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On the Vanishing of Some D-Solutions to the Stationary Magnetohydrodynamics System

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Abstract. In this paper, we study the stationary magnetohydrodynamics system in $\mathbb{R}^2 \times \mathbb{T}$. We prove trivialness of D-solutions (the velocity field u and the magnetic field h) when they are swirl-free. Meanwhile, this Liouville type theorem also holds provided u is swirl-free and h is axially symmetric, or both u and h are axially symmetric. Our method is also valid for certain related boundary value problems in the slab $\mathbb{R}^2 \times [-\pi, \pi]$.

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1. Introduction

In this paper, we consider the stationary magnetohydrodynamics system

$$\begin{cases} u \cdot \nabla u + \nabla p - h \cdot \nabla h - \Delta u = 0, \\ u \cdot \nabla h - h \cdot \nabla u - \Delta h = 0, \\ \nabla \cdot u = \nabla \cdot h = 0, \end{cases} \tag{1.1}$$

in $\mathbb{R}^2 \times \mathbb{T}$ or in the slab $\mathbb{R}^2 \times [-\pi, \pi]$, where $u(x), h(x) \in \mathbb{R}^3$, $p(x) \in \mathbb{R}$ represent the velocity vector, the magnetic field and the scalar pressure respectively. The MHD equations, which describe the state of the fluid flows of plasma, are fundamental partial differential equations in nature. For the background of the MHD system, we refer readers to [10] for more details. We note that if $h \equiv 0$, the MHD system is reduced to the Navier–Stokes system.

In the following, we will carry out our proof in the cylindrical coordinates (r, θ, z). That is, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\begin{cases} r = \sqrt{x_1^2 + x_2^2}, \\ \theta = \arctan \frac{x_2}{x_1}, \\ z = x_3. \end{cases} \tag{1.2}$$

And the solution of the incompressible stationary magnetohydrodynamics system is given as

$$u = u^r(r, \theta, z)e_r + u^\theta(r, \theta, z)e_\theta + u^z(r, \theta, z)e_z,$$
$$h = h^r(r, \theta, z)e_r + h^\theta(r, \theta, z)e_\theta + h^z(r, \theta, z)e_z,$$

where the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

The components $u^r, u^\theta, u^z, h^r, h^\theta, h^z$ satisfy

$$\begin{cases} \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)u^r - \frac{(u^\theta)^2}{r} + \frac{2}{r^2}\partial_\theta u^\theta + \partial_r p = \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)h^r - \frac{(h^\theta)^2}{r} + \left(\Delta - \frac{1}{r^2}\right)u^r, \\ \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)u^\theta + \frac{u^\theta u^r}{r} - \frac{2}{r^2}\partial_\theta u^r + \frac{1}{r}\partial_\theta p = \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)h^\theta + \frac{h^r h^\theta}{r} + \left(\Delta - \frac{1}{r^2}\right)u^\theta, \\ \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)u^z + \partial_z p = \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)h^z + \Delta u^z, \\ \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)h^r - \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)u^r + \frac{2}{r^2}\partial_\theta h^\theta = \left(\Delta - \frac{1}{r^2}\right)h^r, \\ \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)h^\theta - \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)u^\theta + \frac{u^\theta h^r}{r} - \frac{h^\theta u^r}{r} - \frac{2}{r^2}\partial_\theta h^r = \left(\Delta - \frac{1}{r^2}\right)h^\theta, \\ \left(u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z\right)h^z - \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)u^z = \Delta h^z, \\ \nabla\cdot u = \partial_r u^r + \frac{u^r}{r} + \frac{1}{r}\partial_\theta u^\theta + \partial_z u^z = 0, \quad \nabla\cdot h = \partial_r h^r + \frac{h^r}{r} + \frac{1}{r}\partial_\theta h^\theta + \partial_z h^z = 0. \end{cases} \tag{1.3}$$

Here

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{1.4}$$

is the usual Laplacian operator.

The main aim of our paper is to study the Liouville type theorem of D-solutions of the stationary MHD system (1.3). The study is partly motivated by the related Liouville problem of the stationary Navier–Stokes equations, which has attracted much attention in recent years and is still far from being fully understood. See for example [1–5,7–9,15,16] and the reference therein. First, in full 3D case, the Liouville-type theorem holds provided the vanishing of u^θ and h^θ. That is:

Theorem 1.1. *Let (u,h) be a smooth solution to the problem*

$$\begin{cases} u\cdot\nabla u + \nabla p - h\cdot\nabla h - \Delta u = 0, \quad \textit{in} \quad \mathbb{R}^2\times\mathbb{T}, \\ u\cdot\nabla h - h\cdot\nabla u - \Delta h = 0, \quad \textit{in} \quad \mathbb{R}^2\times\mathbb{T}, \\ \nabla\cdot u = \nabla\cdot h = 0, \quad \textit{in} \quad \mathbb{R}^2\times\mathbb{T}, \\ u(x',z) = u(x',z+2\pi); \quad h(x',z) = h(x',z+2\pi), \\ \lim_{|x|\to\infty}|u(x)| = 0; \quad \lim_{|x|\to\infty}|h(x)| = 0, \end{cases} \tag{1.5}$$

with finite Dirichlet integral

$$\int_{\mathbb{T}}\int_{\mathbb{R}^2}|\nabla u(x)|^2dx + \int_{\mathbb{T}}\int_{\mathbb{R}^2}|\nabla h(x)|^2dx < \infty. \tag{1.6}$$

Then $(u,h)\equiv 0$ provided $u^\theta = h^\theta \equiv 0$. □

Remark 1.1. We emphasize here that our assumption of the smoothness of the solution $(u,\,h)$ is reasonable since one can derive the smoothness of any weak solution to (1.5) satisfying the D-condition (1.6) by following the method developed in [5]. □

In the cylinder coordinate, we say a 3 dimensional vector field

$$v(x) = v^r(r,\theta,z)e_r + v^\theta(r,\theta,z)e_\theta + v^z(r,\theta,z)e_z \tag{1.7}$$

is axially symmetric if and only if

$$\partial_\theta v^r = \partial_\theta v^\theta = \partial_\theta v^z \equiv 0. \tag{1.8}$$

Moreover, for axially symmetric magnetic field or axially symmetric velocity and magnetic fields, we derive two further results:

Corollary 1.1. *Let (u,h) be a smooth solution to the problem (1.5) with finite Dirichlet integral (1.6). Then $(u,h)\equiv 0$ provided one of the following two conditions is satisfied:*

(i) *$u^\theta \equiv 0$ and h is axially symmetric;*
(ii) *Both u and h are axially symmetric.*

□

Remark 1.2. Consider the special case that $h\equiv 0$, part (ii) of the above corollary is reduced to Theorem 1.1 in [2]. For part (i) (also Theorem 1.1) with $h\equiv 0$, this Liouville-type theorem do not need to add the axially symmetric condition of u. More precisely, this is a result of the swirl-free full 3-D case. □

Instead of u and h are z-periodic, our method is valid for D-solutions of certain boundary value problems of magnetohydrodynamics system (1.1) in the slab $\mathbb{R}^2 \times [-\pi,\, \pi]$. Here is the corollary:

Corollary 1.2. *Let (u, h) be a smooth solution to the magnetohydrodynamics system*

$$\begin{cases} u\cdot\nabla u + \nabla p - h\cdot\nabla h - \Delta u = 0, \quad \text{in} \quad \mathbb{R}^2\times[-\pi,\pi], \\ u\cdot\nabla h - h\cdot\nabla u - \Delta h = 0, \quad \text{in} \quad \mathbb{R}^2\times[-\pi,\pi], \\ \nabla\cdot u = \nabla\cdot h = 0, \quad \text{in} \quad \mathbb{R}^2\times[-\pi,\pi], \\ \lim_{|x|\to\infty}|u(x)| = 0; \quad \lim_{|x|\to\infty}|h(x)| = 0, \end{cases} \tag{1.9}$$

with finite Dirichlet integral

$$\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}|\nabla u(x)|^2dx + \int_{-\pi}^{\pi}\int_{\mathbb{R}^2}|\nabla h(x)|^2dx < \infty \tag{1.10}$$

in the slab $\mathbb{R}^2\times[-\pi,\, \pi]$ equipped with the boundary conditions

$$(u^z,\, \partial_z u^r,\, \partial_z u^\theta)\Big|_{z\in\{-\pi,\,\pi\}} = 0, \quad h\Big|_{z\in\{-\pi,\,\pi\}} = 0, \tag{1.11}$$

or

$$(u^z,\, \partial_z u^r,\, \partial_z u^\theta)\Big|_{z\in\{-\pi,\,\pi\}} = 0, \quad (h^z,\, \partial_z h^r,\, \partial_z h^\theta)\Big|_{z\in\{-\pi,\,\pi\}} = 0. \tag{1.12}$$

Then $(u, h) \equiv 0$ provided one of the following three conditions is satisfied:

(i) $u^\theta = h^\theta \equiv 0$;
(ii) $u^\theta \equiv 0$ *and h is axially symmetric;*
(iii) *Both u and h are axially symmetric.*

□

We refer readers to the "Appendix" of our paper for some explanation to the reasonableness of the boundary conditions in the Corollary. The proof of Corollary 1.2 is omitted in this paper.

Remark 1.3. In our paper, we do not pursue any vanishing results in a slab with u and h both satisfying homogeneous Dirichlet boundary condition. Reminded by the anonymous referee and by using the methods in [12–14], etc. to solve the stationary Navier–Stokes problems, we think it is possible to derive the Liouville-type theorems of the stationary magnetohydrodynamics systems with Dirichlet boundary condition in a layer-like domain, under certain asymptotic conditions of u and h. Also, the assumption of axial symmetry can be dropped. □

Our proof of the theorem and corollaries are based on the oscillation estimate of the pressure in [2]. Because of the partly "θ-dependent" of the pressure p and some magnetic related terms, we need a careful treatment for getting the boundedness of u and h up to their second order derivatives and oscillation estimate of p in a dyadic annulus. At last, we prove the Liouville type theorems by providing the vanishing of the L^2 norms of ∇u and ∇h.

This paper is organized as follows. In Sect. 2, we give the proof of Theorem 1.1. Section 3 is devoted to proving the part (i) of Corollary 1.1, while Sect. 4 is for the part (ii). Some details of the boundary conditions in Corollary 1.2 could be found in the "Appendix".

Throughout the paper, we use C to denote a generic constant which may be different from line to line. We also apply $A \lesssim B$ to denote $A \le CB$. We denote by $B(x_0, r) := \{x \in \mathbb{R}^d : |x - x_0| < r\}$. We simply denote by $B_r := B(0, r)$ and $B := B_1$. For a domain Ω and $1 \le p \le \infty$, $L^p(\Omega)$ denotes the usual Lebesgue space with norm $\|\cdot\|_{L^p(\Omega)}$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x = (x', x_3)$ or (x', z) for simplicity. The symbol ∂_i stands for $\frac{\partial}{\partial x_i}$, for $i = 1, 2, 3$, while ∂_r, ∂_θ and ∂_z stands for $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$ respectively.

2. Proof of Theorem 1.1

First we see, under the condition $u^\theta = h^\theta \equiv 0$, (1.3) turns to

$$\begin{cases} (u^r\partial_r + u^z\partial_z)u^r + \partial_r p = (h^r\partial_r + h^z\partial_z)h^r + \left(\Delta - \frac{1}{r^2}\right)u^r, \\ -\frac{2}{r^2}\partial_\theta u^r + \frac{1}{r}\partial_\theta p = 0, \\ (u^r\partial_r + u^z\partial_z)u^z + \partial_z p = (h^r\partial_r + h^z\partial_z)h^z + \Delta u^z, \\ (u^r\partial_r + u^z\partial_z)h^r - (h^r\partial_r + h^z\partial_z)u^r = \left(\Delta - \frac{1}{r^2}\right)h^r, \\ (u^r\partial_r + u^z\partial_z)h^z - (h^r\partial_r + h^z\partial_z)u^z = \Delta h^z, \\ \nabla\cdot u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \nabla\cdot h = \partial_r h^r + \frac{h^r}{r} + \partial_z h^z = 0. \end{cases} \tag{2.1}$$

This section is divided into three parts. The first one is to derive the boundedness of u and h up to their second order derivatives. We have applied a result for local solutions in [18]. Second, by integrating the equation of $\partial_r p$, we actually prove the boundedness of the oscillation of p in a bounded dyadic annulus. Finally, by testing the MHD system with standard test functions, we prove the trivialness of u and h.

2.1. Boundedness of the Solution Up to Second Order Derivatives

Lemma 2.1. *Under the assumptions of Theorem 1.1, we have*

$$|\nabla^k u| + |\nabla^k h| \le C_k < \infty, \quad 0 \le k \le 2. \tag{2.2}$$

Here $\nabla^k f$ denotes all the derivatives of f with order k.

Proof. Since u and h are assumed to be smooth functions and converge to 0 as $r \to \infty$, we have that both u and h are bounded. Now we derive the boundedness of their derivatives. By denoting

$$w_1 := u + h; \quad w_2 := u - h, \tag{2.3}$$

(1.5) leads to

$$\begin{cases} w_2\cdot\nabla w_1 - \Delta w_1 + \nabla p = 0, \\ w_1\cdot\nabla w_2 - \Delta w_2 + \nabla p = 0, \\ \nabla\cdot w_1 = \nabla\cdot w_2 = 0. \end{cases} \tag{2.4}$$

A direct application of Theorem 1.7 in [18] shows that, $\exists\, r_0 \le 1$, the gradient of w_1 and w_2 satisfy

$$|\nabla w_i(x)| \le \frac{C}{r_0^3}\int_{B(x,r_0)} |\nabla w_i(y)|dy + \frac{C}{r_0^4}\int_{B(x,r_0)} |w_i - (w_i)_{B(x,r_0)}|dy, \quad i = 1, 2. \tag{2.5}$$

Here, for $i = 1, 2$, $(w_i)_{B(x,r_0)} = \frac{1}{|B(x,r_0)|}\int_{B(x,r_0)} w_i(y)dy$. Applying Cauchy–Schwartz inequality and Poincaré inequality, one find

$$\begin{aligned} |\nabla w_i(x)| &\lesssim \frac{1}{r_0^3}\left(\int_{B(x,r_0)} |\nabla w_i(y)|^2 dy\right)^{1/2}\left(\int_{B(x,r_0)} dy\right)^{1/2} \\ &\quad + \frac{1}{r_0^4}\left(\int_{B(x,r_0)} |w_i - (w_i)_{B(x,r_0)}|^2 dy\right)^{1/2}\left(\int_{B(x,r_0)} dy\right)^{1/2} \\ &\lesssim r_0^{-3/2}\|\nabla w_i\|_{L^2(B(x,1))} \\ &\lesssim r_0^{-3/2}(\|\nabla u\|_{L^2(B(x,1))} + \|\nabla h\|_{L^2(B(x,1))}) \lesssim 1, \quad \text{for} \quad i = 1, 2. \end{aligned} \tag{2.6}$$

Taking the *curl* of the first two equations of (2.4), we then eliminate the terms of pressure. With the boundedness of w_1, w_2, ∇w_1 and ∇w_2, routine elliptic estimates prove the boundedness of $\nabla^2 w_1$ and $\nabla^2 w_2$. This leads to the boundedness of u and h up to their second order derivatives. □

2.2. Boundedness of the Oscillation of p in Dyadic Annulus

Lemma 2.2. *Under the assumptions of Theorem 1.1, for fixed $R > 0$, it follows that*

$$\sup_{r\in[R,\,2R],\,\theta\in[0,\,2\pi],\,z\in[-\pi,\,\pi]} |p(r,\theta,z) - p(R,0,0)| \lesssim 1. \tag{2.7}$$

Proof. First we note that the second equation of (2.1) turns to

$$\partial_\theta p = \frac{2}{r}\partial_\theta u^r. \tag{2.8}$$

Owing to the boundedness of ∇u, we have

$$\left|\frac{1}{r}\partial_\theta u^r\right| = |-\sin\theta\cdot\partial_{x_1}u^r + \cos\theta\cdot\partial_{x_2}u^r| \le |\nabla u|. \tag{2.9}$$

This leads to

$$|\partial_\theta p| \lesssim 1. \tag{2.10}$$

Meanwhile, due to the third equation of (2.1) and Lemma 2.1, it follows

$$|\partial_z p| \lesssim 1. \tag{2.11}$$

In the following part we will show that for any fixed $R > 1$, the estimate

$$\left|\int_{-\pi}^{\pi}\int_0^{2\pi} p(r,\theta,z) - p(R,\theta,z)d\theta dz\right| \lesssim 1 \tag{2.12}$$

holds for all $r \in [R,\,2R]$. Now we integrate the first equation of (3.1) to get

$$\begin{aligned}
\partial_r\int_{-\pi}^{\pi}\int_0^{2\pi} pd\theta dz &= \int_{-\pi}^{\pi}\int_0^{2\pi}\Big(-(u^r\partial_r + u^z\partial_z)u^r\\
&\quad+\Big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2 - \frac{1}{r^2}\Big)u^r + (h^r\partial_r + h^z\partial_z)h^r\Big)d\theta dz\\
&= -\frac{1}{2}\int_{-\pi}^{\pi}\int_0^{2\pi}\partial_r(u^r)^2 d\theta dz - \int_{-\pi}^{\pi}\int_0^{2\pi} u^z\partial_z u^r d\theta dz\\
&\quad+\partial_r^2\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz + \frac{1}{r}\partial_r\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz\\
&\quad-\frac{1}{r^2}\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz + \frac{1}{2}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\partial_r(h^r)^2 d\theta dz\\
&\quad+\int_{-\pi}^{\pi}\int_0^{2\pi} h^z\partial_z h^r d\theta dz.
\end{aligned} \tag{2.13}$$

$\forall\ r_0 \in [R,\, 2R]$, we integrate on r from R to r_0. It follows that

$$
\begin{aligned}
&\left|\int_{-\pi}^{\pi}\int_0^{2\pi} p(r_0,\theta,z)-p(R,\theta,z)d\theta dz\right|\\
&\lesssim \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi}\partial_r(u^r)^2 d\theta dz dr\right| + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi} u^z\partial_z u^r d\theta dz dr\right|\\
&\quad + \left|\int_R^{r_0}\partial_r^2\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz dr\right| + \left|\int_R^{r_0}\frac{1}{r}\partial_r\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz dr\right|\\
&\quad + \left|\int_R^{r_0}\frac{1}{r^2}\int_{-\pi}^{\pi}\int_0^{2\pi} u^r d\theta dz dr\right| + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\partial_r(h^r)^2 d\theta dz dr\right|\\
&\quad + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi} h^z\partial_z h^r d\theta dz dr\right|\\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
\tag{2.14}
$$

In the following, we show that I_1 to I_5 in (2.14) are all bounded. First, due to the boundedness of u, we see

$$
I_1 = \left|\int_{-\pi}^{\pi}\int_0^{2\pi}\big((u^r)^2(r_0,\theta,z)-(u^r)^2(R,\theta,z)\big)\,d\theta dz\right| \lesssim 1. \tag{2.15}
$$

Now we consider term I_2. Using integrating by parts and divergence free condition, we have

$$
\begin{aligned}
I_2 &= \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi} u^r\partial_z u^z d\theta dz dr\right|\\
&= \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi} u^r\left(\partial_r u^r+\frac{1}{r}u^r\right)d\theta dz dr\right|\\
&\lesssim I_1 + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi}\frac{(u^r)^2}{r}d\theta dz dr\right|\\
&\lesssim 1 + \left|\int_R^{2R}\frac{1}{r}dr\right| \lesssim 1.
\end{aligned}
\tag{2.16}
$$

Here the second inequality holds because the boundedness of u. For I_3, it follows

$$
I_3 = \left|\int_{-\pi}^{\pi}\int_0^{2\pi}\left(\partial_r u^r(r_0,\theta,z)-\partial_r u^r(R,\theta,z)\right)d\theta dz\right| \lesssim 1. \tag{2.17}
$$

Meanwhile, I_4 satisfies the following estimate by using integration by parts

$$
\begin{aligned}
I_4 &\lesssim \left|\int_{-\pi}^{\pi}\int_0^{2\pi}\frac{u^r}{r}\Big|_R^{r_0}d\theta dz\right| + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi}\frac{u^r}{r^2}d\theta dz dr\right|\\
&\lesssim 1 + \left|\int_R^{\infty}\frac{1}{r^2}dr\right| \lesssim 1.
\end{aligned}
\tag{2.18}
$$

Here the last two inequalities hold since u^r is bounded. And the related estimate for I_5 holds similarly as the second item above after the first "$\lesssim$". Meanwhile, estimates of I_6 and I_7 hold similarly with that of I_1 and I_2 respectively. Combining those estimates above in this section, (2.12) holds for any $r \in [R,\, 2R]$, i.e.

$$
\left|\int_{-\pi}^{\pi}\int_0^{2\pi} p(r,\theta,z)-p(R,\theta,z)d\theta dz\right| \lesssim 1, \quad \forall r\in[R,\,2R]. \tag{2.19}
$$

Applying the mean value theorem, for a fixed $R>1$ and $r\in[R,\,2R]$, there exist $\theta(r)\in[0,\,2\pi]$ and $z(r)\in[-\pi,\,\pi]$, such that

$$
\big|p(r,\theta(r),z(r))-p(R,\theta(r),z(r))\big| = \left|\int_{-\pi}^{\pi}\int_0^{2\pi} p(r,\theta,z)-p(R,\theta,z)d\theta dz\right| \lesssim 1. \tag{2.20}
$$

Combining this with the uniformly boundedness of $\partial_z p$ and $\partial_\theta p$, it follows that, $\forall\theta \in [0,\, 2\pi]$, $z \in [-\pi,\, \pi]$

$$
\begin{aligned}
|p(r,\theta,z) - p(R,0,0)| &\le |p(r,\theta,z) - p(r,\theta,z(r))| + |p(r,\theta,z(r)) - p(r,\theta(r),z(r))| \\
&\quad + |p(r,\theta(r),z(r)) - p(R,\theta(r),z(r)| \\
&\quad + |p(R,\theta(r),z(r)) - p(R,\theta(r),0)| \\
&\quad + |p(R,\theta(r),0) - p(R,0,0)| \\
&\le |\partial_z p| \cdot |z - z(r)| + |\partial_\theta p| \cdot |\theta - \theta(r)| + C \\
&\quad + |\partial_z p| \cdot |z(r)| + |\partial_\theta p| \cdot |\theta(r)| \\
&\lesssim 1.
\end{aligned} \tag{2.21}
$$

Hence we have

$$
\sup_{r\in[R,\,2R],\,\theta\in[0,\,2\pi],\,z\in[-\pi,\,\pi]} |p(r,\theta,z) - p(R,0,0)| \lesssim 1. \tag{2.22}
$$

□

2.3. Trivialness of u and h

At the beginning, we claim that u^r, $h^r \in L^2(\mathbb{R}^2 \times \mathbb{T})$. We have the following lemma

Lemma 2.3. *Under the assumption of Theorem 1.1, we have*

$$
\|u^r\|_{L^2(\mathbb{R}^2\times\mathbb{T})} + \|h^r\|_{L^2(\mathbb{R}^2\times\mathbb{T})} < +\infty.
$$

Proof. According to the divergence-free condition and $u^\theta \equiv 0$, we see that

$$
\partial_r(ru^r(r,\theta,z)) + \partial_z(ru^z(r,\theta,z)) = 0. \tag{2.23}
$$

Integrating (2.23) on z from $-\pi$ to π, it follows that

$$
\partial_r\left(r\int_{-\pi}^{\pi} u^r(r,\theta,z)dz\right) = -\int_{-\pi}^{\pi} \partial_z(ru^z(r,\theta,z))dz = -ru^z(r,\theta,z)\Big|_{z=-\pi}^{\pi} = 0. \tag{2.24}
$$

Here the last identity follows from the periodic condition of u in z-direction. This leads to

$$
r\int_{-\pi}^{\pi} u^r(r,\theta,z)dz = C(\theta), \tag{2.25}
$$

where $C(\theta)$ is a function depends only on θ. Moreover, we find $C(\theta) \equiv 0$ by choosing $r = 0$. Therefore

$$
\int_{-\pi}^{\pi} u^r(r,\theta,z)dz = 0. \tag{2.26}
$$

Hence we have, by using the Poincaré inequality and the D-solution condition

$$
\begin{aligned}
\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} |u^r|^2 dx &= \int_{\mathbb{R}^2}\int_{-\pi}^{\pi} \left|u^r(x',z) - \frac{1}{2\pi}\int_{-\pi}^{\pi} u^r(x',z')dz'\right|^2 dzdx' \\
&\lesssim \int_{\mathbb{R}^2}\int_{-\pi}^{\pi} |\partial_z u^r(x',z)|^2 dzdx' \\
&\le \int_{\mathbb{R}^2\times\mathbb{T}} |\nabla u(x)|^2 dx < \infty.
\end{aligned} \tag{2.27}
$$

At the same time, the related estimate holds for h^r, that is

$$
\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} |h^r|^2 dx < \infty, \tag{2.28}
$$

since h is divergence-free and $h^\theta = 0$, which means Eq. (2.23) also holds for h. The rest is similar with that of u and the lemma is proved. □

Now let $\phi = \phi(\rho)$ be a smooth cut-off function satisfying

$$\begin{cases} \phi(\rho) = 1, & \rho \in [0,1], \\ \phi(\rho) = 0, & \rho \geq 2, \\ 0 \leq \phi \leq 1, & \forall \rho \in [0,\infty), \end{cases} \tag{2.29}$$

with ϕ' and ϕ'' being bounded. And we set $\phi_R(y') = \phi\left(\frac{|y'|}{R}\right)$ with $y' \in \mathbb{R}^2$ and $R > 0$. Testing the first equation of (1.5)

$$u \cdot \nabla u + \nabla p - h \cdot \nabla h - \Delta u = 0 \tag{2.30}$$

with $u\phi_R$, we achieve that

$$\int_{\mathbb{R}^2\times\mathbb{T}} u\phi_R \Delta u dx = \int_{\mathbb{R}^2\times\mathbb{T}} u\phi_R \left(u \cdot \nabla u - h \cdot \nabla h + \nabla(p - p(R,0,0))\right) dx. \tag{2.31}$$

Direct integrating by parts implies

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{T}} |\nabla u|^2 \phi_R dx - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{T}} |u|^2 \Delta\phi_R dx \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{T}} |u|^2 u \cdot \nabla\phi_R dx + \int_{\mathbb{R}^2\times\mathbb{T}} (p(r,\theta,z) - p(R,0,0)) u \cdot \nabla\phi_R dx \\ &\qquad - \sum_{i,j=1}^{3}\int_{\mathbb{R}^2\times\mathbb{T}} h_i h_j \partial_{x_i} u_j \phi_R dx - \sum_{i,j=1}^{3}\int_{\mathbb{R}^2\times\mathbb{T}} h_i h_j u_j \partial_{x_i}\phi_R dx. \end{aligned} \tag{2.32}$$

Meanwhile, we test the second equation of (1.5)

$$u \cdot \nabla h - h \cdot \nabla u - \Delta h = 0 \tag{2.33}$$

with $h\phi_R$ to get

$$\int_{\mathbb{R}^2\times\mathbb{T}} h\phi_R \Delta h dx = \int_{\mathbb{R}^2\times\mathbb{T}} h\phi_R \left(u \cdot \nabla h - h \cdot \nabla u\right) dx. \tag{2.34}$$

Integrating by parts, (2.34) is equivalent to

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{T}} |\nabla h|^2 \phi_R dx - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{T}} |h|^2 \Delta\phi_R dx \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{T}} |h|^2 u \cdot \nabla\phi_R dx + \sum_{i,j=1}^{3}\int_{\mathbb{R}^2\times\mathbb{T}} h_i h_j \partial_{x_i} u_j \phi_R dx. \end{aligned} \tag{2.35}$$

Therefore, the following equation is achieved by adding (2.32) and (2.35) together:

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{T}} \left(|\nabla u|^2 + |\nabla h|^2\right) \phi_R dx - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{T}} \left(|u|^2 + |h|^2\right) \Delta\phi_R dx \\ &\quad = \int_{\mathbb{R}^2\times\mathbb{T}} \left(\frac{1}{2}|u|^2 + \frac{1}{2}|h|^2 + (p(r,\theta,z) - p(R,0,0))\right) u \cdot \nabla\phi_R dx \\ &\qquad - \int_{\mathbb{R}^2\times\mathbb{T}} (h \cdot u)(h \cdot \nabla\phi_R) dx. \end{aligned} \tag{2.36}$$

We denote $\bar{B}_{2R/R} := \{x' : R \le |x'| \le 2R\}$ the dyadic annulus. Since ϕ_R depends only on r, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{T}} \left(|\nabla u|^2 + |\nabla h|^2\right)\phi_R dx \\
&\le \int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} \left(|u|^2 + |h|^2\right)\cdot|\Delta\phi_R| dx'dz \\
&\quad + \int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |u^r|\cdot|\partial_r\phi_R|\cdot(|u|^2+|h|^2)dx'dz \\
&\quad + \sup_{r\in[R,\,2R],\,\theta\in[0,\,2\pi],\,z\in[-\pi,\,\pi]} |p(r,\theta,z)-p(R,0,0)| \int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |u^r|\cdot|\partial_r\phi_R|dx'dz \\
&\quad + \int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |h|\cdot|u|\cdot|h^r|\cdot|\partial_r\phi_R|dx'dz \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.37}$$

First, I_1 satisfies

$$\begin{aligned}
I_1 &\lesssim \frac{1}{R^2}\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} (|u|^2+|h|^2)dx'dz \\
&\le \frac{1}{R^2}\left(\|u\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h\|^2_{L^\infty(\bar{B}_{2R/R})}\right)\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} dx'dz \\
&\lesssim \|u\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h\|^2_{L^\infty(\bar{B}_{2R/R})} \to 0, \quad \text{as} \quad R\to\infty.
\end{aligned} \tag{2.38}$$

Here we applied the vanishing of both u and h at the far field, and the same as below for the estimates of I_2 and I_4. Using the Hölder inequality, I_2 follows that

$$\begin{aligned}
I_2 &\lesssim \frac{1}{R}\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |u^r|\cdot\left(|u|^2+|h|^2\right)dx'dz \\
&\le \frac{1}{R}\left(\|u\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h\|^2_{L^\infty(\bar{B}_{2R/R})}\right)\left(\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |u^r|^2dx'dz\right)^{1/2}\cdot|\bar{B}_{2R/R}|^{1/2} \\
&\lesssim \left(\|u\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h\|^2_{L^\infty(\bar{B}_{2R/R})}\right)\cdot\|u^r\|_{L^2(\mathbb{R}^2\times\mathbb{T})} \to 0, \quad \text{as} \quad R\to\infty.
\end{aligned} \tag{2.39}$$

Next, for I_3 we have

$$\begin{aligned}
I_3 &\lesssim \sup_{r\in[R,\,2R],\,\theta\in[0,\,2\pi],\,z\in[-\pi,\,\pi]} |p(r,\theta,z)-p(R,0,0)| \\
&\quad \cdot\frac{1}{R}\left(\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |u^r|^2dx'dz\right)^{1/2}\cdot|\bar{B}_{2R/R}|^{1/2} \\
&\lesssim \|u^r\|_{L^2(\bar{B}_{2R/R})} \to 0, \quad \text{as} \quad R\to\infty.
\end{aligned} \tag{2.40}$$

Here we have applied the Hölder inequality and the boundedness of the oscillation of p in dyadic annulus which is achieved in Lemma 2.2. Finally, the following estimate is satisfied by I_4:

$$\begin{aligned}
I_4 &\lesssim \frac{1}{R}\cdot\|u\|_{L^\infty(\bar{B}_{2R/R})}\cdot\|h\|_{L^\infty(\bar{B}_{2R/R})}\cdot\left(\int_{-\pi}^{\pi}\int_{\bar{B}_{2R/R}} |h^r|^2dx'dz\right)^{1/2}\cdot|\bar{B}_{2R/R}|^{1/2} \\
&\lesssim \|u\|_{L^\infty(\bar{B}_{2R/R})}\cdot\|h\|_{L^\infty(\bar{B}_{2R/R})}\cdot\|h^r\|_{L^2(\bar{B}_{2R/R})} \to 0, \quad \text{as} \quad R\to\infty.
\end{aligned} \tag{2.41}$$

Combining those estimates of I_1, I_2, I_3 and I_4, (2.37) implies

$$\int_{\mathbb{R}^2\times\mathbb{T}} \left(|\nabla u|^2 + |\nabla h|^2\right)dx = 0, \tag{2.42}$$

by choosing $R \to \infty$. This means u and h are both constants. Recalling u and h vanish at the far field, we deduce the trivialness of u and h themselves. Now we have finished the proof of the part (i) of Theorem 1.1. $\square$

3. Proof of Corollary 1.1, Part (i)

This section is devoted to the case that $u^\theta \equiv 0$ and h is axially symmetric. First we see, in this situation, (1.3) turns to

$$\begin{cases} (u^r\partial_r + u^z\partial_z)u^r + \partial_r p = (h^r\partial_r + h^z\partial_z)h^r - \frac{(h^\theta)^2}{r} + \left(\Delta - \frac{1}{r^2}\right)u^r, \\ -\frac{2}{r^2}\partial_\theta u^r + \frac{1}{r}\partial_\theta p = (h^r\partial_r + h^z\partial_z)h^\theta + \frac{h^r h^\theta}{r}, \\ (u^r\partial_r + u^z\partial_z)u^z + \partial_z p = (h^r\partial_r + h^z\partial_z)h^z + \Delta u^z, \\ (u^r\partial_r + u^z\partial_z)h^r - \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)u^r = \left(\Delta - \frac{1}{r^2}\right)h^r, \\ (u^r\partial_r + u^z\partial_z)h^\theta - \frac{u^r h^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right)h^\theta, \\ (u^r\partial_r + u^z\partial_z)h^z - \left(h^r\partial_r + \frac{1}{r}h^\theta\partial_\theta + h^z\partial_z\right)u^z = \Delta h^z, \\ \nabla\cdot u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \nabla\cdot h = \partial_r h^r + \frac{h^r}{r} + \partial_z h^z = 0. \end{cases} \tag{3.1}$$

We point out that, in this situation, h^θ is vanishing so that the proof follows from the proof of Theorem 1.1. Here goes the proof of the vanishing of h^θ.

The Vanishing of h^θ

Under the axially symmetric condition of h and the vanishing of u^θ, the equation of h^θ reads

$$(u^r\partial_r + u^z\partial_z)h^\theta - \frac{u^r h^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right)h^\theta. \tag{3.2}$$

Denoting $H = \frac{h^\theta}{r}$, direct calculation shows that H satisfies

$$\left(\Delta + \frac{2}{r}\partial_r\right)H - (u^r\partial_r + u^z\partial_z)H = 0. \tag{3.3}$$

Since h is axially symmetric, the Laplacian operator here write

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.4}$$

Therefore, if we denoting

$$\Delta_5 := \sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial z^2} \tag{3.5}$$

and $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$, (3.3) becomes

$$\Delta_5 H - (u^r\partial_r + u^z\partial_z)H = 0. \tag{3.6}$$

See [7] or [11] for more details about this "dimension lifting method". From the boundedness of h^θ, one find

$$\lim_{r\to\infty} H = 0 \tag{3.7}$$

uniformly for all z. Therefore, $H \equiv 0$ is achieved by the maximum princlple. This leads to the vanishing of h^θ.

4. Proof of Corollary 1.1, Part (ii)

In this section, we consider the case that both u and h are axially symmetric. At the beginning we see (1.3) now turns to

$$\begin{cases} (u^r\partial_r + u^z\partial_z)u^r - \frac{(u^\theta)^2}{r} + \partial_r p = (h^r\partial_r + h^z\partial_z)h^r - \frac{(h^\theta)^2}{r} + \left(\Delta - \frac{1}{r^2}\right)u^r, \\ (u^r\partial_r + u^z\partial_z)u^\theta + \frac{u^r u^\theta}{r} = (h^r\partial_r + h^z\partial_z)h^\theta + \frac{h^r h^\theta}{r} + \left(\Delta - \frac{1}{r^2}\right)u^\theta, \\ (u^r\partial_r + u^z\partial_z)u^z + \partial_z p = (h^r\partial_r + h^z\partial_z)h^z + \Delta u^z, \\ (u^r\partial_r + u^z\partial_z)h^r - (h^r\partial_r + h^z\partial_z)u^r = \left(\Delta - \frac{1}{r^2}\right)h^r, \\ (u^r\partial_r + u^z\partial_z)h^\theta - (h^r\partial_r + h^z\partial_z)u^\theta + \frac{u^\theta h^r}{r} - \frac{h^\theta u^r}{r} = \left(\Delta - \frac{1}{r^2}\right)h^\theta, \\ (u^r\partial_r + u^z\partial_z)h^z - (h^r\partial_r + h^z\partial_z)u^z = \Delta h^z, \\ \nabla\cdot u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \nabla\cdot h = \partial_r h^r + \frac{h^r}{r} + \partial_z h^z = 0. \end{cases} \tag{4.1}$$

The main idea are similar with the proof of Theorem 1.1 so that we only focus on the different portion. First we see, by Sect. 2.1, u and h are bounded up to their second order derivatives. Combining these with the third equation of (4.1), we have the boundedness of $\partial_z p$. Integrating the first equation in (4.1) like Sect. 2, we are ready to prove the boundedness of p in dyadic annulus. Pay attention that, due to the axially symmetric condition for both u and h, p is no longer a function of θ. We only need to prove the boundedness of

$$I := \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi}\frac{(u^\theta)^2}{r}d\theta dz dr\right| + \left|\int_R^{r_0}\int_{-\pi}^{\pi}\int_0^{2\pi}\frac{(h^\theta)^2}{r}d\theta dz dr\right| \tag{4.2}$$

since the boundedness of the rest terms have already proven in Sect. 2. Here goes the boundedness of I:

$$\begin{aligned} I &\lesssim \left(\|u^\theta\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h^\theta\|^2_{L^\infty(\bar{B}_{2R/R})}\right)\int_R^{2R}\frac{1}{r}dr \\ &\lesssim \|u^\theta\|^2_{L^\infty(\bar{B}_{2R/R})} + \|h^\theta\|^2_{L^\infty(\bar{B}_{2R/R})} \\ &\lesssim 1. \end{aligned} \tag{4.3}$$

Then the vanishing of u and h is achieved by following the method in Sect. 2.3. We omit the details here.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

6. Appendix: Some Details of the Boundary Conditions

This Appendix is devoted to some explanations of the boundary conditions in Corollary 1.2. As we mentioned in the Corollary 1.2, instead of the periodic condition for the velocity field, our method is also valid for a certain Navier slip boundary condition with a slight modification. That is

$$u\cdot n = 0, \quad (\mathbb{D}u\cdot n)_\tau = 0, \quad \forall x\in\partial\Omega. \tag{6.1}$$

Here n is the outward unit normal to Ω. $\mathbb{D}$ is the strain tensor

$$\mathbb{D}u = \frac{1}{2}\left(\nabla u + \nabla^T u\right). \tag{6.2}$$

And for a vector filed v, v_τ stands for its tangential part: $v_\tau = v - (v \cdot n)n$. In our case, since $\Omega = \mathbb{R}^2 \times [-\pi,\, \pi]$, we have $n = (0, 0, \pm 1)$. Therefore, (6.1) is reduced to

$$u^z = 0, \quad \partial_z u_1 = 0, \quad \partial_z u_2 = 0, \quad \forall z = -\pi \text{ or } \pi. \tag{6.3}$$

In the cylinder coordinate, (6.3) equals to

$$\begin{cases} u^z = 0, \\ \partial_z u^r \cos\theta - \partial_z u^\theta \sin\theta = 0, \\ \partial_r u^z \sin\theta + \partial_z u^\theta \cos\theta = 0, \end{cases} \qquad \forall z = -\pi \text{ or } \pi. \tag{6.4}$$

That is,

$$\partial_z u^r\Big|_{z=-\pi,\,\pi} = \partial_z u^\theta\Big|_{z=-\pi,\,\pi} = u^z\Big|_{z=-\pi,\,\pi} \equiv 0. \tag{6.5}$$

Meanwhile, for magnetic field h, our method is valid for the Dirichlet condition

$$h = 0, \quad \forall z = -\pi \text{ or } \pi, \tag{6.6}$$

and the following two physical conditions, which are widely used in the research of the boundary value problem or the initial-boundary value problems to the MHD system. See [6,17], etc.

$$[\text{PC1}]\begin{cases} h \cdot n = 0, \\ \nabla \times h \times n = 0, \end{cases} \qquad [\text{PC2}]\begin{cases} h \cdot n = 0, \\ \nabla \times h = 0, \end{cases} \qquad \forall x \in \partial\Omega. \tag{6.7}$$

In our cases, similarly to (6.5) before, *[PC1]* and *[PC2]* are equivalent and both of them can be simplified to

$$\partial_z h^r\Big|_{z=-\pi,\,\pi} = \partial_z h^\theta\Big|_{z=-\pi,\,\pi} = h^z\Big|_{z=-\pi,\,\pi} \equiv 0. \tag{6.8}$$

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