde{J}_t\partial_t^{k}w_x dx.
\end{aligned}
\tag{3.42}
$$

Inserting (3.42) into (3.41), we can get

$$
\begin{aligned}
&\frac{d}{dt}\int\Big[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}J\partial_t^k w_x\Big]dx\\
&\quad+\int\Big[(1+t)^{1-\delta\mathbf{1}_{\lambda<1}}\sigma^\alpha(\partial_t^{k+1}w)^2+(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\sigma^{\alpha+1}(\partial_t^k w_x)^2\Big]dx\\
&\lesssim\sum_{\ell=1}^{k}(1+t)^{1-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^\alpha\partial_t^{k+1-\ell}w\,\partial_t^{k+1}w\,dx\right|\\
&\quad+\sum_{\ell=1}^{k}(1+t)^{-\ell-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^\alpha\partial_t^{k+1-\ell}w\,\partial_t^k w\,dx\right|\\
&\quad+(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^{\alpha+1}\Big[(1+w_x)^{-\gamma-1}\Big]_t\left(\partial_t^k w_x\right)^2dx\right|\\
&\quad+(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^{\alpha+1}\Big[(1+w_x)^{-\gamma-1}\Big]_{tt}\partial_t^{k-1}w_x\,\partial_t^k w_x\,dx\right|\\
&\quad+(1+t)^{\lambda-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^{\alpha+1}J\partial_t^k w_x\,dx\right|\\
&\quad+(1+t)^{1+\lambda-\delta\mathbf{1}_{\lambda<1}}\left|\int\sigma^{\alpha+1}\widetilde{J}_t\partial_t^k w_x\,dx\right|\\
&:=\sum_{i=1}^{6}I_i.
\end{aligned}
\tag{3.43}
$$

Now we estimate I_i $(1\le i\le 6)$ term by term by using (3.38) and the left of (3.43). Next we only consider the case for $\lambda<1$ and the proof below is still valid for $\lambda=1$ by simply replacing $\lambda=1$ and $\delta=0$. Also for simplicity of presentation and notations, sometimes we denote $\beta:=\lambda+1$.

Step two: Estimates of the low order and nonlinear terms

For I_1, by using Cauchy-Schwartz inequality, we have, for a small constant ν,

$$
\begin{aligned}
I_1&\lesssim\nu(1+t)^{1-\delta}\int\sigma^\alpha(\partial_t^{k+1}w)^2dx+C_\nu\sum_{\ell=1}^{k}(1+t)^{1-2\ell-\delta}\int\sigma^\alpha(\partial_t^{k+1-\ell}w)^2dx\\
&\lesssim\nu(1+t)^{1-\delta}\int\sigma^\alpha(\partial_t^{k+1}w)^2dx\\
&\quad+(1+t)^{-2k}C_\nu\sum_{\ell=1}^{k}(1+t)^{2(k+1-\ell)-1-\delta}\int\sigma^\alpha(\partial_t^{k+1-\ell}w)^2dx\\
&\lesssim\nu(1+t)^{1-\delta}\int\sigma^\alpha(\partial_t^{k+1}w)^2dx\\
&\quad+(1+t)^{-2k}C_\nu\sum_{\ell=1}^{k}(1+t)^{2\ell-1-\delta}\int\sigma^\alpha(\partial_t^{\ell}w)^2dx.
\end{aligned}
$$

The first term on the right hand of the above inequality can be absorbed by the left hand of (3.43) and the time integral of the second term can be bounded by the initial data from the induction assumption (3.38).

For I_2, we can estimate it similarly with I_1 as follows

$$
\begin{aligned}
|I_2| \lesssim& (1+t)^{-1-\delta} \int \sigma^\alpha (\partial_t^k w)^2 dx \\
&+ \sum_{\ell=1}^{k} (1+t)^{1-2\ell-\delta} \int \sigma^\alpha (\partial_t^{k+1-\ell} w)^2 dx \\
\lesssim& (1+t)^{-2k} \sum_{\ell=1}^{k} (1+t)^{2\ell-1-\delta} \int \sigma^\alpha (\partial_t^\ell w)^2 dx.
\end{aligned}
$$

For term I_3, using the L^∞ estimate for w_{xt} in (3.2), we have

$$
\begin{aligned}
|I_3| :=& (1+t)^{1+\lambda-\delta} \left| \int \sigma^{\alpha+1} \left[(1+w_x)^{-\gamma-1} \right]_t \left(\partial_t^k w_x \right)^2 dx \right| \\
\lesssim& \|(1+t) w_{xt}\|_{L^\infty} (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} \left(\partial_t^k w_x \right)^2 dx \\
\lesssim& \sqrt{\epsilon_0} (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} \left(\partial_t^k w_x \right)^2 dx,
\end{aligned}
$$

which can be absorbed by the positive term on the righthand of (3.43) if ϵ_0 is small enough.

For term I_4, first using (3.2), we have

$$
\left[(1+w_x)^{-\gamma-1} \right]_{tt} \lesssim |w_{xtt}| + (w_{xt})^2.
$$

Inserting this into I_4 and using Cauchy Schwartz inequality, we have

$$
\begin{aligned}
|I_4| :=& (1+t)^{1+\lambda-\delta} \left| \int \sigma^{\alpha+1} \left[(1+w_x)^{-\gamma-1} \right]_{tt} \partial_t^{k-1} w_x \partial_t^k w_x dx \right| \\
\lesssim& (1+t)^{1+\lambda-\delta} \int \sigma^{\alpha+1} \left(|w_{xtt}| + (w_{xt})^2 \right) |\partial_t^{k-1} w_x| |\partial_t^k w_x| dx \\
\lesssim& \nu (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} \left(\partial_t^k w_x \right)^2 dx \\
&+ C_\nu (1+t)^{2+\lambda-\delta} \left\{ \|\sigma^{1/2} w_{xtt}\|_{L^\infty}^2 \int \sigma^\alpha |\partial_t^{k-1} w_x|^2 dx \right. \\
&\left. + \|w_{xt}\|_{L^\infty}^4 \int \sigma^{\alpha+1} |\partial_t^{k-1} w_x|^2 dx \right\} \\
\lesssim& \nu (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} \left(\partial_t^k w_x \right)^2 dx \\
&+ C_\nu \epsilon_0 (1+t)^{-3} \left\{ \int \sigma^\alpha (\partial_t^{k-1} w_x)^2 dx + \int \sigma^{\alpha+1} (\partial_t^{k-1} w_x)^2 dx \right\}.
\end{aligned}
$$

Remembering the definition of $\mathcal{E}_{j,i}$ and $\mathcal{E}_j$, we have

$$\begin{aligned}|I_4| &\lesssim \nu(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}\left(\partial_t^k w_x\right)^2 dx + C_\nu \epsilon_0 (1+t)^{-3-2(k-1)-1-\lambda+\delta \mathbf{1}_{\lambda<1}}\left(\mathcal{E}_{k-1}+\mathcal{E}_{k-1,1}\right)\\ &\lesssim \nu(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}\left(\partial_t^k w_x\right)^2 dx + C_\nu \epsilon_0 (1+t)^{-2k-2-\lambda+\delta}\sum_{\ell=0}^{k}\mathcal{E}_\ell,\end{aligned} \tag{3.44}$$

where at the last line, we have used Proposition 3.3. Since there appears $\mathcal{E}_k$ on the right hand of the above inequality, which can not either be absorbed by the left hand of (3.43) or be controlled by the induction assumption (3.38). We calculate it further more as follows.

By using the representation of $\mathcal{E}_k(t)$, we have

$$\begin{aligned}\epsilon_0(1+t)^{-2k-2-\lambda+\delta}\mathcal{E}_k \lesssim\, & \epsilon_0(1+t)^{-1}\int\left[\sigma^\alpha(\partial_t^{k+1}w)^2+\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dx\\ &+\epsilon_0(1+t)^{-2-\lambda}\int \sigma^\alpha(\partial_t^k w)^2 dx\\ \lesssim\, & \epsilon_0(1+t)^{1-\delta}\int \sigma^\alpha(\partial_t^{k+1}w)^2 dx+\epsilon_0(1+t)^{\lambda-\delta}\int\sigma^{\alpha+1}(\partial_t^k w_x)^2 dx\\ &+\epsilon_0(1+t)^{-2k-1-\lambda+\delta}(1+t)^{2k-1-\delta}\int\sigma^\alpha(\partial_t^k w)^2 dx.\end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45), we can achieve that

$$\begin{aligned}|I_4| \lesssim\, &(\nu+\epsilon_0)\left\{(1+t)^{1-\delta}\int\sigma^\alpha(\partial_t^{k+1}w)^2 dx+(1+t)^{\lambda-\delta}\int\sigma^{\alpha+1}(\partial_t^k w_x)^2 dx\right\}\\ &+\epsilon_0(1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell+(1+t)^{-2k}(1+t)^{2k-1-\delta}\int\sigma^\alpha(\partial_t^k w)^2 dx,\end{aligned}$$

where 1^+ is a constant bigger than can be arbitrarily close to 1.

Now we come to estimate the terms involving J and $\tilde{J}$, which are a little complicated.

From (3.35), (3.36) and (3.2), we have

$$\begin{aligned}|J| :\lesssim &\sum_{\ell=1}^{k-1}|\partial_t^\ell w_x||\partial_t^{k-\ell}w_x|+\text{l.o.t.}\\ \lesssim &|w_{xt}||\partial_t^{k-1}w_x|+\sum_{\ell=2}^{k-2}|\partial_t^\ell w_x||\partial_t^{k-\ell}w_x|+\text{l.o.t}\\ \lesssim &\sqrt{\epsilon_0}(1+t)^{-\frac{\lambda+1-\delta}{2}-1}|\partial_t^{k-1}w_x|\\ &+\sqrt{\epsilon_0}\sum_{\ell=2}^{[k/2]-1}(1+t)^{-\frac{\lambda+1-\delta}{2}-\ell}\sigma^{-\frac{\ell-1}{2}}|\partial_t^{k-\ell}w_x|.\end{aligned} \tag{3.46}$$

Here and thereafter the notation *l.o.t.* is used to represent the lower-order terms involving $\partial_t^\ell w_x$ with $\ell = 2, \ldots, k-2$. It should be noticed that the second term on the right-hand side of (3.46) only appears as $k-2 \geq 2$.

Using Cauchy Schwartz inequality, we can get

$$\begin{aligned} |I_5| &\lesssim \left| (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} J \partial_t^k w_x dx \right| \\ &\lesssim \nu(1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} \left(\partial_t^k w_x\right)^2 dx + C_\nu (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} J^2 dx. \end{aligned} \tag{3.47}$$

While, from (3.46), the second term on the right hand of the above inequality can be bounded as follows,

$$\begin{aligned} (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} J^2 dx \lesssim & \epsilon_0 (1+t)^{-3} \int \sigma^{\alpha+1} \left(\partial_t^{k-1} w_x\right)^2 dx \\ & + \epsilon_0 \sum_{\ell=2}^{[k/2]-1} (1+t)^{-2\ell-1} \int \sigma^{\alpha+2-\ell} \left(\partial_t^{k-\ell} w_x\right)^2 dx. \end{aligned} \tag{3.48}$$

In view of (3.6), we see that for $\ell = 2, \ldots, [k/2]-1$, $\alpha + 2 - \ell > -1$, then we have

$$\begin{aligned} & \int \sigma^{\alpha+2-\ell} \left|\partial_t^{k-\ell} w_x\right|^2 dx \\ \lesssim & \int \sigma^{\alpha+2-\ell+2} \left|\partial_t^{k-\ell} w_{xx}\right|^2 dx \\ & \underbrace{\cdots\cdots}_{\ell-2\ times} \\ \lesssim & \int \sigma^{\alpha+\ell} \left|\partial_t^{k-\ell} \partial_x^\ell w\right|^2 dx \\ \lesssim & (1+t)^{-2(k-\ell)-(\lambda+1)+\delta} \mathcal{E}_{k-\ell,\ell-1}. \end{aligned} \tag{3.49}$$

Inserting (3.49) into (3.48) indicates that

$$\begin{aligned} & (1+t)^{\lambda-\delta} \int \sigma^{\alpha+1} J^2 dx \\ \lesssim & \epsilon_0 (1+t)^{-2k-2-\lambda+\delta} \mathcal{E}_{k-1} + \epsilon_0 (1+t)^{-2k-2-\lambda+\delta} \sum_{\ell=2}^{[k/2]} \mathcal{E}_{k-\ell,\ell-1} \\ \lesssim & \epsilon_0 (1+t)^{-2k-2-\lambda+\delta} \sum_{\ell=0}^{k-1} \mathcal{E}_\ell. \end{aligned} \tag{3.50}$$

Combining (3.50) and (3.47), we can get

$$|I_5| \lesssim \nu(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}\left(\partial_t^k w_x\right)^2 dx + C_\nu \epsilon_0 (1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell .$$

Next we come to deal with the term I_6 involving $\tilde{J}_t$. First, from (3.36), we have

$$\begin{aligned}
|\tilde{J}_t| :\lesssim & |w_{xtt}|\sum_{\ell=1}^{k-2}|\partial_t^{k-1-\ell}w_x||\partial_t^\ell w_x| + |w_{xt}|\sum_{\ell=1}^{k-1}|\partial_t^{k-\ell}w_x||\partial_t^\ell w_x| \\
& + \sum_{\ell=2}^{k-1}|\partial_t^{k-\ell+1}w_x||\partial_t^\ell w_x| + \text{l.o.t.} \\
\lesssim & |w_{xtt}||\partial_t^{k-2}w_x||\partial_t w_x| + |w_{xt}||\partial_t^{k-1}w_x||\partial_t w_x| \\
& + |w_{xtt}|\sum_{\ell=2}^{k-3}|\partial_t^{k-1-\ell}w_x||\partial_t^\ell w_x| + |w_{xt}|\sum_{\ell=2}^{k-2}|\partial_t^{k-\ell}w_x||\partial_t^\ell w_x| \\
& + \sum_{\ell=2}^{k-1}|\partial_t^{k-\ell+1}w_x||\partial_t^\ell w_x| + \text{l.o.t.}.
\end{aligned}$$

Then the L^∞ estimate in (3.2) implies that

$$\begin{aligned}
|\tilde{J}_t| \lesssim & \epsilon_0(1+t)^{-3-(1+\lambda-\delta)}\sigma^{-1/2}|\partial_t^{k-2}w_x| + \epsilon_0(1+t)^{-2-(1+\lambda-\delta)}|\partial_t^{k-1}w_x| \\
& + \epsilon_0(1+t)^{-2-\frac{1+\lambda-\delta}{2}}\sigma^{-1/2}(1+t)^{-\ell-\frac{1+\lambda-\delta}{2}}\sigma^{-\frac{\ell-1}{2}}\sum_{\ell=2}^{[\frac{k-3}{2}]}|\partial_t^{k-1-\ell}w_x| \\
& + \epsilon_0(1+t)^{-1-\frac{1+\lambda-\delta}{2}}(1+t)^{-\ell-\frac{1+\lambda-\delta}{2}}\sigma^{-\frac{\ell-1}{2}}\sum_{\ell=2}^{[\frac{k-2}{2}]}|\partial_t^{k-\ell}w_x| \\
& + \sqrt{\epsilon_0}(1+t)^{-\ell-\frac{1+\lambda-\delta}{2}}\sigma^{-\frac{\ell-1}{2}}\sum_{\ell=2}^{[\frac{k-1}{2}]}|\partial_t^{k+1-\ell}w_x|.
\end{aligned}$$

If we view the $\ell+1$ and $\ell-1$ at second line and fourth line of the above inequality to be the new ℓ and combine the second and third line together, we can achieve that

$$\begin{aligned}
|\tilde{J}_t| :\lesssim & \epsilon_0(1+t)^{-3-(1+\lambda-\delta)}\sigma^{-1/2}|\partial_t^{k-2}w_x| + \epsilon_0(1+t)^{-2-(1+\lambda-\delta)}|\partial_t^{k-1}w_x| \\
& + \epsilon_0(1+t)^{-\ell-1-(1+\lambda-\delta)}\sigma^{-\frac{\ell-1}{2}}\sum_{\ell=2}^{[\frac{k-1}{2}]}|\partial_t^{k-\ell}w_x| \\
& + \sqrt{\epsilon_0}(1+t)^{-\ell-1-\frac{1+\lambda-\delta}{2}}\sigma^{-\frac{\ell}{2}}\sum_{\ell=1}^{[\frac{k-3}{2}]}|\partial_t^{k-\ell}w_x|.
\end{aligned} \tag{3.51}$$

Using Cauchy Schwartz inequality,

$$|I_6| \lesssim \nu(1+t)^{\lambda-\delta} \int \sigma^{\alpha+1}(1+t)\left(\partial_t^k w_x\right)^2 dx + C_\nu(1+t)^{2+\lambda-\delta} \int \sigma^{\alpha+1}|\tilde{J}_t|^2 dx. \tag{3.52}$$

And inserting (3.51) into the second term on the right hand of (3.52), we have

$$\begin{aligned} \int \sigma^{\alpha+1}(1+t)^{2+\lambda-\delta}|\tilde{J}_t|^2 dx \lesssim & \epsilon_0^2(1+t)^{-6-\lambda+\delta} \int \sigma^{\alpha}\left(\partial_t^{k-2} w_x\right)^2 dx \\ & + \epsilon_0^2(1+t)^{-4-\lambda+\delta} \int \sigma^{\alpha+1}\left(\partial_t^{k-1} w_x\right)^2 dx \\ & + \epsilon_0^2 \sum_{\ell=2}^{[\frac{k-1}{2}]} (1+t)^{-2(\ell+1)-\lambda+\delta} \int \sigma^{\alpha+2-\ell}\left(\partial_t^{k-\ell} w_x\right)^2 dx \\ & + \epsilon_0 \sum_{\ell=1}^{[\frac{k-3}{2}]} (1+t)^{-2(\ell+1)+1} \int \sigma^{\alpha+1-\ell}\left(\partial_t^{k-\ell} w_x\right)^2 dx. \end{aligned} \tag{3.53}$$

Using (3.6) repeatedly as that in (3.49), we can have

$$\begin{aligned} \int \sigma^{\alpha}\left(\partial_t^{k-2} w_x\right)^2 dx \lesssim & \int \sigma^{\alpha+\ell}\left(\partial_t^{k-2} w_{xx}\right)^2 dx \\ \lesssim & (1+t)^{-2(k-2)-(1+\lambda)+\delta} \mathcal{E}_{k-2,1}, \\ \int \sigma^{\alpha+2-\ell}\left(\partial_t^{k-\ell} w_x\right)^2 dx \lesssim \cdots \lesssim & \int \sigma^{\alpha}\left(\partial_t^{k-\ell}\partial_x^{\ell} w\right)^2 dx \\ \lesssim & (1+t)^{-2(k-\ell)-(1+\lambda)+\delta} \mathcal{E}_{k-\ell,\ell-1}, \end{aligned}$$

and

$$\begin{aligned} \int \sigma^{\alpha+1-\ell}\left(\partial_t^{k-\ell} w_x\right)^2 dx \lesssim \cdots \lesssim & \int \sigma^{\alpha-1}\left(\partial_t^{k-\ell}\partial_x^{\ell} w\right)^2 dx \\ \lesssim & (1+t)^{-2(k-\ell)-(1+\lambda)+\delta} \mathcal{E}_{k-\ell,\ell}. \end{aligned}$$

Inserting the above three inequalities into (3.53) and using Proposition 3.3, we can obtain

$$\begin{aligned} & \int \sigma^{\alpha+1}(1+t)^{2+\lambda-\delta}|\tilde{J}_t|^2 dx \\ \lesssim & \epsilon_0^2(1+t)^{-2k-1^+}\left(\mathcal{E}_{k-2,1}+\mathcal{E}_{k-1}\right) + \epsilon_0^2(1+t)^{-2k-1^+}\mathcal{E}_{k-\ell,\ell-1} \\ & + \epsilon_0(1+t)^{-2k-2-\lambda+\delta}\mathcal{E}_{k-\ell,\ell} \\ \lesssim & \epsilon_0^2(1+t)^{-2k-1^+}\left(\mathcal{E}_{k-2,1}+\mathcal{E}_{k-1}\right) + \epsilon_0^2(1+t)^{-2k-1^+}\mathcal{E}_{k-\ell,\ell-1} \\ & + \epsilon_0(1+t)^{-2k-2-\lambda+\delta} \sum_{\ell=0}^{k} \mathcal{E}_{\ell}. \end{aligned} \tag{3.54}$$

The same as (3.45), we have

$$
\begin{aligned}
&(1+t)^{-2k-2-\lambda+\delta}\sum_{\ell=0}^{k}\mathcal{E}_\ell \\
\lesssim&(1+t)^{1-\delta}\int \sigma^\alpha(\partial_t^{k+1}w)^2dx+(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}(\partial_t^k w_x)^2dx \\
&+(1+t)^{-2k}(1+t)^{2k-1-\delta}\int \sigma^\alpha(\partial_t^k w)^2dx+(1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell.
\end{aligned}
\tag{3.55}
$$

Inserting (3.55) in (3.54) then into (3.52), we obtain

$$
\begin{aligned}
I_6\lesssim&\epsilon_0(1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell+(1+t)^{-2k}(1+t)^{2k-1-\delta}\int \sigma^\alpha(\partial_t^k w)^2dx \\
&+(\nu+\epsilon_0)\left((1+t)^{1-\delta}\int \sigma^\alpha(\partial_t^{k+1}w)^2dx+(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}(\partial_t^k w_x)^2dx\right).
\end{aligned}
$$

Step 3: Finishing proof of Lemma 3.8

From all the above estimates for terms I_1 to I_6, we get that

$$
\begin{aligned}
&\frac{d}{dt}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx \\
&+(1+t)^{1-\delta}\int \sigma^\alpha(\partial_t^{k+1}w)^2dx+(1+t)^{\lambda-\delta}\int \sigma^{\alpha+1}(\partial_t^k w_x)^2dx \\
\lesssim&(\nu+\epsilon_0)\int\left[(1+t)^{1-\delta}\sigma^\alpha(\partial_t^{k+1}w)^2+(1+t)^{\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dx \\
&+(1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(t)+(1+t)^{-2k}\sum_{\ell=0}^{k-1}\int\Big[(1+t)^{2\ell+1-\delta}\sigma^\alpha(\partial_t^{\ell+1}w)^2 \\
&\qquad+(1+t)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^\ell w_x)^2\Big]dx.
\end{aligned}
$$

Then we get by choosing small ν, for some large N, to be determined later,

$$
\begin{aligned}
&\frac{d}{dt}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx \\
&+N\int\left[(1+t)^{1-\delta}\sigma^\alpha(\partial_t^{k+1}w)^2+(1+t)^{\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dx \\
\lesssim&(1+t)^{-2k-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(t)+(1+t)^{-2k}\sum_{\ell=0}^{k-1}\int\Big[(1+t)^{2\ell+1-\delta}\sigma^\alpha(\partial_t^{\ell+1}w)^2 \\
&\qquad+(1+t)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^\ell w_x)^2\Big]dx.
\end{aligned}
$$

Multiplying the above inequality by $(1+t)^{2k}$, we can get

$$\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{2k}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx\right\}\\
&-2k(1+t)^{2k-1}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx\\
&+N\int\left[\sigma^{\alpha}(1+t)^{2k+1-\delta}(\partial_t^{k+1}w)^2+(1+t)^{2k+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dx\\
\lesssim&(1+t)^{-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(t)\\
&+\sum_{\ell=0}^{k-1}\int\left[(1+t)^{2\ell+1-\delta}\sigma^{\alpha}(\partial_t^{\ell+1}w)^2+(1+t)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^{\ell}w_x)^2\right]dx.
\end{aligned}\tag{3.56}$$

For the term $\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x$, the same estimate as (3.50) implies that

$$\begin{aligned}
&\left|(1+t)^{1+\lambda-\delta}\int\sigma^{\alpha+1}J\partial_t^k w_x dx\right|\\
&\lesssim\nu(1+t)^{1+\lambda-\delta}\int\sigma^{\alpha+1}\left(\partial_t^k w_x\right)^2dx\\
&+(1+t)^{-2k-1-\lambda+\delta}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell.
\end{aligned}$$

From this, we have

$$\begin{aligned}
&(1+t)^{2k-1}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx\\
&\lesssim\int\left\{\left[(1+t)^{2k+\lambda-\delta}\sigma^{\alpha}(\partial_t^{k+1}w)^2+(1+t)^{2k+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]\right.\\
&\left.+\ (1+t)^{2k-1-\delta}\sigma^{\alpha}(\partial_t^k w)^2\right\}dx\\
&+(1+t)^{-2-\lambda+\delta}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell,
\end{aligned}\tag{3.57}$$

and

$$(1+t)^{2k}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx\gtrsim\mathcal{E}_k-\sum_{\ell=0}^{k-1}\mathcal{E}_\ell.\tag{3.58}$$

Inserting (3.57) into (3.56), by choosing sufficiently large N and using the induction assumption (3.38), we can get

$$
\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{2k}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx\right\}\\
&\quad+\int\left[(1+t)^{2k+1-\delta}\sigma^{\alpha}(\partial_t^{k+1}w)^2+(1+t)^{2k+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]\\
&\lesssim(1+t)^{-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(t)\\
&\quad+\sum_{\ell=0}^{k-1}\int\left[(1+t)^{2\ell+1-\delta}\sigma^{\alpha}(\partial_t^{\ell+1}w)^2+(1+t)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^\ell w_x)^2\right]dx\\
&\lesssim(1+t)^{-1^+}\sum_{\ell=0}^{k-1}\mathcal{E}_\ell(0)\\
&\quad+\sum_{\ell=0}^{k-1}\int\left[(1+t)^{2\ell+1-\delta}\sigma^{\alpha}(\partial_t^{\ell+1}w)^2+(1+t)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^\ell w_x)^2\right]dx.
\end{aligned}
$$

Integrating the above inequality from 0 to t and remembering (3.58) and (3.38), we can get

$$
\begin{aligned}
&\mathcal{E}_k+\int_0^t\left[\int(1+\tau)^{2k+1-\delta}\sigma^{\alpha}(\partial_t^{k+1}w)^2+(1+\tau)^{2k+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dxd\tau\\
&\lesssim(1+t)^{2k}\int\left[\mathfrak{E}_k(t)+\sigma^{\alpha+1}(1+t)^{1+\lambda-\delta}J\partial_t^k w_x\right]dx+\sum_{\ell=0}^{k}\mathcal{E}_\ell(0)\\
&\quad+\int_0^t\left[\int(1+\tau)^{2k+1-\delta}\sigma^{\alpha}(\partial_t^{k+1}w)^2+(1+\tau)^{2k+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^k w_x)^2\right]dxd\tau\\
&\lesssim\sum_{\ell=0}^{k}\mathcal{E}_\ell(0)\\
&\quad+\sum_{\ell=0}^{k-1}\int_0^t\int\left[(1+\tau)^{2\ell+1-\delta}\sigma^{\alpha}(\partial_t^{\ell+1}w)^2+(1+\tau)^{2\ell+\lambda-\delta}\sigma^{\alpha+1}(\partial_t^\ell w_x)^2\right]dxd\tau\\
&\lesssim\sum_{\ell=0}^{k}\mathcal{E}_\ell(0).
\end{aligned}
$$

This finishes the proof of Lemma 3.8. □

Then Proposition 3.3 and Proposition 3.6 together imply (3.5), which proves Theorem 2.2 by continuation argument.

4. Proof of Theorem 2.3

Proof. In this section, we prove Theorem 2.3. First, it follows from (2.4) and (2.7), and that for $(x,t) \in \mathcal{I} \times [0,\infty)$

$$\rho(\eta(x,t),t) - \bar{\rho}(\bar{\eta}(x,t),t) = \frac{\bar{\rho}_0(x)}{\eta_x(x,t)} - \frac{\bar{\rho}_0(x)}{\bar{\eta}_x(x,t)} = -\bar{\rho}_0(x)\frac{w_x(x,t)}{(1+w_x)},$$

and

$$u(\eta(x,t),t) - \bar{u}(\bar{\eta}(x,t),t) = w_t(x,t).$$

Hence, by virtue of (2.11), we have, for $(x,t) \in \mathcal{I} \times [0,\infty)$,

$$|\rho(\eta(x,t),t) - \bar{\rho}(\bar{\eta}(x,t),t)| \lesssim x^{\frac{1}{\gamma-1}}(1+t)^{-\frac{1+\lambda}{2}+\frac{\delta}{2}\mathbf{1}_{\lambda<1}},$$

and

$$|u(\eta(x,t),t) - \bar{u}(\bar{\eta}(x,t),t)| \lesssim (1+t)^{-1+\frac{\delta}{2}\mathbf{1}_{\lambda<1}}.$$

Then (2.12) and (2.13) follow. It follows from (2.3) and (2.7) that

$$\begin{aligned} x_b(t) = \eta\left(\bar{x}_b(0),t\right) = \eta\left(0,t\right) &= \left(\bar{\eta}+w\right)(0,t) \\ &\approx -(1+t)^{\lambda+1} + w(0,t) \\ &\approx -(1+t)^{\lambda+1}. \end{aligned}$$

For $k = 1, 2$

$$\frac{d^k x_b(t)}{dt^k} = \partial_t^k \bar{\eta}(0,t) + \partial_t^k w(0,t).$$

So using (2.11) and the representation of $\bar{\eta}$, we get (2.15). □

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Appendix A. Gain of the solution (1.2) and (1.3)

First, we assume that the velocity is independent of the space variable x. So system (1.1) is simplified as

$$\begin{cases} \bar{\rho}_t + \bar{u}\bar{\rho}_x = 0, \\ \bar{u}_t + \dfrac{(p(\bar{\rho}))_x}{\rho} = -\dfrac{\mu}{(1+t)^\lambda}\bar{u}. \end{cases} \tag{A.1}$$

Since $p(\bar{\rho}) = \frac{1}{\gamma}\bar{\rho}^\gamma$, from (A.1)$_2$, we see that

$$\bar{u}_t + \left(\frac{\bar{\rho}^{\gamma-1}}{\gamma-1}\right)_x = -\frac{\mu}{(1+t)^\lambda}\bar{u}. \tag{A.2}$$

Let $\frac{\bar{\rho}^{\gamma-1}}{\gamma-1} = x + f(t)$ for some function $f(t)$, to be determined later. Then (A.2) and (A.1)$_1$ imply that

$$\begin{cases} \left(\frac{\bar{\rho}^{\gamma-1}}{\gamma-1}\right)_t + \bar{u}(t) = 0 \\ \bar{u}_t + 1 = -\frac{\mu}{(1+t)^\lambda}\bar{u}. \end{cases} \tag{A.3}$$

Solving the ordinary differential equation (A.3)$_2$ with the initial data $\bar{u}(t)|_{t=0} = 0$, we can get (1.2). Then From (A.3)$_1$, we can get (1.3). □

Appendix B. Verification of the a priori assumption Lemma 3.1

Proof. For a one dimensional domain Ω, we have the following embedding: $H^{1/2+\nu}(\Omega) \hookrightarrow L^\infty(\Omega)$ with the estimate

$$\|F\|_{L^\infty(\Omega)} \le C_\nu \|F\|_{H^{1/2+\nu}(\Omega)} \tag{B.1}$$

for $\nu > 0$. This will be used in the rest of the proof.

We separate the proof by dividing E_1 and E_2 with and without the space weight, respectively.

Case 1: E_1 with $0 \le j \le 3$ and E_2 with $i = 1$, $j = 0, 1$.

The L^∞ estimates will be done in the intervals $[0, 1]$ and $[1, +\infty)$ separately.

It follows from (3.8) that for $j \le 3 \le 5 + [\alpha] - \alpha$

$$\begin{aligned} \left\|\partial_t^j w\right\|^2_{H^{\frac{5-j+[\alpha]-\alpha}{2}}([0,1])} &= \left\|\partial_t^j w\right\|^2_{H^{m-j+1-\frac{m-j+1+\alpha}{2}}([0,1])} \\ &\lesssim \left\|\partial_t^j w\right\|^2_{H^{m-j+1+\alpha,m-j+1}([0,1])} \\ &\lesssim \sum_{k=0}^{m-j+1}\int_0^1 \sigma^{\alpha+1+m-j}\left(\partial_x^k\partial_t^j w\right)^2 dx \\ &\lesssim \sum_{k=0}^{m-j+1}\int_0^\infty \sigma^{\alpha+k}\left(\partial_x^k\partial_t^j w\right)^2 dx \\ &\lesssim (1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\left(\mathcal{E}_j + \sum_{k=1}^{m-j}\mathcal{E}_{j,k-1}(t)\right) \\ &\lesssim (1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t). \end{aligned} \tag{B.2}$$

Also for $j \le 1 \le 3 + [\alpha] - \alpha$,

$$\begin{aligned}\left\| \partial_t^j w_x \right\|^2_{H^{\frac{3-j+[\alpha]-\alpha}{2}}([0,1])} &= \left\| \partial_t^j w_x \right\|^2_{H^{m-j-\frac{m-j+1+\alpha}{2}}([0,1])} \\ &\lesssim \left\| \partial_t^j w_x \right\|^2_{H^{m-j+1+\alpha,m-j}([0,1])} \\ &\lesssim \sum_{k=0}^{m-j} \int_0^1 \sigma^{\alpha+1+m-j} \left(\partial_x^{k+1} \partial_t^j w \right)^2 dx \\ &\lesssim \sum_{k=0}^{m-j} \int_0^\infty \sigma^{\alpha+1+k} \left(\partial_x^{k+1} \partial_t^j w \right)^2 dx \\ &\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \sum_{k=0}^{m-j} \mathcal{E}_{j,k}(t) \\ &\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}(t).\end{aligned} \tag{B.3}$$

The above two inequality (B.2) and (B.3) together indicate that

$$\begin{aligned}&\sum_{j=0}^{3} (1+t)^{2j-\delta \mathbf{1}_{\lambda<1}} \left\| \partial_t^j w \right\|^2_{L^\infty([0,1])} \\ &+ \sum_{j=0}^{1} (1+t)^{2j+(\lambda+1)-\delta \mathbf{1}_{\lambda<1}} \left\| \partial_t^j w_x \right\|^2_{L^\infty([0,1])} \lesssim \mathcal{E}(t).\end{aligned} \tag{B.4}$$

Besides, we have for $0 \le j \le 3$

$$\begin{aligned}\|\partial_t^j w\|^2_{H^1([1,+\infty))} &\lesssim \int_1^\infty \left(\sigma^\alpha (\partial_t^j w)^2 + \sigma^{\alpha+1} (\partial_t^j w_x)^2 \right) dx \\ &\lesssim (1+t)^{-2j+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}_j \\ &\le (1+t)^{-2j+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}(t),\end{aligned} \tag{B.5}$$

and for $0 \le j \le 1$

$$\begin{aligned}\|\partial_t^j w_x\|^2_{H^1([1,+\infty))} &\lesssim \int_1^\infty \left(\sigma^\alpha (\partial_t^j w_x)^2 + \sigma^{\alpha+2} (\partial_t^j w_{xx})^2 \right) dx \\ &\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}_{j,1} \\ &\le (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}(t).\end{aligned} \tag{B.6}$$

Then combining the above (B.4), (B.5) and (B.6), we can get

$$\sum_{j=0}^{3}(1+t)^{2j-\delta\mathbf{1}_{\lambda<1}}\left\|\partial_t^j w\right\|_{L^\infty}^2+\sum_{j=0}^{1}(1+t)^{2j+(\lambda+1)-\delta\mathbf{1}_{\lambda<1}}\left\|\partial_t^j w_x\right\|_{L^\infty}^2\lesssim\mathcal{E}(t).$$

Case 2: E_1 with $j\geq 4$ and E_2 with $i\geq 1, 2i+j\geq 4$.

We denote

$$\psi:=\sigma^{\frac{2i+j-3}{2}}\partial_t^j\partial_x^i w,$$

here $i\geq 0$. It is easy to see that

$$\begin{aligned}
&\|\psi\|_{L^\infty([1,+\infty))}^2\lesssim\|\psi\|_{H^1([1,+\infty))}^2\\
&\lesssim\int_1^\infty\sigma^{2i+j-3}(\partial_t^j\partial_x^i w)^2dx+\int_1^\infty\left(\partial_x\left(\sigma^{\frac{2i+j-3}{2}}\partial_t^j\partial_x^i w\right)\right)^2dx\\
&\lesssim\int_1^\infty\sigma^{2i+j-3}(\partial_t^j\partial_x^i w)^2dx+\int_1^\infty\sigma^{2i+j-5}(\partial_t^j\partial_x^i w)^2\\
&+\sigma^{2i+j-3}\left(\partial_t^j\partial_x^{i+1}w\right)^2dx\\
&\lesssim\int_0^\infty\sigma^{\alpha+i}(\partial_t^j\partial_x^i w)^2dx+\sigma^{\alpha+i+1}\left(\partial_t^j\partial_x^{i+1}w\right)^2dx.
\end{aligned}$$

Here we have used the fact that for $x>1$ and $i+j\leq m-1=[\alpha]+3$,

$$2i+j-5\leq 2i+j-3\leq\alpha+i\leq\alpha+i+1.$$

Then when $i=0$, we have

$$\begin{aligned}
\|\psi\|_{L^\infty([1,+\infty))}^2\lesssim&(1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}_j\\
\leq&(1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t),
\end{aligned}$$

and when $i\geq 1$, we have

$$\begin{aligned}
\|\psi\|_{L^\infty([1,+\infty))}^2\lesssim&(1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\left(\mathcal{E}_{j,0}+\mathcal{E}_{j,1}\right)\\
\leq&(1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t).
\end{aligned}\tag{B.7}$$

In the following, we prove that

$$\begin{aligned}
&\|\psi\|_{L^\infty([0,1])}^2\lesssim(1+t)^{-2j+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t)\quad\text{for } i=0,\\
&\|\psi\|_{L^\infty([0,1])}^2\lesssim(1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t)\quad\text{for } i\geq 1.
\end{aligned}$$

Without loss of generality, we only show the case of $i\geq 1$. The case for $i=0$ follows the same line.

Since $\alpha - [\alpha] \in [0, 1)$, we can choosing a small ν such that $\alpha - [\alpha] + \nu \in [\nu, 1)$, so it follows from (B.1) and (3.8), we have that

$$\begin{aligned}
\|\psi\|^2_{L^\infty([0,1])} &\lesssim \|\psi\|^2_{H^{1-\frac{\alpha-[\alpha]+\nu}{2}}} \\
&= \|\psi\|^2_{H^{m+1-i-j-\frac{2(m-i-j)+\alpha-[\alpha]+\nu}{2}}} \\
&\lesssim \|\psi\|^2_{H^{2(m-i-j)+\alpha-[\alpha]+\nu, m+1-i-j}} \\
&\le \sum_{k=0}^{m+1-i-j} \int_0^1 \sigma^{2(m-i-j)+\alpha-[\alpha]+\nu} \left|\partial_x^k \psi\right|^2 dx.
\end{aligned} \tag{B.8}$$

Now we make some calculations for the right terms in (B.8)

A simple calculation yields

$$\left|\partial_x^k \psi\right| \leqslant \sum_{p=0}^{k} \left|\sigma^{\frac{2i+j-3}{2}-p} \partial_t^j \partial_x^{i+k-p} w\right| \quad \text{for } k = 1, 2, \dots, m+1-j-i. \tag{B.9}$$

It follows from (B.9) that for $1 \le k \le m+1-i-j$

$$\begin{aligned}
\int_0^1 \sigma^{2(m-i-j)+\alpha-[\alpha]+\nu} \left|\partial_x^k \psi\right|^2 dx &\lesssim \int_0^1 \sum_{p=0}^{k} \sigma^{\alpha+m-j+1-2p+\nu} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\lesssim \int_0^1 \sigma^{m-i-j+1-k+\nu} \sum_{p=0}^{1} \sigma^{\alpha+i+k-2p} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\quad + \int_0^1 \sum_{p=2}^{k} \sigma^{\alpha+m-j+1-2p+\nu} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\lesssim \int_0^1 \sum_{p=0}^{1} \sigma^{\alpha+i+k-2p} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\quad + \int_0^1 \sum_{p=2}^{k} \sigma^{\alpha+m-j+1-2p+\nu} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}_{j,i+k-1} \\
&\quad + \int_0^1 \sum_{p=2}^{k} \sigma^{\alpha+m-j+1-2p+\nu} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx.
\end{aligned}$$

To bound the second term on the right-hand side of the inequality above, notice that

$$\alpha + m - j + 1 - 2p + \nu = 2(m + 1 - i - j - k) + 2(k - p) + (\alpha - [\alpha]) + \nu + (2i + j - 3) - 2 > -1,$$

for $p \in [2, k]$, and $2i + j \geq 4$. We then have, with the aid of (3.7), that for $p \in [2, k]$,

$$\begin{aligned}
&\int_0^1 \sigma^{\alpha+m-j+1-2p} \left|\partial_t^j \partial_x^{i+k-p} w\right|^2 dx \\
&\lesssim \int_0^1 \sigma^{\alpha+m-j+1-2p+2} \sum_{\ell=0}^{1} \left|\partial_t^j \partial_x^{i+k-p+\ell} w\right|^2 dx \\
&\vdots \\
&\lesssim \int_0^1 \sigma^{\alpha+m-j+1} \sum_{\ell=0}^{p} \left|\partial_t^j \partial_x^{i+k-p+\ell} w\right|^2 dx \\
&= \int_0^1 \sum_{\ell=0}^{p} \sigma^{(m+1-i-j-k)+(p-\ell)} \sigma^{\alpha+i+k-p+\ell} \left|\partial_t^j \partial_x^{i+k-p+\ell} w\right|^2 dx \\
&\lesssim \sum_{\ell=0}^{p} \int_0^1 \sigma^{\alpha+i+k-p+\ell} \left|\partial_t^j \partial_x^{i+k-p+\ell} w\right|^2 dx \\
&\lesssim \sum_{\ell=i+k-p}^{i+k-1} \mathcal{E}_{j,\ell-1}.
\end{aligned}$$

That yields, for $k = 1, 2, \ldots, m + 1 - j - i$,

$$\begin{aligned}
\int_0^1 \sigma^{2(m-i-j)+\alpha-[\alpha]+\nu} \left|\partial_x^k \psi\right|^2 dx &\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}_{j,i+k-1} \\
&\quad + \sum_{p=2}^{k} \sum_{\ell=i+k-p}^{i+k-1} (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \mathcal{E}_{j,\ell-1} \\
&\lesssim (1+t)^{-2j-(\lambda+1)+\delta \mathbf{1}_{\lambda<1}} \sum_{\ell=i-1}^{m-j} \mathcal{E}_{j,\ell}.
\end{aligned} \tag{B.10}$$

Therefore, it follows from (B.8) and (B.10) that

$$\begin{aligned}\|\psi\|_{L^\infty([0,1])}^2 &\lesssim \int_0^1 \sigma^{2(m-i-j)+\alpha-[\alpha]+\delta}|\psi|^2dx + (1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\sum_{\ell=i-1}^{m-j}\mathcal{E}_{j,\ell} \\ &\lesssim \int_0^1 \sigma^{\alpha+i+m+1-i-j}|\partial_t^j\partial_x^i w|^2dx + (1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t) \\ &\lesssim \int_0^1 \sigma^{\alpha+i}|\partial_t^j\partial_x^i w|^2dx + (1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t) \\ &\lesssim (1+t)^{-2j-(\lambda+1)+\delta\mathbf{1}_{\lambda<1}}\mathcal{E}(t).\end{aligned}$$

This and (B.7) completes the L^∞ estimate for Case 2.

Combining results in case 1 and case 2, we finish the proof of Lemma 3.1. □

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