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The combined quasineutral and low Mach number limit of the Navier–Stokes–Poisson system

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Abstract. In this paper, the quasineutral limit of the compressible Navier–Stokes–Poisson system *in the critical L^p-type Besov space* is considered. More precisely, we will show that the solution of compressible Navier–Stokes–Poisson equations will converge to that of incompressible Navier–Stokes equations in the L^p framework when the *Debye length* is proportional to the Mach number and tends to zero. Moreover, the convergence rate will be obtained.

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Keywords. Quasineutral limit, Navier–Stokes–Poisson, Debye length.

1. Introduction

In this paper, we investigate the quasineutral limit for the following compressible Navier–Stokes–Poisson equations

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \tilde{\mu}\Delta u - (\tilde{\mu}+\tilde{\nu})\nabla \mathrm{div} u + \nabla P = \rho\nabla\Phi, \\ \lambda^2\Delta\Phi = \rho - \bar{\rho}, \end{cases} \tag{1.1}$$

where (ρ, u, Φ) represent the electron density, the electron velocity and the electrostatic potential, respectively. The pressure P is a smooth function of ρ with $P'(\rho) > 0$ for $\rho > 0$, and the viscosity coefficients $\tilde{\mu}, \tilde{\nu}$ are constants and satisfy $\tilde{\nu} > 0$ and $n\tilde{\nu} + 2\tilde{\mu} > 0$. Such a condition ensures ellipticity for the operator $\tilde{\mu}\Delta + (\tilde{\mu}+\tilde{\nu})\nabla\mathrm{div}$ and is satisfied in the physical cases. $\bar{\rho} > 0$ describes the background doping profile, and in this paper, for simplicity, we set $\bar{\rho} = 1$ and suppose that $P'(1) = 1$. The parameter λ is the so-called Debye length (up to a constant factor). The Navier–Stokes–Poisson system is a simplified model to describe the dynamics of a plasma where the compressible electron fluid interacts with its own electric field against a constant charged ion background. See [1] for more details.

For simplicity, we will use the abbreviation of "NSP" for "Navier–Stokes–Poisson" later on throughout the paper.

The quasineutral limit of the NSP has already been studied by many authors. Wang [2] studied the quasineutral limit for the smooth solution with well-prepared initial data. Jiang and Wang [3] studied the combined quasineutral and inviscid limit of the compressible Navier–Stokes–Poisson system for weak solution. Gasser and Marcati [4] studied the NSP system in the context of a combined quasineutral and relaxation time limit. Donatelli and Marcati [5] investigated the quasineutral limit of the NSP system by means of dispersive estimates of Strichartz's type under the assumption that the Mach number is related to the Debye length. See also [6–12] and references therein for more details on this topic.

We shall prove rigorously that, as the Debye length $\lambda \to 0$, the solution of the compressible Navier–Stokes Poisson system converges strongly to the strong solution of the incompressible Navier–Stokes equations. In the present paper, we shall consider the general ill-prepared initial data system (1.1), so the fast oscillating singular term will be produced by the non-divergence-free part of initial momentum and has to be described carefully in order to pass into the quasineutral limit.

Formally, if we set $\lambda = 0$, then we obtain $\rho = 1$, which is the so-called quasineutrality regime in plasma physics, and the behaviour of the fluid can be described by the incompressible Navier–Stokes system

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \pi = 0, \\ \nabla \cdot v = 0. \end{cases} \tag{1.2}$$

The present limit analysis has a very strong analogy with the theory of incompressible limits widely investigated on mathematical fluid dynamics. In particular, low Mach number limits have been studied by several authors, among which we recall [13–17]. The quasineutral limit yields to the introduction of a time scaling because of the incompressible limit regime; in addition, there is an electric potential scaling which is responsible for a very singular term which requires a more careful analysis of the acoustic waves. Recall that the Mach number for the compressible flow (1.1) is defined as

$$M = \frac{|u|}{\sqrt{P'(\rho)}}.$$

Then letting $M \to 0$, we hope that ρ keeps a size of 1 and u is of size ε, where ε is a small parameter. The incompressible limit scaling is given by

$$\rho(t,x) = \rho^\varepsilon(\varepsilon t, x), \quad u(t.x) = \varepsilon u^\varepsilon(\varepsilon t, x), \quad \Phi(t,x) = \Phi^\varepsilon(\varepsilon t, x),$$

and the viscosity coefficients are given as

$$\tilde{\mu} = \varepsilon \mu, \;\; \tilde{\nu} = \varepsilon \nu.$$

with $\mu, \;\nu$ being positive and $2\mu + n\nu > 0$. Then the system for $(\rho^\varepsilon, u^\varepsilon, \Phi^\varepsilon)$ satisfies

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mu \Delta u^\varepsilon - (\mu + \nu) \nabla \mathrm{div} u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} = \rho^\varepsilon \dfrac{\nabla \Phi^\varepsilon}{\varepsilon^2}, \\ \lambda^2 \Delta \Phi^\varepsilon = \rho^\varepsilon - 1, \end{cases} \tag{1.3}$$

where $P^\varepsilon \triangleq P(\rho^\varepsilon)$. Our analysis is performed under the assumption that the previous small parameter ε is related to the Debye length λ as in [5] by

$$\lambda = \varepsilon. \tag{1.4}$$

The heuristics of quasineutral limit of the compressible NSP system has already been justified rigorously in different contexts as motioned above. However, we remark that all the above results were carried out in the framework of Sobolev spaces. We strive to study the quasineutral limit for critical regularity assumption of system (1.1) consistent with those of the well-posedness for system (1.2) in Besov spaces. For the compressible NSP system, strictly speaking, it does not have any scaling invariance. However, if we neglect the pressure and the electrostatic potential (the lower-order terms), then it is clear that if (ρ, u) solves system (1.1), so does the couple $(a^c(t,x), u^c(t,x))$ with $(a^c, u^c) = (a(c^2t, cx), cu(c^2t, x))$, for any $c > 0$. This motivates us to introduce the critical space, whose norm is invariant under the transformation $(a, u) \to (a^c, u^c)$ (up to a constant independent of c).

In this paper, we focus on the case of ill-prepared data of the form $(\rho_0^\varepsilon = 1 + \varepsilon a_0^\varepsilon, u_0^\varepsilon)$ so that acoustic waves have to be considered, where $(a_0^\varepsilon, u_0^\varepsilon)$ are bounded in L^p critical Besov spaces. Set $\rho^\varepsilon = 1 + \varepsilon a^\varepsilon$, then $(a^\varepsilon, u^\varepsilon, \Phi^\varepsilon)$ satisfies

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = -\mathrm{div}(a^\varepsilon u^\varepsilon), \\ \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \mathcal{A} u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon} \\ \qquad = -K(\varepsilon a^\varepsilon)\dfrac{\nabla a^\varepsilon}{\varepsilon} - I(\varepsilon a^\varepsilon)\mathcal{A} u^\varepsilon + \dfrac{\nabla \Phi^\varepsilon}{\varepsilon^2}, \\ \Delta \Phi^\varepsilon = \dfrac{a^\varepsilon}{\varepsilon}, \end{cases} \tag{1.5}$$

where $I(\varepsilon a^\varepsilon) = \frac{\varepsilon a^\varepsilon}{1+\varepsilon a^\varepsilon}$, $K(\varepsilon a^\varepsilon) = \frac{P'(1+\varepsilon a^\varepsilon)}{1+\varepsilon a^\varepsilon} - 1$, and $\mathcal{A} u^\varepsilon = \mu \Delta u^\varepsilon + (\nu + \mu)\nabla \mathrm{div} u^\varepsilon$.

In order to be more specific, let us pause for a while and introduce the notation and function spaces that will be used throughout the paper. We will denote a generic constant by C which may be different from line to line and denote $A \leq CB$ by $A \lesssim B$. The notation $A \approx B$ means $A \leq CB$ and $B \leq CA$.

Littlewood–Paley theory and Besov spaces First, we introduce the Littlewood–Paley decomposition. There exist two radial smooth functions $\varphi(x), \chi(x)$ supported in the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ and the ball $B = \{\xi \in \mathbb{R}^n : |\xi| \leq 4/3\}$, respectively, such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n.$$

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogenous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f, \quad \dot{S}_j f = \sum_{k \leq j-1} \dot{\Delta}_k f = \chi(2^{-j}D)f.$$

With our choice of φ, it is easy to see that

$$\begin{aligned} \dot{\Delta}_j \dot{\Delta}_k f = 0 \quad \text{if } |j-k| \geq 2, \\ \dot{\Delta}_j(\dot{S}_{k-1} f \dot{\Delta}_k f) = 0 \quad \text{if } |j-k| \geq 5. \end{aligned} \tag{1.6}$$

The next Bernstein-type inequality will be repeatedly used through the paper.

Lemma 1.1 (Lemma2.1 of [18]). *Let $\mathcal{C}$ be an annulus and B a ball. A constant C exists such that for any non-negative integer k, any couple (p,q) in $[1,\infty]^2$ with $q \geq p \geq 1$, and any function u of L^p, we have*

$$Supp \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+n\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p},$$

$$Supp \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1}\lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k \|u\|_{L^p}.$$

Denote $\mathcal{Z}'(\mathbb{R}^n)$ by the dual space of

$$\mathcal{Z}(\mathbb{R}^n) \triangleq \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\alpha \hat{f}(0) = 0, \forall \alpha \in (\mathbb{N} \cup 0)^n\}.$$

Then the definition of the homogeneous Besov space is the following.

Definition 1.1. Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \Big(\sum_{j \in \mathbb{Z}} 2^{jsr}\|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} < \infty.$$

Since we work with time-dependent functions valued in Besov spaces, we introduce the norms

$$\|u\|_{L^q_T \dot{B}^s_{p,r}} := \big\| \|u(t,\cdot)\|_{\dot{B}^s_{p,r}} \big\|_{L^q(0,T)}.$$

Also, when performing the parabolic estimate, it is natural to use the following quantity

$$\|u\|_{\tilde{L}^q_T \dot{B}^s_{p,r}} \triangleq \Big(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^q_T(L^p)} \Big)^{\frac{1}{r}}.$$

The index T will be omitted if $T=+\infty$, and we shall denote by $\tilde{\mathcal{C}}_b(\dot{B}^s_{p,r})$ the subset of functions $\tilde{L}^\infty(\dot{B}^s_{p,r})$ which are continuous from $\mathbb{R}_+$ to $\dot{B}^s_{p,r}$.

An important estimate for the heat equation in Besov spaces is expressed in the following.

Lemma 1.2. *Let $p,q,r\in[1,\infty]$, $s\in\mathbb{R}$. Assume that $u_0\in \dot{B}^{s-1}_{p,r}$, $f\in \tilde{L}^1_T\dot{B}^{s-1}_{p,r}$. Let u be a solution of the equation*

$$\partial_t u - \mu\Delta u = f, \quad u|_{t=0}=u_0. \tag{1.7}$$

Then for $t\in[0,T]$, there holds

$$\|u\|_{\tilde{L}^q_T\dot{B}^{s-1+2/q}_{p,r}} \le C\big(\|u_0\|_{\dot{B}^{s-1}_{p,r}} + \|f\|_{\tilde{L}^1_T\dot{B}^{s-1}_{p,r}}\big). \tag{1.8}$$

Moreover, $u\in\mathcal{C}((0,T];\dot{B}^{s-1}_{p,r})$ if $r<\infty$. Readers can see [18] and [19] for its proof.

Restricting to the case of small-data global solutions, the corresponding well-posedness result for (1.2) reads as follows:

Theorem 1.1. *Assume $v_0\in\big(\dot{B}^{n/p-1}_{p,r}\big)^n$ with $\nabla\cdot v_0=0$, $p<+\infty$, and $r\in[1,+\infty]$. Then there exists a $\eta\sim\eta(\mu)$ such that when*

$$\|v_0\|_{\dot{B}^{n/p-1}_{p,r}} \le \eta,$$

then system (1.2) has a unique global solution

$$v\in\Big(\tilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-1}_{p,r})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}+1}_{p,r})\Big)^n \tag{1.9}$$

which is also in $\tilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-1}_{p,r})$ if $r<+\infty$ and satisfies

$$\|v\|_{\tilde{L}^\infty\dot{B}^{\frac{n}{p}-1}_{p,r}\cap L^1\dot{B}^{\frac{n}{p}+1}_{p,r}} \le C\|v_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,r}} \le C\eta,$$

where the constant C depends only on μ, n and p.

Remark 1.1. The above statement has been proved by J.-Y. Chemin in [19]. Data in general critical Besov spaces, with a slightly different solution space, have been considered by Kozono and Yamazaki in [20], and by Cannone, Meyer and Planchon in [21].

We can see that Theorem 1.1 is not related to energy arguments, but to our knowledge, all the present results about proving the limit of (1.1) to (1.2) strongly rely on the L^2-type norms estimates in order to get rid of the dependence on ε. This is due to the presence of the singular first skew symmetric terms (which disappear when performing L^2 estimates) in the following linearized equations of (1.5)

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\mathrm{div} u^\varepsilon}{\varepsilon} = f^\varepsilon, \\ \partial_t u^\varepsilon - \mathcal{A}u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon} + \dfrac{\nabla(-\Delta)^{-1}a^\varepsilon}{\varepsilon^3} = g^\varepsilon. \end{cases} \tag{1.10}$$

However, the singular terms do not affect the divergence-free part $\mathcal{P}u^\varepsilon$ of the velocity for the divergence-free part satisfies the heat equation (1.7). Thus, we can expect to deal with $\mathcal{P}u^\varepsilon$ by means of a L^p-type approach the same as that in Theorem 1.1. At the same time, for low frequencies, the singular terms dominate the evolution of a^ε and of the potential $\mathcal{P}^\perp u^\varepsilon$ of the velocity, which prevent us to use a $L^p(p\neq 2)$-type approach and restrict us to handle it only in L^2-type spaces as the wave equation is ill-posed in the $L^p(p\neq 2)$-type space. For the high frequencies, we will see that a^ε and $\mathcal{P}^\perp u^\varepsilon$ tend to behave like the solution of a damped equation and of a heat equation, respectively, and are tractable in L^p-type spaces.

Now let us recall some results related to the equations of Navier–Stokes–Poisson. The global existence of weak solutions of the compressible NSP system subject to large initial data is shown [22,23]. Hao and Li [24] gave the global existence and uniqueness of strong solutions in the L^2 Besov framework, while Chikami and Danchin [25] study the global existence and decay for the NSP system in the critical regularity framework. See similar arguments in [26,27]. Wu and Wang [28] give the pointwise estimates for bipolar compressible Navier–Stokes–Poisson system in dimension three. Tan et al. in [29] give the global solutions to the one-dimensional compressible Navier–Stokes–Poisson equations with large data. Li et al. [30] give an optimal decay rate of the solution of the NPS system in Sobolev spaces, while Wang [31] established the same time decay rates via a refined pure energy method. See also Bie *et al.* [32] in the context of Besov spaces.

When the electrostatic potential Φ is a constant, system (1.1) becomes the classical barotropic Navier–Stokes equations; Danchin [33] proved the global well-posedness of isentropic Navier–Stokes equations in the critical Besov spaces near equilibrium. Later, this result was extended to more general Besov spaces in [34–36]. In [13,14], the zero Mach number limit of the isentropic Navier–Stokes equation was studied in the whole space or torus in the L^2 critical Besov Space. Recently, Danchin and He [16] extended the result to the full Navier–Stokes–Fourier system in the L^p critical Besov space.

Our paper is arranged as follows. In Sect. 2, we give the main result. In Sect. 3, the global existence of solutions to system (1.5) is established, while Sect. 4 is devoted to proving the quasineutral limit of system (1.1) to system (1.2). Some useful estimates in Besov spaces are given in Appendix A.

2. Main results

First, we introduce some notations. For $z\in\mathcal{S}'(\mathbb{R}^n)$, the low-frequency and high-frequency parts of z with respect to a parameter ε are defined as

$$z^{\ell,\varepsilon}:=\sum_{2^j\varepsilon\le R_0}\dot{\Delta}_j z\quad\text{and}\quad z^{h,\varepsilon}:=\sum_{2^j\varepsilon>R_0}\dot{\Delta}_j z,$$

where R_0 is a sufficiently large constant depending on $n,\ \nu$ and μ. The corresponding semi-norms are

$$\|z\|^{\ell,\varepsilon}_{\dot{B}^s_{p,r}}:=\Big(\sum_{2^j\varepsilon\le R_0}2^{jsr}\|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac1r}\quad\text{and}\quad \|z\|^{h,\varepsilon}_{\dot{B}^s_{p,r}}:=\Big(\sum_{2^j\varepsilon>R_0}\|\dot{\Delta}_j z\|^r_{L^p}\Big)^{\frac1r}.$$

We consider a family of initial data $(a_0^\varepsilon,u_0^\varepsilon,\Phi_0^\varepsilon)$ so that

- $(a_0^\varepsilon)^{\ell,\varepsilon}\in\dot{B}^{n/2-2}_{2,1}$, $(\mathcal{P}^\perp u_0^\varepsilon)^{\ell,\varepsilon}\in\dot{B}^{n/2-1}_{2,1}$,
- $(a_0^\varepsilon)^{h,\varepsilon}\in\dot{B}^{n/p}_{p,1}$, $(\mathcal{P}^\perp u_0^\varepsilon)^{h,\varepsilon}\in\dot{B}^{n/p-1}_{p,1}$,
- $\mathcal{P}u_0^\varepsilon\in\dot{B}^{n/p-1}_{p,1}$,

with $\Phi_0^\varepsilon=-\varepsilon^{-1}(-\Delta)^{-1}a_0^\varepsilon$.

Our assumptions on the data induce us to look for a solution $(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)$ of (1.5) in the following space X_p^ε

- $(a^\varepsilon)^{\ell,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/2-2}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/2}_{2,1})$, $(\mathcal{P}^\perp u^\varepsilon)^{\ell,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/2+1}_{2,1})$,
- $(a^\varepsilon)^{h,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/p}_{p,1})$, $(\mathcal{P}^\perp u^\varepsilon)^{h,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/p+1}_{p,1})$,
- $\mathcal{P}u^\varepsilon\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/p+1}_{p,1})$,
- $(\Phi^\varepsilon)^{\ell,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/2}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/2+2}_{2,1})$, $(\Phi^\varepsilon)^{h,\varepsilon}\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{n/p+2}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{n/p+2}_{p,1})$,

which is endowed with the norm

$$
\begin{aligned}
&\|(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)\|_{X_p^\varepsilon}\\
&:=\varepsilon^{-1}\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B^{n/2-2}_{2,1}\cap L^1\dot B^{n/2}_{2,1}}+\varepsilon\|a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B^{n/p}_{p,1}}+\varepsilon^{-1}\|a^\varepsilon\|^{h,\varepsilon}_{L^1\dot B^{n/p}_{p,1}}\\
&\quad+\|\mathcal P^\perp u^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B^{n/2-1}_{2,1}\cap L^1\dot B^{n/2+1}_{2,1}}+\|\mathcal P^\perp u^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B^{n/p-1}_{p,1}\cap L^1\dot B^{n/p+1}_{p,1}}\\
&\quad+\|\mathcal P u^\varepsilon\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}\cap L^1\dot B^{n/p+1}_{p,1}}\\
&\quad+\|\Phi^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B^{n/2}_{2,1}\cap L^1\dot B^{n/2+2}_{2,1}}+\varepsilon^2\|\Phi^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B^{n/p+2}_{p,1}}+\|\Phi^\varepsilon\|^{h,\varepsilon}_{L^1\dot B^{n/p+2}_{p,1}}.
\end{aligned}\tag{2.1}
$$

We state our main result as follows.

Theorem 2.1. *Let $n\ge 2$ and p satisfy $2\le p\le\min\big(4,2n/(n-2)\big)$ and $p\ne 4$ if $n=2$. Assume that the initial data $(a_0^\varepsilon,u_0^\varepsilon,\Phi_0^\varepsilon)$ are as above. Then there there exist two positive constants, η and M, depending only on n, ν, μ and the function K, such that if*

$$
\begin{aligned}
C_0^\varepsilon\ &:=\varepsilon^{-1}\|a_0^\varepsilon\|^{\ell,\varepsilon}_{\dot B^{n/2-2}_{2,1}}+\varepsilon\|a_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{n/p}_{p,1}}\\
&\qquad+\|\mathcal P^\perp u_0^\varepsilon\|^{\ell,\varepsilon}_{\dot B^{n/2-1}_{2,1}}+\|\mathcal P^\perp u_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{n/p-1}_{p,1}}+\|\mathcal P u_0^\varepsilon\|_{\dot B^{n/p-1}_{p,1}}\le\eta,
\end{aligned}
$$

then system (1.5) has a unique global solution $(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)$ in X_p^ε such that

$$
\|(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)\|_{X_p^\varepsilon}\le MC_0^\varepsilon.\tag{2.2}
$$

In addition, $(a^\varepsilon,\mathcal P^\perp u^\varepsilon)$ converges to 0 when ε goes to zero and $\mathcal P u^\varepsilon$ converges to the solution of system (1.2) in the following function spaces if $\mathcal P u_0^\varepsilon\to v_0$ in the corresponding spaces.

Case $n\ge 3$:

$$
\begin{aligned}
&\|a^\varepsilon\|_{\tilde L^2\dot B^{\frac np-\frac12}_{p,1}}\le MC_0^\varepsilon\varepsilon^{\frac12},\quad \|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\le MC_0^\varepsilon\varepsilon^{\frac12-\frac1p}\\
&\|\mathcal P u^\varepsilon-v\|_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac32}_{p,1}\cap L^1\dot B^{\frac{n+1}{p}+\frac12}_{p,1}}\le M\Big(\|\mathcal P u_0^\varepsilon-v_0\|_{\dot B^{\frac{n+1}{p}-\frac32}_{p,1}}+C_0^\varepsilon\varepsilon^{\frac12-\frac1p}\Big).
\end{aligned}\tag{2.3}
$$

Case $n=2$:

$$
\begin{aligned}
&\|a^\varepsilon\|_{\tilde L^2\dot B^{\frac np-\frac12}_{p,1}}\le MC_0^\varepsilon\varepsilon^{\frac12},\quad \|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\le MC_0^\varepsilon\varepsilon^{\delta(\frac12-\frac1p)}\\
&\|\mathcal P u^\varepsilon-v\|_{\tilde L^\infty\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}\cap L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2+1}_{p,1}}\le M\Big(\|\mathcal P u_0^\varepsilon-v_0\|_{\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}}+C_0^\varepsilon\varepsilon^{\delta(\frac12-\frac1p)}\Big),
\end{aligned}\tag{2.4}
$$

where the constant δ satisfies $0\le\delta\le\frac12$ and $\delta<\frac{8-2p}{p-2}$.

3. Global existence for fixed ε in system (1.5)

Making change of unknowns

$$
(a,u,\Phi)(t,x):=(\varepsilon a^\varepsilon,\varepsilon u^\varepsilon,\Phi^\varepsilon)(\varepsilon^2t,\varepsilon x)\tag{3.1}
$$

and the change of data

$$
(a_0,u_0)(x):=(\varepsilon a_0^\varepsilon,\varepsilon u_0^\varepsilon)(\varepsilon x),\tag{3.2}
$$

then we note that $(a^\varepsilon,u^\varepsilon)$ solves (1.5) if and only if (a,u,Φ) solves the following system

$$
\begin{cases}
\partial_t a+\nabla\cdot u=-\nabla\cdot(au),\\
\partial_t u+u\cdot\nabla u-\mathcal A u+\nabla a\\
\qquad=-K(a)\nabla a-I(a)\mathcal A u+\nabla\Phi,\\
\Delta\Phi=a,
\end{cases}\tag{3.3}
$$

where $I(a)=\frac{a}{1+a}$ and $K(a)=\frac{P'(1+a)}{1+a}-1$.

For simplicity, we denote

$$z^\ell := z^{\ell,1} \quad \text{and} \quad z^h = z^{h,1},$$

$$\|z\|^\ell_{\dot B^s_{p,r}} := \Big(\sum_{2^j\le R_0} 2^{jsr}\|\dot\Delta_j z\|^r_{L^p}\Big)^{\frac1r} \quad \text{and} \quad \|z\|^h_{\dot B^s_{p,r}} := \Big(\sum_{2^j> R_0} \|\dot\Delta_j z\|^r_{L^p}\Big)^{\frac1r}.$$

Up to a harmless constant, we have

$$\begin{aligned}
&\varepsilon^{-1}\|a_0^\varepsilon\|^{\ell,\varepsilon}_{\dot B^{n/2-2}_{2,1}} + \|\mathcal P^\perp u_0^\varepsilon\|^{\ell,\varepsilon}_{\dot B^{n/2-1}_{2,1}} + \varepsilon\|a_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{n/p}_{p,1}} + \|\mathcal P^\perp u_0^\varepsilon\|^{h,\varepsilon}_{\dot B^{n/p-1}_{p,1}} + \|\mathcal P u_0^\varepsilon\|_{\dot B^{n/p-1}_{p,1}},\\
&= \|a_0\|^{\ell}_{\dot B^{n/2-2}_{2,1}} + \|\mathcal P^\perp u_0\|^{\ell}_{\dot B^{n/2-1}_{2,1}} + \|a_0\|^{h}_{\dot B^{n/p}_{p,1}} + \|\mathcal P^\perp u_0\|^{h}_{\dot B^{n/p-1}_{p,1}} + \|\mathcal P u_0\|_{\dot B^{n/p-1}_{p,1}}\\
&:= \|(a_0,u_0)\|_{X_p(0)}
\end{aligned}$$

and

$$\|(a,u,\Phi)\|_{X_p} := \|(a,u,\Phi)\|_{X^1_p} = \|(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)\|_{X^\varepsilon_p}.$$

3.1. A priori estimates to the solution of system (3.3)

Step 1: the incompressible part of the velocity. Applying $\mathcal P$ to the momentum equation $(3.3)_2$, we can have

$$\Big\{\, \partial_t\mathcal P u - \mu\Delta\mathcal P u = \mathcal P\Big(-u\cdot\nabla u - I(a)\mathcal A u\Big).$$

Using estimate (1.8) in Lemma 1.2 for the heat equation, we have

$$\begin{aligned}
\|\mathcal P u\|_{\tilde L^\infty \dot B^{\frac np-1}_{p,1}\cap L^1\dot B^{\frac np+1}_{p,1}} &\lesssim \|\mathcal P u_0\|_{\dot B^{\frac np-1}_{p,1}}\\
&\quad + \Big\|\mathcal P\Big(-u\cdot\nabla u - I(a)\mathcal A u\Big)\Big\|_{L^1\dot B^{\frac np-1}_{p,1}}.
\end{aligned}\tag{3.4}$$

Now we come to estimate the right-hand side of (3.4). Using (A.9) and (4.10), we have

$$\begin{aligned}
\|I(a)\nabla^2 u\|_{\tilde L^1\dot B^{\frac np-1}_{p,1}} &\lesssim \|I(a)\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}}\|\nabla^2 u\|_{L^1\dot B^{\frac np-1}_{p,1}}.\\
&\lesssim (1+\|a\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}})^{n/2+1}\|a\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}}\|u\|_{L^1\dot B^{\frac np+1}_{p,1}},\\
\|u\nabla u\|_{L^1\dot B^{\frac np-1}_{p,1}} &\lesssim \|u\|_{\tilde L^\infty\dot B^{\frac np-1}_{p,1}}\|u\|_{L^1\dot B^{\frac np+1}_{p,1}}.
\end{aligned}$$

By Bernstein inequality,

$$\begin{aligned}
\|a\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}} &\lesssim \|a\|^\ell_{\tilde L^\infty\dot B^{\frac np}_{p,1}} + \|a^h\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}}\\
&\lesssim R_0^2\|a\|^\ell_{\tilde L^\infty\dot B^{\frac n2-2}_{2,1}} + \|a^h\|_{\tilde L^\infty\dot B^{\frac np}_{p,1}}.
\end{aligned}$$

So, we can get

$$\|\mathcal P u\|_{\tilde L^\infty \dot B^{\frac np-1}_{p,1}\cap L^1\dot B^{\frac np+1}_{p,1}} \lesssim \|\mathcal P u_0\|_{\dot B^{\frac np-1}_{p,1}} + \big(1+\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\tag{3.5}$$

Step 2: the low-frequency part of $(a,\mathcal P^\perp u)$. Now we come to estimate the low frequencies of $(a,\mathcal P^\perp u)$ which satisfies

$$\begin{cases}
\partial_t a + \mathrm{div}\mathcal P^\perp u = -\mathrm{div}(au),\\
\partial_t\mathcal P^\perp u - (\nu+2\mu)\Delta\mathcal P^\perp u + \nabla a + \nabla(-\Delta)^{-1}a\\
\qquad = -\mathcal P^\perp\Big(u\cdot\nabla u + K(a)\nabla a + I(a)\mathcal A u\Big).
\end{cases}\tag{3.6}$$

Without loss of generality, we set $\nu + 2\mu = 1$.

Denote $a_j = \dot{\Delta}_j a$ and $u_j = \dot{\Delta}_j u$ for simplicity. By applying $\dot{\Delta}_j$ to (3.6), taking L^2 inner product of $(3.6)_1$ with $\Lambda^{-2} a_j$ and a_j, $(3.6)_2$ with $\mathcal{P}^\perp u_j$, we arrive at

$$\begin{aligned}
&\partial_t\Big(\|\mathcal{P}^\perp u_j\|_{L^2}^2 + \|a_j\|_{L^2}^2 - \|\Lambda^{-1} a_j\|_{L^2}^2\Big) + \|\nabla \mathcal{P}^\perp u_j\|_{L^2}^2 \\
&\quad = \Big(\Lambda^{-1}\mathrm{div}\dot{\Delta}_j(au)|\Lambda^{-1}a_j\Big) - \Big(\mathrm{div}\dot{\Delta}_j(au)|a_j\Big) \\
&\qquad - \Big(\dot{\Delta}_j\big(\mathcal{P}^\perp\big(u\cdot\nabla u + K(a)\nabla a + I(a)\mathcal{A}u\big)\big)|\mathcal{P}^\perp u_j\Big).
\end{aligned} \tag{3.7}$$

By applying $\nabla\dot{\Delta}_j$ to $(3.6)_1$ and $\dot{\Delta}_j$ to $(3.6)_2$, and then taking L^2 inner product of the resulting equations with $\mathcal{P}^\perp u_j$ and ∇a_j, respectively, we arrive at

$$\begin{aligned}
&\partial_t\Big(\nabla a_j|\mathcal{P}^\perp u_j\Big) - \|\mathrm{div}\mathcal{P}^\perp u_j\|_{L^2}^2 - \Big(\nabla a_j|\Delta\mathcal{P}^\perp u_j\Big) \\
&\quad + \|\nabla a_j\|_{L^2}^2 + \|a_j\|_{L^2}^2 \\
&\quad = \Big(\mathrm{div}\dot{\Delta}_j(au)|\mathrm{div}\mathcal{P}^\perp u_j\Big) \\
&\qquad - \Big(\dot{\Delta}_j\big(\mathcal{P}^\perp\big(u\cdot\nabla u + K(a)\nabla a + I(a)\mathcal{A}u\big)\big)|\nabla a_j\Big).
\end{aligned} \tag{3.8}$$

Multiplying (3.8) by a small constant $\nu(\sim R_0)$ and adding the resulting equation to (3.7), and then by using Cauchy–Schwarz inequality, we can arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathcal{P}^\perp u_j\|_{L^2} + 2^{-j}\|a_j\|_{L^2}\Big) + 2^{2j}\|\mathcal{P}^\perp u_j\|_{L^2} + 2^j\|a_j\|_{L^2} \\
&\quad \lesssim \|\dot{\Delta}_j(au)\|_{L^2} + \Big\|\dot{\Delta}_j\big(\mathcal{P}^\perp\big(u\cdot\nabla u + K(a)\nabla a + I(a)\mathcal{A}u\big)\big)\Big\|_{L^2}
\end{aligned} \tag{3.9}$$

The standard energy estimates for the barotropic linearized equations (see [18] or [33]) indicate

$$\begin{aligned}
&\|a\|^\ell_{\tilde{L}^\infty \dot{B}^{\frac{n}{2}-2}_{2,1}\cap L^1\dot{B}^{\frac{n}{2}}_{2,1}} + \|\mathcal{P}^\perp u\|^\ell_{\tilde{L}^\infty \dot{B}^{\frac{n}{2}-1}_{2,1}\cap L^1\dot{B}^{\frac{n}{2}+1}_{2,1}} \\
&\quad \lesssim \|a_0\|^\ell_{\dot{B}^{\frac{n}{2}-2}_{2,1}} + \|\mathcal{P}^\perp u_0\|^\ell_{\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|(au)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}} \\
&\qquad + \|\mathcal{P}^\perp(u\cdot\nabla u)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\mathcal{P}^\perp\big(K(a)\nabla a\big)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}} + \|\mathcal{P}^\perp\big(I(a)\mathcal{A}u\big)\|^\ell_{L^1\dot{B}^{\frac{n}{2}-1}_{2,1}}.
\end{aligned} \tag{3.10}$$

Let us bound the right-hand side of the above inequality term by term. By using $f = u, g = a, r_3 = \infty, r_4 = 1, \gamma = 0, r_1 = r_2 = 2$ in (A.7) and interpolations, we have

$$\begin{aligned}
\|(au)\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} &\lesssim \|u\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|a\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}} + \|u\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|a\|_{L^1\dot{B}^{n/p}_{p,1}} \\
&\lesssim \|(a,u,\Phi)\|^2_{X_p}.
\end{aligned} \tag{3.11}$$

Now let us bound $u\cdot\nabla u$ in $L^1(\dot{B}^{n/2-1}_{2,1})$. Using $f = u, g = \nabla u, r_3 = \infty, r_4 = 1, \gamma = 0, r_1 = r_2 = 2$ in (A.7) and interpolations, we have

$$\begin{aligned}
\|(u\cdot\nabla u)\|^\ell_{L^1\dot{B}^{n/2-1}_{2,1}} &\lesssim \|u\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|\nabla u\|_{L^1\dot{B}^{n/p}_{p,1}} + \|u\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla u\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}} \\
&\lesssim \|(a,u,\Phi)\|^2_{X_p}.
\end{aligned} \tag{3.12}$$

In order to bound $K(a)\nabla a$, we apply $f = K(a), g = \nabla a, r_1 = r_2 = 2, \gamma = 0, r_3 = \infty, r_4 = 1, \gamma = -1$ in (A.6) and interpolations, and then we get

$$\begin{aligned}\|K(a)\cdot\nabla a\|^{\ell}_{L^1\dot B^{n/2-1}_{2,1}} &\lesssim \|K(a)\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla a\|_{\tilde L^2\dot B^{n/p-1}_{p,1}}\\ &\quad+\|K(a)\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|\nabla a\|^{\ell}_{\tilde L^2\dot B^{n/2-1}_{2,1}}+R_0\|K(a)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}\|\nabla a\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\\ &\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}. \end{aligned}\tag{3.13}$$

To handle the term $I(a)\mathcal{A}u$, we apply $f=I(a), g=\nabla^2u, r_1=r_3=\infty, r_2=r_4=1, \gamma=-1$ in (A.7) and interpolations. Then we get

$$\begin{aligned}\|I(a)\mathcal{A}u\|^{\ell}_{L^1\dot B^{n/2-1}_{2,1}} &\lesssim \big(R_0\|I(a)\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}+\|I(a)\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\big)\|\nabla^2u\|_{L^1\dot B^{n/p-1}_{p,1}}\\ &\lesssim R_0\big(1+\|a\|_{L^\infty\dot B^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}. \end{aligned}\tag{3.14}$$

Estimates (3.10) to (3.14) indicate that

$$\begin{aligned}&\|a\|^{\ell}_{\tilde L^\infty\dot B^{\frac n2-2}_{2,1}\cap L^1\dot B^{\frac n2}_{2,1}}+\|\mathcal{P}^\perp u\|^{\ell}_{\tilde L^\infty\dot B^{\frac n2-1}_{2,1}\cap L^1\dot B^{\frac n2+1}_{2,1}}\\ &\quad\lesssim \|a_0\|^{\ell}_{\dot B^{\frac n2-2}_{2,1}}+\|\mathcal{P}^\perp u_0\|^{\ell}_{\dot B^{\frac n2-1}_{2,1}}+\big(1+\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}. \end{aligned}\tag{3.15}$$

Step 3: Effective velocity. We follow the approach in [36] to estimate the high frequencies of $\mathcal{P}^\perp u$. Introduce the following "effective velocity":

$$w:=\mathcal{P}^\perp u+(-\Delta)^{-1}\nabla a+(-\Delta)^{-2}\nabla a.$$

Then from (3.6), we get

$$\begin{aligned}\partial_t w-\Delta w=&-\big(Id+(-\Delta)^{-1}\big)\mathcal{P}^\perp(au)-\big(Id+(-\Delta)^{-1}\big)\mathcal{P}^\perp w\\ &-\big(Id+(-\Delta)^{-1}+(-\Delta)^{-2}\big)\nabla(-\Delta)^{-1}a\\ &-\mathcal{P}^\perp\Big(u\cdot\nabla u+K(a)\nabla a+I(a)\mathcal{A}u\Big).\end{aligned}$$

Applying the high-frequency estimate (1.8) of the heat equation to the above w equation, we obtain

$$\begin{aligned}\|w\|^{h}_{\tilde L^\infty\dot B^{n/p-1}_{p,1}\cap L^1\dot B^{n/p+1}_{p,1}} &\lesssim \|w_0\|^{h}_{\dot B^{n/p-1}_{p,1}}+(1+R_0^{-2})\|w\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\\ &\quad+(1+R_0^{-4})\|a\|^{h}_{L^1\dot B^{n/p-2}_{p,1}}+(1+R_0^{-2})\|(au)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\\ &\quad+\|(u\cdot\nabla u,K(a)\nabla a,I(a)\mathcal{A}u)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}},\end{aligned}\tag{3.16}$$

where we have used the 0-order homogeneity of operator $\mathcal{P}^\perp$, and when $2^j>R_0$, we have

$$\|(-\Delta)^{-1}f\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\lesssim R_0^{-2}\|w\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\quad\text{and}\quad \|(-\Delta)^{-2}f\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}\lesssim R_0^{-4}\|f\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}.$$

If R_0 is sufficiently large, the term $\|w\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}$ can be absorbed by the left-hand side of (3.16). The other terms satisfy the quadratic estimates. We proceed as follows.

To bound $\|(au)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=a, g=u, r_1=r_2=2, \gamma=1$ in (A.8), we obtain

$$\begin{aligned}\|au\|^{h}_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim R_0^{-1}\|a\|_{\tilde L^2\dot B^{n/p}_{p,1}}\|u\|_{\tilde L^2\dot B^{n/p}_{p,1}}\\ &\lesssim R_0^{-1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned}\tag{3.17}$$

For the term $\|(u\cdot\nabla u)\|^{h}_{L^1\dot B^{n/p-1}_{p,1}}$, applying $f=\nabla u, g=u, \gamma=0, r_1=1, r_2=\infty$ in (A.8), we get

$$\begin{aligned}\|u\nabla u\|^{h}_{L^1\dot B^{n/p-1}_{p,1}} &\lesssim \|u\|_{\tilde L^\infty\dot B^{n/p-1}_{p,1}}\|\nabla u\|_{\tilde L^2\dot B^{n/p}_{p,1}}\\ &\lesssim \|(a,u,\Phi)\|^2_{X_p}.\end{aligned}\tag{3.18}$$

For the term $\|(K(a)\nabla a)\|^h_{L^1\dot{B}^{n/p-1}_{p,1}}$, applying $f=K(a), g=\nabla a, \gamma=0, r_1=r_2=2$ in (A.8), we get

$$\begin{aligned}\|K(a)\nabla a\|^h_{L^1\dot{B}^{n/p-1}_{p,1}} &\lesssim \|K(a)\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla a\|_{\tilde{L}^2\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim \big(1+\|a\|_{L^\infty\dot{B}^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned} \tag{3.19}$$

For the term $\|(I(a)\mathcal{A}u)\|^h_{L^1\dot{B}^{n/p-1}_{p,1}}$, applying $f=I(a), g=\nabla^2 a, \gamma=0, r_1=\infty, r_2=1$ in (A.8), we get

$$\begin{aligned}\|I(a)\mathcal{A}u\|^h_{L^1\dot{B}^{n/p-1}_{p,1}} &\lesssim \|I(a)\|_{\tilde{L}^\infty\dot{B}^{n/p}_{p,1}}\|\nabla^2 u\|_{\tilde{L}^1\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim R_0\big(1+\|a\|_{L^\infty\dot{B}^{n/p}_{p,1}}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned} \tag{3.20}$$

Combining the estimates from (3.16) to (3.20), we conclude that

$$\begin{aligned}\|w\|^h_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}\cap L^1\dot{B}^{n/p+1}_{p,1}} &\lesssim \|w_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + R_0^{-2}\|a\|^h_{L^1\dot{B}^{n/p}_{p,1}}\\ &\quad + R_0\big(1+R_0\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned} \tag{3.21}$$

Step 4: The high frequency of the density. We find that a satisfies

$$\partial_t a + u\cdot\nabla a + a = -a\mathrm{div}u - \mathrm{div}w - (-\Delta)^{-1}a.$$

To bound the high frequency of a, for $2^j > R_0$,

$$\partial_t\dot{\Delta}_j a + u\cdot\nabla\dot{\Delta}_j a + \dot{\Delta}_j a = -\dot{\Delta}_j(a\mathrm{div}u + (-\Delta)^{-1}a + \mathrm{div}w) + R_j, \tag{3.22}$$

where $R_j := [u\cdot\nabla, \dot{\Delta}_j]a$.

Multiplying (3.22) by $\dot{\Delta}_j a|\dot{\Delta}_j a|^{p-2}$, and then integrating on $\mathbb{R}^n\times[0,t]$, we can get

$$\begin{aligned}\|\dot{\Delta}_j a(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_j a\|_{L^p}ds &\lesssim \|\dot{\Delta}_j a_0\|_{L^p} + \int_0^t \|\mathrm{div}u\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^p}ds\\ &\quad + \int_0^t \|\dot{\Delta}_j(a\mathrm{div}u + (-\Delta)^{-1}a + \mathrm{div}w)\|_{L^p}ds + \int_0^t \|R_j\|_{L^p}ds.\end{aligned} \tag{3.23}$$

Using (A.9), we have

$$\|a\mathrm{div}u\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|a\|_{\dot{B}^{n/p}_{p,1}}\|\mathrm{div}u\|_{\dot{B}^{n/p}_{p,1}}.$$

Commutator estimates in [18] give that

$$\sum_{j\in\mathbb{Z}}\|R_j\|_{L^p} \lesssim \|\nabla u\|_{\dot{B}^{n/p}_{p,1}}\|a\|_{\dot{B}^{n/p}_{p,1}}.$$

Now multiplying (3.23) by $2^{jn/p}$, using the above two estimates and summing over $2^j > R_0$, then we get

$$\begin{aligned}\|a\|^h_{\tilde{L}^\infty_t\dot{B}^{n/p}_{p,1}} + \int_0^t \|a\|^h_{\dot{B}^{n/p}_{p,1}}ds &\lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}}\\ &\quad + \int_0^t \|\nabla u\|_{\dot{B}^{n/p}_{p,1}}\|a\|_{\dot{B}^{n/p}_{p,1}}ds + \|w\|^h_{L^1\dot{B}^{n/p+1}_{p,1}}.\end{aligned}$$

Therefore, we get

$$\|a\|^h_{\tilde{L}^\infty_t\dot{B}^{n/p}_{p,1}\cap L^1_t\dot{B}^{n/p}_{p,1}} \lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + R_0\|(a,u,\Phi)\|^2_{X_p} + \|w\|^h_{L^1\dot{B}^{n/p+1}_{p,1}}. \tag{3.24}$$

Step5: Close of the a priori estimate. For a suitable small δ, multiply (3.24) by δ and then add it to (3.21). By choosing R_0 sufficiently large, we can get

$$\begin{aligned}&\|a\|^h_{\tilde L^\infty_t \dot B^{n/p}_{p,1}\cap L^1_t \dot B^{n/p}_{p,1}} + \|w\|^h_{\tilde L^\infty_t \dot B^{n/p-1}_{p,1}\cap L^1_t \dot B^{n/p+1}_{p,1}}\\ &\lesssim \|w_0\|^h_{\dot B^{n/p-1}_{p,1}} + \|a_0\|^h_{\dot B^{n/p}_{p,1}} + R_0\big(1+R_0\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned}$$

Since $\mathcal{P}^\perp u^h = w^h - (-\Delta)^{-1}\nabla a^h$, the above estimate still holds for a^h and $\mathcal{P}^\perp u^h$. So we have

$$\begin{aligned}&\|a\|^h_{\tilde L^\infty_t \dot B^{n/p}_{p,1}\cap L^1_t \dot B^{n/p}_{p,1}} + \|\mathcal{P}^\perp u\|^h_{\tilde L^\infty_t \dot B^{n/p-1}_{p,1}\cap L^1_t \dot B^{n/p+1}_{p,1}}\\ &\lesssim \|\mathcal{P}^\perp u_0\|^h_{\dot B^{n/p-1}_{p,1}} + \|a_0\|^h_{\dot B^{n/p}_{p,1}} + R_0\big(1+R_0\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned} \tag{3.25}$$

Finally, combining estimates (3.5), (3.15) and (3.25) and the relation $\Phi = -(-\Delta)^{-1}a$, we have

$$\begin{aligned}\|(a,u,\Phi)\|_{X_p} &\lesssim \|a_0\|^\ell_{\dot B^{n/2-2}_{2,1}} + \|\mathcal{P}^\perp u_0\|^\ell_{\dot B^{n/2-1}_{2,1}} + \|a_0\|^h_{\dot B^{n/p}_{p,1}} + \|\mathcal{P}^\perp u_0\|^h_{\dot B^{n/p-1}_{p,1}}\\ &\quad + \|\mathcal{P} u_0\|_{\dot B^{n/p-1}_{p,1}} + R_0\big(1+R_0\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|^2_{X_p}.\end{aligned} \tag{3.26}$$

3.2. Global existence of the solution to system (3.3)

Now we come to give the global existence of the solution to system (3.3). Define

$$\begin{aligned}X_2(T) = \big\{(a,u,\Phi) : \;& a^\ell \in \tilde L^\infty(0,T;\dot B^{n/2-2}_{2,1})\cap L^1(0,T;\dot B^{n/2}_{2,1});\\ & a^h \in \tilde L^\infty(0,T;\dot B^{n/2}_{2,1})\cap L^1(0,T;\dot B^{n/2}_{2,1});\\ & u \in \tilde L^\infty(0,T;\dot B^{n/2-1}_{2,1})\cap L^1(0,T;\dot B^{n/2+1}_{2,1});\\ & \Phi^\ell \in \tilde L^\infty(0,T;\dot B^{n/2}_{2,1})\cap L^1(0,T;\dot B^{n/2+2}_{2,1}); \qquad (3.27)\\ & \Phi^h \in \tilde L^\infty(0,T;\dot B^{n/2+2}_{2,1})\cap L^1(0,T;\dot B^{n/2+2}_{2,1})\big\}, \qquad (3.28)\end{aligned}$$

with norm

$$\begin{aligned}&\|(a,u,\Phi)\|_{X_2}\\ &= \|a\|^\ell_{\tilde L^\infty \dot B^{n/2-2}_{2,1}\cap L^1 \dot B^{n/2}_{2,1}} + \|a\|^h_{\tilde L^\infty \dot B^{n/2}_{2,1}\cap L^1 \dot B^{n/2}_{2,1}}\\ &\quad + \|u\|_{\tilde L^\infty \dot B^{n/2-1}_{2,1}\cap L^1 \dot B^{n/2+1}_{2,1}} + \|\Phi\|^\ell_{\tilde L^\infty \dot B^{n/2}_{2,1}\cap L^1 \dot B^{n/2+2}_{2,1}} + \|\Phi\|^h_{\tilde L^\infty \dot B^{n/2+2}_{2,1}\cap L^1 \dot B^{n/2+2}_{2,1}}.\end{aligned}$$

Set

$$\|(a_0,u_0)\|_{X_2(0)} := \|a_0\|^\ell_{\dot B^{n/2-2}_{2,1}} + \|a_0\|^h_{\dot B^{n/2}_{2,1}} + \|u_0\|_{\dot B^{n/2-1}_{2,1}}.$$

Duplicating the proof of the a priori estimate (3.26), we can prove that

$$\begin{aligned}\|(a,u,\Phi)\|_{X_2} &\lesssim \|(a_0,u_0)\|_{X_2(0)}\\ &\quad + R_0\big(1+R_0\|(a,u,\Phi)\|_{X_p}\big)^{n/2+1}\|(a,u,\Phi)\|_{X_p}\|(a,u,\Phi)\|_{X_2}.\end{aligned} \tag{3.29}$$

Now by using the a priori estimates (3.26) and (3.29), we sketch the proof of the global existence of system (3.3). The simplest way is to smooth out the initial data (a_0,u_0) into a sequence of initial data $(a_{0,k},u_{0,k})_{k\in\mathbb{N}}$ with

$$a^\ell_{0,k} \in \dot B^{n/2-2}_{2,1},\, a^h_{0,k} \in \dot B^{n/2}_{2,1},\, u_{0,k} \in \dot B^{n/2-1}_{2,1}$$

and

$$\|a_{0,k}-a_0\|^{\ell}_{\dot B^{n/2-2}_{2,1}}+\|\mathcal{P}^{\perp}u_{0,k}-\mathcal{P}^{\perp}u_0\|^{\ell}_{\dot B^{n/2-1}_{2,1}}+\|a_{0,k}-a_0\|^{h}_{\dot B^{n/p}_{p,1}}$$
$$+\|\mathcal{P}^{\perp}u_{0,k}-\mathcal{P}^{\perp}u_0\|^{h}_{\dot B^{n/p-1}_{p,1}}+\|\mathcal{P}u_{0,k}-\mathcal{P}u_0\|_{\dot B^{n/p-1}_{p,1}}\longrightarrow 0 \quad \text{as} \quad k\to\infty. \tag{3.30}$$

Estimate (3.30) implies that there exists a constant C_0 such that

$$\|(a_{0,k},u_{0,k})\|_{X_p(0)}\le C\|(a_0,u_0)\|_{X_p(0)}\le C_0\eta. \tag{3.31}$$

Now it is not hard to use a Fridrich's method as in [18] to get that there exists a maximal existence time $T_k>0$ such that system (3.3) have a local unique-in-time solution $(a_k,u_k,\Phi_k)\in X_2(T_k)$ with initial data $(a_{0,k},u_{0,k})$. Using the definition of Besov space, it is easy to see that $(a_k,u_k,\Phi_k)\in X_p(T_k)$. The a priori estimates (3.26) and (3.31) imply that there exists a M such that

$$\|(a_k,u_k,\Phi_k)\|_{X_p(T_k)}\le M\eta. \tag{3.32}$$

Actually, by choosing η small and using the a priori estimates (3.29) and (3.32), we can get

$$\|(a_k,u_k,\Phi_k)\|_{X_2(T_k)}\lesssim\|(a_k,u_k,\Phi_k)\|_{X_2(0)},$$

which implies that $T_k=\infty$.

At last, we get that

$$\|(a_k,u_k,\Phi_k)\|_{X_p(\infty)}\lesssim\|(a_0,u_0)\|_{X_p(0)}\le M\eta. \tag{3.33}$$

Next, compactness arguments similar to those of [18] allow us to conclude (a_k,u_k,Φ_k) weakly converges (up to extraction) to some function (a,u,Φ) which is a solution of (3.3) with the desired regularity properties and satisfies (2.2) with $\varepsilon=1$. Scaling back to the original unknowns $(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)$ completes the proof of the global existence part of Theorem 2.1.

4. The incompressible limit: Strong convergence of the solution

In this section, we combine Strichartz estimates for the following linear system of acoustics:

$$\begin{cases}\partial_t b+\Lambda v=F\\ \partial_t v-(\Lambda+\Lambda^{-1})b=G\\ (b,v)|_{t=0}=(b_0,v_0),\end{cases} \tag{4.1}$$

and the uniform estimates in (2.2) for the global solution $(a^\varepsilon,u^\varepsilon,\Phi^\varepsilon)$ so as to establish the strong convergence for $(a^\varepsilon,\mathcal{P}^{\perp}u^\varepsilon)$ to zero and for $\mathcal{P}u^\varepsilon$ to the solution v of system (1.3) in proper function spaces.

4.1. Convergence to zero of the compressible part

First, we give an estimate for the solution of the linear system (4.1) whose proof can be founded in Proposition 10.30 of [18].

Proposition 4.1. *Let (b,v) be a solution of (4.1). Then, for any $s\in\mathbb{R}$ and positive T (possibly infinite), the following estimate holds:*

$$\|(b,v)\|_{\tilde L^r_T\dot B^{s+n(\frac1q-\frac12)+\frac1r}_{q,1}}\lesssim\|(b_0,v_0)\|_{\dot B^s_{2,1}}+\|(F,G)\|_{\tilde L^{\bar r'}_T\dot B^{s+n(\frac1{\bar q'}-\frac12)+\frac1{\bar r'}}_{\bar q',1}} \tag{4.2}$$

with

$$q\ge2,\frac2r\le\min\{1,\gamma(q)\},\quad (r,q,n)\ne(2,\infty,3),$$
$$\bar q\ge2,\frac2{\bar r}\le\min\{1,\gamma(\bar q)\},\quad (\bar r,\bar q,n)\ne(2,\infty,3),$$

where $\gamma(q):=(n-1)(\frac12-\frac1q)$, $\quad\frac1{\bar q}+\frac1{\bar q'}=1$, *and* $\frac1{\bar r}+\frac1{\bar r'}=1$.

Proof. Although there is an extra term $\Lambda^{-1}b$ in $(4.1)_2$ comparing with the equations W_1 in Proposition 10.30 of [18], however, by going through the proof there, we can still get the same result (4.2) with no much difference. So, we omit the details for convenience.

Remark 4.1. Actually, the above inequality (4.2) still holds for the low frequency, which means that

$$\|(b,v)\|^{\ell}_{\tilde{L}^r_T\dot{B}^{s+n(\frac{1}{q}-\frac{1}{2})+\frac{1}{r}}_{q,1}} \lesssim \|(b_0,v_0)\|^{\ell}_{\dot{B}^s_{2,1}} + \|(F,G)\|^{\ell}_{\tilde{L}^{\bar{r}'}_T\dot{B}^{s+n(\frac{1}{\bar{p}'}-\frac{1}{2})+\frac{1}{\bar{r}'}}_{\bar{p}',1}}. \tag{4.3}$$

In order to prove the convergence to 0 for the $(a^\varepsilon, \mathcal{P}^\perp u^\varepsilon)$, we use the fact that

$$\begin{cases} \partial_t a^\varepsilon + \varepsilon^{-1}\mathrm{div}\mathcal{P}^\perp u^\varepsilon = F^\varepsilon, \\ \partial_t \mathcal{P}^\perp u^\varepsilon + \varepsilon^{-1}\nabla a^\varepsilon + \varepsilon^{-3}\nabla(-\Delta)^{-1}a^\varepsilon = G^\varepsilon, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} F^\varepsilon &:= -\mathrm{div}(a^\varepsilon u^\varepsilon) \\ G^\varepsilon &:= \mathcal{P}^\perp\Big(-u^\varepsilon\cdot\nabla u^\varepsilon - K(\varepsilon a^\varepsilon)\frac{\nabla a^\varepsilon}{\varepsilon} + \frac{1}{1+\varepsilon a^\varepsilon}\mathcal{A}u^\varepsilon\Big). \end{aligned}$$

Doing the same scaling as in (3.1), we can get

$$\begin{cases} \partial_t a + \mathrm{div}\mathcal{P}^\perp u = F, \\ \partial_t \mathcal{P}^\perp u + \nabla a + \nabla(-\Delta)^{-1}a = G, \end{cases} \tag{4.5}$$

with

$$\begin{aligned} F &:= -\mathrm{div}(au) \\ G &:= \mathcal{P}^\perp\Big(-u\cdot\nabla u - K(a)\nabla a + \frac{1}{1+a}\mathcal{A}u\Big). \end{aligned}$$

Obviously, estimate (4.3) stated in Proposition 4.1 also holds for system (4.5) since a and $\Lambda^{-1}\mathrm{div}\mathcal{P}^\perp u$ satisfy (4.1) with source terms F, $\Lambda^{-1}\mathrm{div}G$ and $\Lambda^{-1}\mathrm{div}$ is a homogeneous multiplier of degree 0.

Hence, by taking $\bar{q}=2, \bar{r}=\infty, s=n/2-1$ and

— $\frac{1}{r}=\frac{1}{2}-\frac{1}{q}$ for $q\in[2,\frac{2(n-1)}{n-3}]$ if $n\geq 3$,

— $\frac{1}{r}=\delta(\frac{1}{2}-\frac{1}{q})$ for $q\in[2,\infty]$ if $n=2$, where $\delta\in[0,\frac{1}{2}]$ will be determined later on, we have

Case $n\geq 3$:

$$\|(a,\mathcal{P}^\perp u)\|^{\ell}_{\tilde{L}^{\frac{2q}{q-2}}\dot{B}^{\frac{n-1}{q}-\frac{1}{2}}_{q,1}} \lesssim \|(a_0,\mathcal{P}^\perp u_0)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2}_{2,1}}.$$

Case $n=2$:

$$\|(a,\mathcal{P}^\perp u)\|^{\ell}_{\tilde{L}^{\frac{2q}{\delta(q-2)}}\dot{B}^{\frac{n-\delta}{q}-1+\frac{\delta}{2}}_{q,1}} \lesssim \|(a_0,\mathcal{P}^\perp u_0)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}} + \|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2}_{2,1}}.$$

Duplicating the estimate of (3.10), we can get for $n\geq 2$,

$$\|(F,G)\|^{\ell}_{L^1\dot{B}^{n/2}_{2,1}} \lesssim (1+\|(a,u,\Phi)\|_{X_p})^{n/2+1}\|(a,u,\Phi)\|_{X_p} \lesssim C_0^1.$$

Then we have the following estimates

$$\begin{aligned} &\|(a,\mathcal{P}^\perp u)\|^{\ell}_{\tilde{L}^{\frac{2q}{q-2}}\dot{B}^{\frac{n-1}{q}-\frac{1}{2}}_{q,1}} \lesssim C_0^1, \text{ for } n\geq 3; \\ &\|(a,\mathcal{P}^\perp u)\|^{\ell}_{\tilde{L}^{\frac{2q}{\delta(q-2)}}\dot{B}^{\frac{n-\delta}{q}-1+\frac{\delta}{2}}_{q,1}} \lesssim C_0^1 \text{ for } n=2. \end{aligned} \tag{4.6}$$

We also have the estimate

$$\|a\|^{\ell}_{L^1\dot{B}^{n/2}_{2,1}} + \|\mathcal{P}^\perp u\|^{\ell}_{L^1\dot{B}^{n/2+1}_{2,1}} \lesssim C_0^1, \text{ for } n\geq 2. \tag{4.7}$$

Recall the complex interpolation inequality

$$\|f\|_{\tilde{L}^{r_\theta}\dot{B}^{s_\theta}_{p_\theta,1}} \lesssim \|f\|^{1-\theta}_{\tilde{L}^{r_0}\dot{B}^{s_0}_{p_0,1}} \|f\|^{\theta}_{\tilde{L}^{r_1}\dot{B}^{s_1}_{p_1,1}}, \tag{4.8}$$

with $\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s_\theta = (1-\theta)s_0 + \theta s_1$, and $0 < \theta < 1, s_0 < s_1, p_0 < p_1, r_0 < r_1$.

When $n \geq 3$, set $q = 2p-2$, which satisfies the condition $q \in [2, \frac{2(n-1)}{n-3}]$ if $p \in \min\{4, \frac{2n}{n-2}\}$ and when $n = 2$, set $q = 2\frac{(p-2)\delta+p}{(p-2)\delta+4-p}$, which satisfies $q \geq 2$ if $2 \leq p < 4$.

Using interpolation (4.8) between (4.6) and (4.7) with q chosen as above, we can get

$$\begin{aligned} &\|a\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} + \|\mathcal{P}^{\perp}u\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} \lesssim C_0^1 \text{ for } n \geq 3, \\ &\|a\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} + \|\mathcal{P}^{\perp}u\|^{\ell}_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} \lesssim C_0^1 \text{ for } n = 2. \end{aligned} \tag{4.9}$$

Back to the original variables in (4.4), we can have for any $\varepsilon > 0$,

$$\begin{aligned} &\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \lesssim \varepsilon^{\frac{1}{2}}C_0^\varepsilon, \quad \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0^\varepsilon, \quad \text{for} \quad n \geq 3, \\ &\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \lesssim \varepsilon^{\frac{1}{2}}C_0^\varepsilon, \quad \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} \lesssim \varepsilon^{\delta(\frac{1}{2}-\frac{1}{p})}C_0^\varepsilon, \quad \text{for} \quad n = 2, \delta \in [0, 1/2]. \end{aligned}$$

Now combining the high-frequency cut-off in (2.2), we obtain

Case $n \geq 3$:

$$\begin{aligned} \|a^\varepsilon\|_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} &\lesssim \|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} + \|a^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} \\ &\lesssim \|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}} + \varepsilon^{\frac{1}{2}}\|(a^\varepsilon, \mathcal{P}^{\perp}u^\varepsilon)\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}}_{p,1}} \\ &\lesssim \varepsilon^{\frac{1}{2}}C_0^\varepsilon. \\ \|\mathcal{P}^{\perp}u^\varepsilon\|_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} &\lesssim \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} + \|\mathcal{P}^{\perp}u^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} \\ &\lesssim \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+1}{p}-\frac{1}{2}}_{p,1}} + \varepsilon^{\frac{1}{2}-\frac{1}{p}}\|\mathcal{P}^{\perp}u^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}}_{p,1}} \\ &\lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0^\varepsilon. \end{aligned}$$

Case $n = 2$: for $\delta \in [0, \frac{1}{2}]$,

The estimate of $\|a^\varepsilon\|_{\tilde{L}^2\dot{B}^{\frac{n}{p}-\frac{1}{2}}_{p,1}}$ is the same as the case $n \geq 3$, while

$$\begin{aligned} \|\mathcal{P}^{\perp}u^\varepsilon\|_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} &\lesssim \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} + \|\mathcal{P}^{\perp}u^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} \\ &\lesssim \|\mathcal{P}^{\perp}u^\varepsilon\|^{\ell,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n+\delta}{p}-\frac{\delta}{2}}_{p,1}} + \varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{p}\right)}\|\mathcal{P}^{\perp}u^\varepsilon\|^{h,\varepsilon}_{\tilde{L}^2\dot{B}^{\frac{n}{p}}_{p,1}} \\ &\lesssim \varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{p}\right)}C_0^\varepsilon. \end{aligned}$$

This completes the strong convergence of $(a^\varepsilon, \mathcal{P}^{\perp}u^\varepsilon)$ to 0 with an explicit rate in suitable Besov spaces.

4.2. Convergence of the incompressible part

Let us now give the convergence of $\mathcal{P}u^\varepsilon$ to the solution v of (1.2). Set $v^\varepsilon = \mathcal{P}u^\varepsilon - v$. Applying $\mathcal{P}$ to $(1.5)_2$, $(1.2)_1$ and subtracting the latter from the former yield the following equations

$$\begin{cases} \partial_t v^\varepsilon - \mu\Delta v^\varepsilon = J^\varepsilon, \\ v^\varepsilon|_{t=0} = \mathcal{P}u_0^\varepsilon - v_0, \end{cases} \tag{4.10}$$

with

$$J^\varepsilon := -\mathcal{P}\big(u^\varepsilon\cdot\nabla v^\varepsilon + \mathcal{P}^\perp u^\varepsilon\cdot\nabla v + v^\varepsilon\cdot\nabla v + u^\varepsilon\cdot\nabla\mathcal{P}^\perp u^\varepsilon + I(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\big).$$

In what follows, we aim at estimating v^ε in the space

$$\left(\tilde{L}^\infty\left(\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}\right)\cap L^1\left(\dot{B}_{p,1}^{\frac{n+1}{p}+\frac{1}{2}}\right)\right)^n (n\geq 3),$$

and

$$\left(\tilde{L}^\infty\left(\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}\right)\cap L^1\left(\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{\delta}{2}+1}\right)\right)^n (n=2).$$

Set

$$Y_{p,n} := \|v^\varepsilon\|_{\tilde{L}^\infty\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}\cap L^1\dot{B}_{p,1}^{\frac{n+1}{p}+\frac{1}{2}}} \quad \text{if } n\geq 3,$$

and

$$Y_{p,n} := \|v^\varepsilon\|_{\tilde{L}^\infty\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}\cap L^1\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}+1}} \quad \text{if } n=2.$$

We claim that for any p, satisfying the assumption stated in Theorem 2.1, we have

$$Y_{p,n} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0^\varepsilon + \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} \quad \text{if } n\geq 3. \tag{4.11}$$

And

$$Y_{p,n} \lesssim \varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{p}\right)}C_0^\varepsilon + \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}} \quad \text{if } n=2, \tag{4.12}$$

with $\delta\in[0,1/2]$ and $\delta<\frac{8-2p}{p-2}$.

Actually, by virtue of inequality (1.8), we have

$$Y_{p,n} \lesssim \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} + \|J^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} \quad \text{if } n\geq 3, \tag{4.13}$$

and

$$Y_{p,n} \lesssim \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}} + \|J^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}} \quad \text{if } n=2. \tag{4.14}$$

Next, we deal with J^ε term by term by using product and composition estimates in the spirit of the previous sections.

Case $n\geq 3$:

It is easy to see that $\frac{n+1}{p}-\frac{1}{2}\leq\frac{n}{p}$ and $\frac{n+1}{p}-\frac{3}{2}+\frac{n}{p}>0$, we will repeatedly use Proposition A.2.

$$\begin{aligned}
\|u^\varepsilon\cdot\nabla v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|u^\varepsilon\|_{\tilde{L}^\infty\dot{B}_{p,1}^{\frac{n}{p}-1}}\|\nabla v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}} \\
&\lesssim C_0^\varepsilon\|v^\varepsilon\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}+\frac{1}{2}}} \lesssim \eta Y_{n,p}, \\
\|\mathcal{P}^\perp u^\varepsilon\cdot\nabla v\|_{L^1\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{3}{2}}} &\lesssim \|\mathcal{P}^\perp u^\varepsilon\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n+1}{p}-\frac{1}{2}}}\|\nabla v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}-1}} \\
&\lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0^\varepsilon\|v\|_{\tilde{L}^2\dot{B}_{p,1}^{\frac{n}{p}}} \lesssim \eta^2\varepsilon^{\frac{1}{2}-\frac{1}{p}},
\end{aligned}$$

$$
\begin{aligned}
\|v^\varepsilon\cdot\nabla v\|_{L^1\dot B^{\frac{n+1}{p}-\frac32}_{p,1}} &\lesssim \|v^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\|\nabla v\|_{\tilde L^2\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\lesssim Y_{p,n}\|v\|_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\lesssim \eta Y_{p,n},\\
\|u^\varepsilon\cdot\nabla\mathcal P^\perp u^\varepsilon\|_{L^1\dot B^{\frac{n+1}{p}-\frac32}_{p,1}} &\lesssim \|u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\|\nabla\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+1}{p}-\frac32}_{p,1}}\\
&\lesssim \varepsilon^{\frac12-\frac1p}(C_0^\varepsilon)^2\lesssim \eta^2\varepsilon^{\frac12-\frac1p},\\
\|I(\varepsilon a^\varepsilon)\mathcal A u^\varepsilon\|_{L^1\dot B^{\frac{n+1}{p}-\frac32}_{p,1}} &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\|\nabla^2 u^\varepsilon\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\lesssim \left(1+\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\right)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\|u^\varepsilon\|_{L^1\dot B^{\frac{n}{p}+1}_{p,1}},
\end{aligned}
$$

Besides, we have

$$
\begin{aligned}
\|\varepsilon a^\varepsilon\|_{L^\infty\dot B^{\frac{n+1}{p}-\frac12}_{p,1}} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}+\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B^{\frac{n+1}{p}-\frac12}_{p,1}}\\
&\lesssim \varepsilon^{1/2-1/p}\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty\dot B^{\frac{n}{2}-1}_{p,1}}+\varepsilon^{1/2-1/p}\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty\dot B^{\frac{n}{p}}_{p,1}}\\
&\lesssim \varepsilon^{1/2-1/p}C_0^\varepsilon.
\end{aligned}
$$

Then we get

$$
\|J^\varepsilon\|_{L^1\dot B^{\frac{n+1}{p}-\frac32}_{p,1}}\lesssim \eta Y_{p,n}+\varepsilon^{\frac12-\frac1p}\eta^2. \tag{4.15}
$$

Plugging estimate (4.15) into (4.13) and remembering that η is sufficiently small, then we obtain (4.11).

Case $n=2$:

In order to use Proposition A.2, we need $\frac{2+\delta}{p}-\frac{\delta}{2}-1+\frac{2}{p}>0$ which indicates that $\delta<\frac{8-2p}{p-2}$.

$$
\begin{aligned}
\|u^\varepsilon\cdot\nabla v^\varepsilon\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}} &\lesssim \|u^\varepsilon\|_{\tilde L^\infty\dot B^{\frac{n}{p}-1}_{p,1}}\|\nabla v^\varepsilon\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\\
&\lesssim C_0^\varepsilon\|v^\varepsilon\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2+1}_{p,1}}\lesssim \eta Y_{n,p},\\
\|\mathcal P^\perp u^\varepsilon\cdot\nabla v\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}} &\lesssim \|\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\|\nabla v\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\lesssim \varepsilon^{\delta(\frac12-\frac1p)}C_0^\varepsilon\|v^\varepsilon\|_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\lesssim \varepsilon^{\delta(\frac12-\frac1p)}\eta^2,\\
\|v^\varepsilon\cdot\nabla v\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}} &\lesssim \|v^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\|\nabla v\|_{\tilde L^2\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\lesssim Y_{p,n}\|v\|_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\lesssim \eta Y_{p,n},\\
\|u^\varepsilon\cdot\nabla\mathcal P^\perp u^\varepsilon\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}} &\lesssim \|u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n}{p}}_{p,1}}\|\nabla\mathcal P^\perp u^\varepsilon\|_{\tilde L^2\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}}\\
&\lesssim \varepsilon^{\delta(\frac12-\frac1p)}(C_0^\varepsilon)^2\lesssim \varepsilon^{\delta(\frac12-\frac1p)}\eta^2,\\
\|I(\varepsilon a^\varepsilon)\mathcal A u^\varepsilon\|_{L^1\dot B^{\frac{n+\delta}{p}-\frac\delta2-1}_{p,1}} &\lesssim \|I(\varepsilon a^\varepsilon)\|_{\tilde L^\infty\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\|\nabla^2 u^\varepsilon\|_{L^1\dot B^{\frac{n}{p}-1}_{p,1}}\\
&\lesssim \left(1+\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B^{n/p}_{p,1}}\right)^{n/2+1}\|\varepsilon a^\varepsilon\|_{\tilde L^\infty\dot B^{\frac{n+\delta}{p}-\frac\delta2}_{p,1}}\|u^\varepsilon\|_{L^1\dot B^{\frac{n}{p}+1}_{p,1}}.
\end{aligned}
$$

Besides, we have

$$\begin{aligned}\|\varepsilon a^\varepsilon\|_{L^\infty \dot B_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}}} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\varepsilon}_{L^\infty \dot B_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}}} + \|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{L^\infty \dot B_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}}}\\ &\lesssim \varepsilon^{\delta(1/2-1/p)}\|a^\varepsilon\|^{\ell,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac{n}{2}-1}} + \varepsilon^{\delta(1/2-1/p)}\|\varepsilon a^\varepsilon\|^{h,\varepsilon}_{\tilde L^\infty \dot B_{p,1}^{\frac{n}{p}}}\\ &\lesssim \varepsilon^{\delta(1/2-1/p)} C_0^\varepsilon.\end{aligned}$$

Then we get

$$\|J^\varepsilon\|_{L^1 \dot B_{p,1}^{\frac{n+\delta}{p}-\frac{\delta}{2}-1}} \lesssim \eta Y_{p,n} + \varepsilon^{\delta(\frac{1}{2}-\frac{1}{p})}\eta^2. \tag{4.16}$$

Plugging estimate (4.16) into (4.14) and remembering that η is sufficiently small, then we obtain (4.12). □

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Appendix

A Some estimates in Besov spaces

In what follows, we denote the characteristic function defined in $\mathbb{Z}$ by $\chi\{\cdot\}$ and by $\{c(j)\}_{j\in\mathbb{Z}}$ a sequence on ℓ^1 with the norm $\|\{c(j)\}\|_{\ell^1} = 1$.

Lemma A.1. *Let $s, t, \sigma, \tau \in \mathbb{R}$, $2 \le p \le 4$ and $1 \le r, r_1, r_2, r_3, r_4 \le \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}$. Then we have the following:*

1. For $2^j \le R_0$, if $s \le n/p$ and $\sigma \le 2n/p - n/2$, then

$$\begin{aligned}&\|\dot\Delta_j(T_f g)\|_{L^r_T L^2}\\ &\quad\le Cc(j)2^{j(n/p-s-t)}\|f\|_{\tilde L^{r_1}_T \dot B^s_{p,1}}\|g\|^{\ell}_{\tilde L^{r_2}_T \dot B^t_{2,1}}\\ &\qquad + C\chi_{\{2^j\sim R_0\}}2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde L^{r_3}_T \dot B^\sigma_{p,1}}\|g\|^h_{\tilde L^{r_4}_T \dot B^\tau_{p,1}};\end{aligned} \tag{A.1}$$

or if $\sigma \le 2n/p - n/2$, then

$$\begin{aligned}&\|\dot\Delta_j(T_f g)\|_{L^r_T L^2}\\ &\quad\le Cc(j)2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde L^{r_1}_T \dot B^\sigma_{p,1}}\|g\|_{\tilde L^{r_2}_T \dot B^\tau_{p,1}}.\end{aligned} \tag{A.2}$$

2. For $2^j > R_0$, if $\sigma \le n/p$, then

$$\begin{aligned}&\|\dot\Delta_j(T_f g)\|_{L^r_T L^p}\\ &\quad\le Cc(j)2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde L^{r_1}_T \dot B^\sigma_{p,1}}\|g\|_{\tilde L^{r_2}_T \dot B^\tau_{p,1}}.\end{aligned} \tag{A.3}$$

Proof. First, we decompose $T_f g$ into $T_f g^\ell + T_f g^h$. Thanks to (1.6), we have

$$\begin{aligned}\dot{\Delta}_j(T_f g^\ell) &= \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1} f \dot{\Delta}_k g^\ell)\\ &= \sum_{|k-j|\le 4}\sum_{k'\le k-2} \dot{\Delta}_j(\dot{\Delta}_{k'} f \dot{\Delta}_k g^\ell).\end{aligned}$$

Denote $J := \{(k,k') : |k-j| \le 4, k' \le k-2\}$, and then for $2^j \le R_0$,

$$\begin{aligned}\|\dot{\Delta}_j(T_f g^\ell)\|_{L^r_T L^2} &\le \sum_J \|\dot{\Delta}_j(\dot{\Delta}_{k'} f \dot{\Delta}_k g^\ell)\|_{L^r_t L^2}\\ &\lesssim \sum_J 2^{k's}\|\dot{\Delta}_{k'} f\|_{L^{r_1}_t L^p} 2^{k'(n/p-s)} 2^{kt}\|\dot{\Delta}_k g^\ell\|_{L^{r_2}_t L^2} 2^{-kt}\\ &\lesssim c(j) 2^{j(n/p-s-t)}\|f\|_{\tilde{L}^{r_1}\dot{B}^s_{p,1}}\|g\|^\ell_{\tilde{L}^{r_2}\dot{B}^t_{2,1}}.\end{aligned}$$

And

$$\begin{aligned}\|\dot{\Delta}_j(T_f g^h)\|_{L^r_T L^2} &\le \sum_J \|\dot{\Delta}_j(\dot{\Delta}_{k'} f \dot{\Delta}_k g^h)\|_{L^r_t L^2}\\ &\lesssim \sum_{2^k\sim 2^j\sim R_0} 2^{k's}\|\dot{\Delta}_{k'} f\|_{L^{r_3}_t L^p} 2^{k'(2n/p-n/2-\sigma)} 2^{k\tau}\|\dot{\Delta}_k g^h\|_{L^{r_4}_t L^p} 2^{-k\tau}\\ &\lesssim c(j)\chi_{\{2^j\sim R_0\}} 2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde{L}^{r_3}\dot{B}^s_{p,1}}\|g\|^h_{\tilde{L}^{r_4}\dot{B}^\tau_{p,1}}.\end{aligned}$$

The above two estimates for $\dot{\Delta}_j(T_f g^\ell)$ and $\dot{\Delta}_j(T_f g^h)$ indicate (A.1). While the proof of (A.2) is essentially the same with the estimate of $\dot{\Delta}_j(T_f g^h)$, we omit the detail. Now we come to prove (A.3). For $2^j > R_0$ and $\sigma \le n/p$

$$\begin{aligned}\|\dot{\Delta}_j(T_f g)\|_{L^r_T L^p} &\le \sum_J \|\dot{\Delta}_j(\dot{\Delta}_{k'} f \dot{\Delta}_k g)\|_{L^r_t L^p}\\ &\lesssim \sum_J \|\dot{\Delta}_{k'} f\|_{L^{r_1}_t L^\infty}\|\dot{\Delta}_k g\|_{L^{r_2}_t L^p}\\ &\lesssim \sum_J 2^{k'\sigma}\|\dot{\Delta}_{k'} f\|_{L^{r_1}_t L^p} 2^{k'(n/p-\sigma)} 2^{k\tau}\|\dot{\Delta}_k g\|_{L^{r_2}_t L^p} 2^{-k\tau}\\ &\lesssim c(j) 2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}\dot{B}^\sigma_{p,1}}\|g\|_{\tilde{L}^{r_2}\dot{B}^\tau_{p,1}}.\end{aligned}$$

□

Lemma A.2. *Let $\sigma,\tau \in \mathbb{R}$, $2 \le p \le 4$ and $1 \le r, r_1, r_2 \le \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Assume that $\sigma + \tau > 0$. Then we have*

$$\begin{aligned}&\|\dot{\Delta}_j R(f,g)\|_{L^r_T L^2}\\ &\quad\le C c(j) 2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^\sigma_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^\tau_{p,1}},\end{aligned}\tag{A.4}$$

and

$$\begin{aligned}&\|\dot{\Delta}_j R(f,g)\|_{L^r_T L^p}\\ &\quad\le C c(j) 2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde{L}^{r_1}_T\dot{B}^\sigma_{p,1}}\|g\|_{\tilde{L}^{r_2}_T\dot{B}^\tau_{p,1}}.\end{aligned}\tag{A.5}$$

Proof. Thanks to (1.6), we have

$$\dot{\Delta}_j R(f,g) = \sum_{k\ge j-3}\sum_{|k-k'|\le 1} \dot{\Delta}_j(\dot{\Delta}_k f \dot{\Delta}_{k'} g).$$

Denote $J := \{(k,k') : k \geq j-3, |k-k'| \leq 1\}$. Then when $\sigma+\tau>0$ and $2\leq p\leq 4$, we have

$$\begin{aligned}\|\dot\Delta_j R(f,g)\|_{L^r_tL^2} &\lesssim 2^{j(2n/p-n/2)}\sum_{(k,k')\in J}\|\dot\Delta_k f\dot\Delta_{k'}g\|_{L^r_tL^{p/2}}\\ &\lesssim 2^{j(2n/p-n/2)}\sum_{(k,k')\in J}\|\dot\Delta_k f\|_{L^{r_1}_tL^p}\|\dot\Delta_{k'}g\|_{L^{r_2}_tL^p}\\ &\lesssim 2^{j(2n/p-n/2)}\sum_{(k,k')\in J}2^{k\sigma}\|\dot\Delta_k f\|_{L^{r_1}_tL^p}2^{-k\sigma}2^{k\tau}\|\dot\Delta_{k'}g\|_{L^{r_2}_tL^p}2^{-k'\tau}\\ &\lesssim c(j)2^{j(2n/p-n/2-\sigma-\tau)}\|f\|_{\tilde L^{r_1}_t\dot B^{\sigma}_{p,1}}\|g\|_{\tilde L^{r_2}_t\dot B^{\tau}_{p,1}},\end{aligned}$$

and

$$\begin{aligned}\|\dot\Delta_j R(f,g)\|_{L^r_tL^p} &\lesssim 2^{jn/p}\sum_{(k,k')\in J}\|\dot\Delta_k f\dot\Delta_{k'}g\|_{L^r_tL^{p/2}}\\ &\lesssim 2^{jn/p}\sum_{(k,k')\in J}\|\dot\Delta_k f\|_{L^{r_1}_tL^p}\|\dot\Delta_{k'}g\|_{L^{r_2}_tL^p}\\ &\lesssim 2^{jn/p}\sum_{(k,k')\in J}2^{k\sigma}\|\dot\Delta_k f\|_{L^{r_1}_tL^p}2^{-k\sigma}2^{k\tau}\|\dot\Delta_{k'}g\|_{L^{r_2}_tL^p}2^{-k'\tau}\\ &\lesssim c(j)2^{j(n/p-\sigma-\tau)}\|f\|_{\tilde L^{r_1}_t\dot B^{\sigma}_{p,1}}\|g\|_{\tilde L^{r_2}_t\dot B^{\tau}_{p,1}}.\end{aligned}$$

This finishes the proof of Lemma A.2. □

Proposition A.1. *Let* $2\leq p\leq \min\{4,\frac{2n}{n-2}\}$ *and* $p\neq 4$ *if* $n=2$. $1\leq r,r_1,r_2,r_3,r_4\leq\infty$ *with* $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}$. *Then we have*

For $2^j\leq R_0,\ \gamma\in\mathbb{R}$

$$\begin{aligned}\sum_{2^j\leq R_0}2^{j(n/2-1)}\|\dot\Delta_j(fg)\|_{L^r_tL^2} &\lesssim \|f\|_{\tilde L^{r_1}_t\dot B^{n/p}_{p,1}}\|g\|_{\tilde L^{r_2}_t\dot B^{n/p-1}_{p,1}}\\ &\quad+\|f\|_{\tilde L^{r_1}_t\dot B^{n/p}_{p,1}}\|g\|^{\ell}_{\tilde L^{r_2}_t\dot B^{n/2-1}_{2,1}}+R_0^{-\gamma}\|f\|_{\tilde L^{r_3}_t\dot B^{n/p-1}_{p,1}}\|g\|^{h}_{\tilde L^{r_4}_t\dot B^{n/p+\gamma}_{p,1}}; \qquad (A.6)\end{aligned}$$

or $\gamma\leq 0$,

$$\begin{aligned}\sum_{2^j\leq R_0}2^{j(n/2-1)}\|\dot\Delta_j(fg)\|_{L^r_tL^2} &\lesssim \|f\|_{\tilde L^{r_1}_t\dot B^{n/p}_{p,1}}\|g\|_{\tilde L^{r_2}_t\dot B^{n/p-1}_{p,1}}\\ &\quad+R_0^{-\gamma}\|f\|_{\tilde L^{r_3}_t\dot B^{n/p-1}_{p,1}}\|g\|_{\tilde L^{r_4}_t\dot B^{n/p+\gamma}_{p,1}}. \qquad (A.7)\end{aligned}$$

For $2^j>R_0$, $0\leq\gamma\leq 1$

$$\sum_{2^j>R_0}2^{j(n/p-1)}\|\dot\Delta_j(fg)\|_{L^r_tL_p}\lesssim R_0^{-\gamma}\|f\|_{\tilde L^{r_1}_t\dot B^{n/p}_{p,1}}\|g\|_{\tilde L^{r_2}_t\dot B^{n/p-1+\gamma}_{p,1}}; \qquad (A.8)$$

Proof. Using Bony decomposition $fg=T_fg+T_gf+R(f,g)$. For the low frequency, we choose $s=n/p, t=n/2-1, \sigma=n/p-1, \tau=n/p+\gamma(\gamma\in\mathbb{R})$ in (A.1) for T_fg; $\sigma=n/p-1,\tau=n/p$ in (A.2) for T_gf and $\sigma=n/p,\tau=n/p-1$ in (A.4) for $R(f,g)$. Then summing over j for $2^j\leq R_0$ indicates (A.6).

Also we can choose $\sigma=n/p-1,\tau=n/p+\gamma(\gamma\leq 0)$ in (A.2) for T_fg; $\sigma=n/p-1,\tau=n/p$ in (A.2) for T_gf and $\sigma=n/p,\tau=n/p-1$ in (A.4) for $R(f,g)$. Summing over j indicates (A.7).

For the high frequency. Applying $\sigma=n/p,\tau=n/p-1+\gamma$ in (A.3) for T_fg and in (A.5) $R(f,g)$; applying $\sigma=n/p-1-\gamma,\tau=n/p$ in (A.3) for T_gf and summing over j for $2^j>R_0$ indicates (A.8).

All these finish the proof of the proposition. □

Proposition A.2 ([37]). *Let* $1 \leq p, r, r_1, r_2 \leq \infty$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, *and* $s_1, s_2 \in \mathbb{R}$ *satisfying* $s_1, s_2 \leq \frac{n}{p}, s_1 + s_2 > n\max(0, \frac{2}{p} - 1)$. *If* $f \in \tilde{L}^{r_1}_T(\dot{B}^{s_1}_{p,1})$ *and* $g \in \tilde{L}^{r_2}_T(\dot{B}^{s_2}_{p,1})$, *then we have*

$$\|fg\|_{\tilde{L}^r_T(\dot{B}^{s_1+s_2-n/p}_{p,1})} \lesssim \|f\|_{\tilde{L}^{r_1}_T \dot{B}^{s_1}_{p,1}} \|g\|_{\tilde{L}^{r_2}_T \dot{B}^{s_2}_{p,1}}. \tag{A.9}$$

Proposition A.3 ([35]). *Assume that* $F \in W^{[s]+2,\infty}_{loc}$ *with* $F(0) = 0$. *Then for any* $s > 0, p, r \in [1, \infty]$, *there holds*

$$\begin{aligned} \|F(f)\|_{\tilde{L}^r_T \dot{B}^s_{p,1}} &\leq C(1 + \|f\|_{L^\infty_T L^\infty})^{[s]+1} \|f\|_{\tilde{L}^r_T \dot{B}^s_{p,1}} \\ &\leq C(1 + \|f\|_{\tilde{L}^\infty_T \dot{B}^{n/p}_{p,1}})^{[s]+1} \|f\|_{\tilde{L}^r_T \dot{B}^s_{p,1}}. \end{aligned} \tag{4.10}$$

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