DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **39**, Number **4**, April **2019**

doi:10.3934/dcds.2019085

pp. **2021–2057**

GLOBAL EXISTENCE AND OPTIMAL DECAY ESTIMATES OF THE COMPRESSIBLE VISCOELASTIC FLOWS IN L^p CRITICAL SPACES

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(Communicated by Irena Lasiecka)

Abstract. In this paper, we are concerned with the compressible viscoelastic flows in whole space $\mathbb{R}^n$ with $n \geq 2$. We aim at extending the global existence in energy spaces (see [18] by Hu & Wang and [30] by Qian & Zhang) such that it holds in more general L^p critical spaces, which allows to the case of *large highly oscillating* initial velocity. Precisely, We define "*two effective velocities*" which are used to eliminate the coupling between the density, velocity and deformation tensor. Consequently, the global existence in the L^p critical framework is constructed by elementary energy approaches. In addition, the optimal time-decay estimates of strong solutions are firstly shown in the L^p framework, which improve recent decay efforts for compressible viscoelastic flows.

1. **Introduction.**

Consider the following equations of multi-dimensional compressible viscoelastic flows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu D(u) + \lambda \operatorname{div} u \mathrm{Id}) + \nabla P = \alpha \operatorname{div}(\rho F F^T), \\ \partial_t F + u \cdot \nabla F = \nabla u F, \end{cases} \tag{1.1}$$

where $\rho \in \mathbb{R}_+$ is the density, $u \in \mathbb{R}^n$ is the velocity and $F \in \mathbb{R}^{n\times n}$ is the deformation gradient. F^T stands for the transpose matrix of F. The pressure P depends only upon the density and the function will be taking suitably smooth. Notations div, $\otimes$ and ∇ denote the divergence operator, Kronecker tensor product and gradient operator, respectively. $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain tensor. The density-dependent viscosity coefficients μ, λ are assumed to be smooth and to satisfy $\mu > 0$, $\nu \triangleq \lambda + 2\mu > 0$. For simplicity, the elastic energy $W(F)$ in System (1.1) has been taken to be the special form of the Hookean linear elasticity:

$$W(F) = \frac{\alpha}{2}|F|^2, \quad \alpha > 0,$$

which does not reduce the essential difficulties in analysis. Our methods and results may be applied to more general Hookean elasticity law.

2010 *Mathematics Subject Classification.* Primary: 35Q35, 35B40; Secondary: 35L60.

Key words and phrases. Compressible viscoelastic flows, critical Besov space, global existence, optimal decay.

In this paper, we focus on the Cauchy problem of System (1.1), so the corresponding initial data are supplemented by

$$(\rho, F; u)|_{t=0} = (\rho_0(x), F_0(x); u_0(x)), \quad x \in \mathbb{R}^n. \tag{1.2}$$

It is well known that there are some fluids which do not satisfy the classical Newtonian law. So far there have been many attempts to capture different phenomena for non-Newtonian fluids, see for example [13, 14, 26, 28] and so on. System (1.1) simulates the compressible viscoelastic flow of Oldroyd type exhibiting the elastic behavior, which belongs to a class of non-Newtonian fluids. We are interested in the well-posedness and stability of solutions to the Cauchy problem (1.1)-(1.2), at least under the perturbation of constant equilibrium $(1, I, 0)$.

Let us first recall mathematical efforts related to viscoelastic flows. For the incompressible viscoelastic flows, there has been much important progress on classical solutions. Lin-Liu-Zhang [26], Chen-Zhang [6], Lei-Liu-Zhou[22] and Lin-Zhang [25] established the local and global well-posedness with small data in Sobolev space H^s. Hu-Wu [21] proved the long-time behavior and weak-strong uniqueness of solutions. Chemin-Masmoudi [4] proved the local existence of solution and a global small solution in critical Besov spaces, where the Cauchy-Green strain tensor is available in the evolution equation. Qian [29] proved the well-posedness of the incompressible viscoelastic system in critical spaces. Subsequently, Zhang-Fang [33] proved the global well-posedness in the critical L^p Besov space. On the other hand, the global existence of weak solutions is still an open problem. Lions and Masmoudi [27] considered the special case that the contribution of the strain rate is neglected, and constructed the global-in-time weak solution for general initial data.

For compressible viscoelastic flows, Lei-Zhou [25] proved the global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. The local existence of strong solutions was obtained by Hu-Wang [19]. Shortly, Hu-Wang [18] and Qian-Zhang [30] independently proved the global existence in the critical L^2 Besov space, provided initial data are close to constant equilibrium. For convenience of reader, we would like to state their results as follows.

Theorem 1.1. *([18, 30]) Assume that $P'(1) > 0$. Then there exists two constant η and M such that if*

$$(\rho_0 - 1, F_0 - I; u_0) \in \left(\dot{\mathcal{B}}^{n/2-1,n/2}_{2,2}\right)^{1+n^2} \times \left(\dot{B}^{n/2-1}_{2,1}\right)^n$$

satisfying

$$\|(\rho_0 - 1, F_0 - I)\|_{\dot{\mathcal{B}}^{n/2-1,n/2}_{2,2}} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}} \le \eta,$$

then there exists a global unique solution $(\rho, F; u) \in \tilde{E}^{n/2}$ to the Cauchy problem (1.1)−(1.2) such that

$$\|(\rho - 1, F - I; u)\|_{\tilde{E}^{n/2}} \le M\big(\|(\rho_0 - 1, F_0 - I)\|_{\dot{\mathcal{B}}^{n/2-1,n/2}_{2,2}} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}}\big),$$

where

$$\tilde{E}^{n/2} \triangleq \left(\tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{\mathcal{B}}^{n/2-1,n/2}_{2,2}) \cap L^1(\mathbb{R}_+; \dot{\mathcal{B}}^{n/2+1,n/2}_{2,2})\right)^{1+n^2} \times \left(\tilde{\mathcal{C}}_b(\mathbb{R}_+; \dot{B}^{n/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/2+1}_{2,1})\right)^n.$$

Concerning those norm notations for hybrid Besov spaces $\tilde{L}^q\dot{\mathcal{B}}^{s,\sigma}_{2,p}(p \ge 2)$ and $\tilde{\mathcal{C}}_b(\dot{\mathcal{B}}^{s,\sigma}_{2,p})$, the reader is referred to Section 2 below. In fact, those functional spaces to investigate (1.1) enjoy the scaling invariance. Precisely, observe that (1.1) is scaling invariant by the transformation: for any constant $\kappa > 0$,

$$(\rho_0(x), F_0(x); u_0(x)) \to (\rho_0(\kappa x), F_0(\kappa x); \lambda u_0(\kappa x)),$$
$$(\rho(t,x), F(t,x); u(t,x)) \to (\rho(\kappa^2 t, \kappa x), F(\kappa^2 t, \kappa x); \kappa u(\kappa^2 t, \kappa x)),$$

up to changes of the pressure P into $\kappa^2 P$ and the constant α into $\kappa^2\alpha$. This inspires the definition of the critical space.

Definition 1.1. A functional space is called the critical one if the associated norm is invariant under the transformation

$$(\rho(t,x), F(t,x); u(t,x)) \to (\rho(\kappa^2 t, \kappa x), F(\kappa^2 t, \kappa x); \kappa u(\kappa^2 t, \kappa x))$$

(up to a constant independent of κ).

Obviously, it is easy to check that $\big(\dot{B}^{n/2}_{2,1}\big)^{1+n^2} \times \big(\dot{B}^{n/2-1}_{2,1}\big)^n$ is the critical functional setting in the sense of Definition 1.1. It should be mentioned that Danchin [8] first applied the basic idea to the study of compressible Navier-Stokes equations. He established the global well-posedness of strong solutions in the critical L^2 spaces. Compared to [8], there is an outstanding difficulty for the compressible viscoelastic system. *How to capture the damping effect of the deformation tensor arising from the nonlinear coupling between the density, velocity and deformation tensor?* Hu-Wang [18] and Qian-Zhang [30] independently explored some intrinsic properties for (1.1) and established uniform estimate for complicated linearized hyperbolic-parabolic systems, which eventually leads to Theorem 1.1.

The goal of this paper is twofold: firstly, we aim at extending Theorem 1.1 to the critical L^p Besov space, which allows highly large oscillating initial velocity. Secondly, we shall establish the large-time behavior of the constructed solution.

Denote

$$\begin{aligned}\mathcal{E}^{n/p} \quad \triangleq \quad & \Big\{(a,O;v)\big|(a,O;v) \in \Big(\tilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}) \cap L^1(\mathbb{R}_+;\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p})\Big)^{1+n^2} \\ & \times\Big(\tilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}) \cap L^1(\mathbb{R}_+;\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p})\Big)^n\Big\},\end{aligned}$$

with its norm

$$\begin{aligned}&\|(a,O;v)\|_{\mathcal{E}^{n/p}} \\ =&\|(a,O)\|_{\tilde{L}^\infty\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}\cap L^1\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}} + \|v\|_{\tilde{L}^\infty\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}\cap L^1\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}.\end{aligned}$$

Now, we state the first result as follows.

Theorem 1.2. *Assume that $P'(1) > 0$. Let p satisfying $2 \le p \le \min\big(4, 2n/(n-2)\big)$ and, additionally, $p \ne 4$ if $n = 2$. If there exists two constant η and M such that if*

$$(\rho_0 - 1, F_0 - I; u_0) \in \Big(\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}\Big)^{1+n^2} \times \Big(\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}\Big)^n.$$

and

$$\|(\rho_0 - 1, F_0 - I)\|_{\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}} + \|u_0\|_{\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}} \le \eta, \tag{1.3}$$

then the Cauchy problem (1.1)-(1.2) has a global unique solution $(\rho, F; u)$ such that $(\rho - 1, F - I; u) \in \mathcal{E}^{n/p}$ and

$$\|(\rho - 1, F - I; u)\|_{\mathcal{E}^{n/p}} \le M\big(\|(\rho_0 - 1, F_0 - I)\|_{\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}} + \|u_0\|_{\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\big).$$

Let us point out new ingredients in the proof of Theorem 1.2. For usual compressible N-S equations (see for example [3, 5]), the major difficulty stems from the convection term in the density equation, as it may cause a loss of one derivative of the density. To overcome it, previous proofs heavily relied on a paralinearized version combined with a Lagrangian change of variables. For the viscoelastic system (1.1), the situation becomes more complicated. As shown by [30], the damping effect of F can be produced by some intrinsic conditions (see Proposition 3.1), however, similar to the density, it also has no smoothing effect at high frequencies. In order to avoid cumbersome estimates and solve (1.1) globally, we follow from an elementary energy approach in terms of *effective velocity*. This argument has been developed by Haspot [15, 16] for compressible Navier-Stokes equations, which is based on the use of Hoff's viscous effective flux as in [17]. Precisely, we introduce the following "*two effective velocities*",

$$w = \nabla(-\Delta)^{-1}(2a - \mathrm{div} v), \qquad \Omega^{ij} = e^{ij} + \frac{1}{\mu_0}\Lambda(-\Delta)^{-1}O^{ij}.$$

Indeed, the definition of w is almost same as that in [15, 16]. The unique difference lies in the coefficient of a, which indicates the contribution arising from the coupling of F. Another effective velocity with respect to Ω^{ij} is totally new, which allows to cancel the coupling between e^{ij} and O^{ij} at high frequencies (see Sections 4 and 5 for more details). In physical dimensions $n = 2, 3$, the value of p enables us to consider the case $p > n$ for which the velocity regularity exponent $n/p - 1$ becomes negative. Consequently, Theorem 1.2 can be applied to *large* highly oscillating initial velocities (see [3, 5] for more explanation).

Another interesting question follows after Theorem 1.2. One may wonder how the global strong solutions constructed above look like for the large time. Although providing an accurate long-time asymptotic description is still out of reach, a number of results concerning the time decay rates of global solutions, sometimes referred to as $L^q - L^r$ decay rates are available. For example, Hu-Wu [20] proved the global existence of strong solutions to (1.1) as initial data are the small perturbation $(1, I; 0)$ in $H^2(\mathbb{R}^3)$. Furthermore, with the extra assumption of $L^1(\mathbb{R}^3)$, it was shown that those solutions converged to equilibrium state at the following speed

$$\|(\rho - 1, F - I; u)\|_{L^p} \le C\langle t\rangle^{-\frac{3}{2}(1-\frac{1}{p})}. \tag{1.4}$$

The time decay rate in (1.4) turns out to be the same one for the heat kernel, which is sometime referred as the optimal decay rate. Next, we state a decay result for those solutions constructed in Theorem 1.2. Precisely, one has

Theorem 1.3. *Let $n \ge 2$ and p satisfies $2 \le p \le \min\big(4, 2n/(n-2)\big)$ and $p \ne 4$ if $n = 2$. Let (ρ_0, u_0, F_0) fulfill the assumptions of Theorem 1.2 and (ρ, u, F) be the global solution of System (1.1). Then there exists a constant $\sigma = \sigma(p, n, \lambda, \mu, \alpha, P)$ such that if additionally*

$$\mathcal{G}_{p,0} \triangleq \|(\rho_0 - 1, F_0 - I; u_0)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} \le \sigma \quad \textit{with} \quad s_0 \triangleq n(2/p - 1/2), \tag{1.5}$$

then we have for $t \ge 0$,

$$\mathcal{G}_p(t) \lesssim \Big(\mathcal{G}_{p,0} + \|(\nabla\rho_0, \nabla F_0; u_0)\|^{h}_{\dot{B}^{n/p-1}_{p,1}}\Big), \tag{1.6}$$

where $\mathcal{G}_p(t)$ is defined by

$$\mathcal{G}_p(t) \triangleq \sup_{s \in [\varepsilon - s_0, \frac{n}{2}+1]} \|\langle\tau\rangle^{\frac{s_0+s}{2}}(\rho - 1, F - I; u)\|^{\ell}_{\tilde{L}^{\infty}_t \dot{B}^{s}_{2,1}}$$

$$+\|\langle\tau\rangle^{\alpha}(\nabla a,\nabla F;u)\|^{h}_{\tilde{L}^{\infty}_{t}\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\tau\nabla u\|^{h}_{\tilde{L}^{\infty}_{t}\dot{B}^{\frac{n}{p}}_{p,1}}, \tag{1.7}$$

with $\alpha \triangleq n/p+1/2-\varepsilon$ *for* $\varepsilon>0$ *sufficiently small.*

Here and below, $\|f\|^{\ell}_{\bullet}$ and $\|f\|^{h}_{\bullet}$ represent the low and high frequency part of some norm $\|f\|_{\bullet}$ to a tempered distribution f whose exact definition will be given in Section 2.

Some comments are in order.

1. Due to the Sobolev imbedding properties $L^1\hookrightarrow \dot{B}^{0}_{1,\infty}\hookrightarrow \dot{B}^{-n/2}_{2,\infty}$, $\dot{H}^{-n/2}\hookrightarrow \dot{B}^{-n/2}_{2,\infty}$, our low-frequency assumption is less restrictive. Actually, the assumption is also relevant in other contexts like the Boltzmann equation (see [31]), or hyperbolic systems with dissipation (see [32]).
2. The decay result remains true in the case of *large* highly oscillating initial velocities, since the case $p>n$ occurs in physical dimensions $n=2,3$, which was not shown by recent efforts (see [20]).
3. Likewise, "*two effective velocities*" play a key role in establishing the nonlinear time-weighted inequality (1.7). Furthermore, the optimal decay estimates of L^q-L^r type can be derived from the definition of $\mathcal{G}_p(t)$ by using standard interpolation tricks. The interested reader is referred to [7] for similar details.

The rest of this paper is arranged as follows: In Section 2, we first recall the Littlewood-Paley theory and present the definition and properties for the hybrid-Besov space. In Section 3, we reformulate our system into a hyperbolic-parabolic system coupled by the density, velocity and deformation gradient. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we prove the decay estimate in Theorem 1.3. Some analysis properties in the hybrid Besov space will be given in the Appendix.

2. **The Littlewood-Paley theory and hybrid Besov space.**

Throughout the paper, we denote by C a generic constant which may be different from line to line. The notation $A\lesssim B$ means $A\leq CB$ and $A\approx B$ indicates $A\leq CB$ and $B\leq CA$.

2.1. **The Littlewood-Paley decomposition.**

Let's begin with the Littlewood-Paley decomposition. There exists two radial smooth functions $\varphi(x)$ and $\chi(x)$, which are supported in the annulus $\mathcal{C}=\{\xi\in\mathbb{R}^n: 3/4\leq|\xi|\leq 8/3\}$ and the ball $B=\{\xi\in\mathbb{R}^n:|\xi|\leq 4/3\}$, respectively, such that

$$\chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1\quad \forall\xi\in\mathbb{R}^n$$

and

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad \forall\xi\in\mathbb{R}^n\setminus\{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homegeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j\in\mathbb{Z}$ by

$$\dot{\Delta}_j u=\varphi(2^{-j}D)f,\quad \dot{S}_j f=\sum_{k\leq j-1}\dot{\Delta}_k f=\chi(2^{-j}D)f.$$

The following Bernstein inequality will be repeatedly used throughout the paper.

Lemma 2.1 ([2]). *A constant C exists such that for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $1\le p\le q\le\infty$, and any function u of L^p, we have*

$$\text{Supp}\hat{u}\subset\lambda B\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q}\le C^{k+1}\lambda^{k+n(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

$$\text{Supp}\hat{u}\subset\lambda\mathcal{C}\Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p}\le\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p}\le C^{k+1}\lambda^k\|u\|_{L^p}.$$

2.2. **The hybrid Besov space.** We denote by $\mathcal{Z}'(\mathbb{R}^n)$ the dual space of

$$\mathcal{Z}(\mathbb{R}^n)\triangleq\{f\in\mathcal{S}(\mathbb{R}^n):\partial^\alpha\hat{f}(0)=0,\forall\alpha\in(\mathbb{N}\cup 0)^n\}.$$

Firstly, we give the definition of the homogeneous Besov space.

Definition 2.1. Let s be a real number and (p,r) be in $[1,\infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u\in\mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}}\triangleq\Big(\sum_{j\in\mathbb{Z}}2^{jsr}\|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}}<\infty.$$

Secondly, we introduce the hybrid Besov space that will be used in this paper.

Definition 2.2. Let $s,\sigma\in\mathbb{R}$, $1\le p\le+\infty$. The hybrid Besov space $\dot{\mathcal{B}}^{s,\sigma}_{2,p}$ is defined by

$$\dot{\mathcal{B}}^{s,\sigma}_{2,p}\triangleq\{f\in\mathcal{Z}'(\mathbb{R}^n):\|f\|_{\dot{\mathcal{B}}^{s,\sigma}_{2,p}}<\infty\},$$

with

$$\|f\|_{\dot{\mathcal{B}}^{s,\sigma}_{2,p}}\triangleq\sum_{2^k\le R_0}2^{ks}\|\dot{\Delta}_k f\|_{L^2}+\sum_{2^k>R_0}2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p},$$

Where R_0 is a fixed and sufficiently large constant which may depend on $\lambda(1)$, $\mu(1),p$ and n.

Since we are concerned with time-dependent functions valued in Besov spaces, the following space-time mixed norm is usually mentioned:

$$\|u\|_{L^q_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}}:=\big\|\|u(t,\cdot)\|_{\dot{\mathcal{B}}^{s,\sigma}_{2,p}}\big\|_{L^q(0,T)}.$$

Here, we introduce another space-time mixed Besov norm, which is referred to Chemin-Lerner's spaces. The definition is given by in such way

$$\|u\|_{\tilde{L}^q_T\dot{B}^{s,\sigma}_{2,p}}\triangleq\sum_{2^k\le R_0}2^{ks}\|\dot{\Delta}_k u\|_{L^q(0,T;L^2)}+\sum_{2^k>R_0}2^{k\sigma}\|\dot{\Delta}_k u\|_{L^q(0,T;L^2)}.$$

The index T will be omitted if $T=+\infty$ and we shall denote by $\tilde{\mathcal{C}}_b(\dot{B}^{s,\sigma}_{2,p})$ the subset of functions $\tilde{L}^\infty(\dot{B}^{s,\sigma}_{2,p})$ which are continuous from $\mathbb{R}_+$ to $\dot{B}^{s,\sigma}_{2,p}$. It is easy to check that $\tilde{L}^1_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}=L^1_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}$ and $\tilde{L}^q_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}\subseteq L^q_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}$ for $q>1$.

Also, for a tempered distribution f, we denote

$$f^\ell\triangleq\sum_{2^k\le R_0}\dot{\Delta}_k f,\quad f^h\triangleq f-f^\ell,$$

and

$$\|f\|^\ell_{\dot{B}^s_{p,1}}=\sum_{2^k\le R_0}2^{ks}\|\dot{\Delta}_k f\|_{L^p},\quad\|f\|^h_{\dot{B}^s_{p,1}}=\sum_{2^k>R_0}2^{k\sigma}\|\dot{\Delta}_k f\|_{L^p},$$

$$\|f\|^\ell_{\tilde{L}^q_T\dot{B}^s_{p,1}}=\sum_{2^k\le R_0}2^{ks}\|\dot{\Delta}_k f\|_{L^q(0,T;L^p)},\quad\|f\|^h_{\tilde{L}^q_T\dot{B}^s_{p,1}}=\sum_{2^k>R_0}2^{k\sigma}\|\dot{\Delta}_k f\|_{L^q(0,T;L^p)},$$

$$\|f\|^{\ell}_{\dot{B}^s_{2,\infty}} = \sup_{2^k\le R_0} 2^{ks}\|\dot{\Delta}_k f\|_{L^2}$$

for $s\in\mathbb{R}$.

Next, we collect the following properties in Besov spaces.

Lemma 2.2. *Let $s,\sigma\in\mathbb{R}$ and $1\le p\le +\infty$. Then we have*

- $\dot{\mathcal{B}}^{s_2,\sigma}_{2,p}\subseteq \dot{\mathcal{B}}^{s_1,\sigma}_{2,p}$ *for $s_1\ge s_2$ and* $\dot{\mathcal{B}}^{s,\sigma_2}_{2,p}\subseteq \dot{\mathcal{B}}^{s,\sigma_1}_{2,p}$ *for $\sigma_1\le\sigma_2$.*
- *Interpolation: For $s_1,s_2,\sigma_1,\sigma_2\in\mathbb{R}$ and $\theta\in[0,1]$, we have*

$$\|f\|_{\dot{\mathcal{B}}^{\theta s_1+(1-\theta)s_2,\theta\sigma_1+(1-\theta)\sigma_2}_{2,p}} \le \|f\|^{\theta}_{\dot{\mathcal{B}}^{s_1,\sigma_1}_{2,p}}\|f\|^{(1-\theta)}_{\dot{\mathcal{B}}^{s_2,\sigma_2}_{2,p}}.$$

- *Embedding: $L^\infty\hookrightarrow \dot{\mathcal{B}}^{n/2,n/p}_{2,p}$;*
  $\dot{B}^s_{2,1}\hookrightarrow \dot{\mathcal{B}}^{s,s-n/2+n/p}_{2,p}\hookrightarrow \dot{B}^{s-n/2+n/p}_{p,1}$ *for $p\ge 2$.*

Lemma 2.3 ([11])**.** *Let $1\le p,q,q_1,q_2\le\infty$ with $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Then we have*

- *If $s_1,s_2\le n/p$ and $s_1+s_2>n\max(0,2/p-1)$, then*

$$\|fg\|_{\tilde{L}^q_T(\dot{B}^{s_1+s_2-n/p}_{p,1})}\le C\|f\|_{\tilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\tilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,1})}.$$

- *If $s_1\le n/p,\ s_2<n/p$ and $s_1+s_2>n\max(0,2/p-1)$, then*

$$\|fg\|_{\tilde{L}^q_T\dot{B}^{s_1+s_2-n/p}_{p,\infty}}\le C\|f\|_{\tilde{L}^{q_1}_T(\dot{B}^{s_1}_{p,1})}\|g\|_{\tilde{L}^{q_2}_T(\dot{B}^{s_2}_{p,\infty})}.$$

Remark 2.1. Lemma 2.3 still remains true in usual homogenous Besov spaces. For example, we have

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}}\le C\|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

Lemma 2.4 ([7])**.** *Let $\sigma>0$ and $1\le p,r\le\infty$. Then $\dot{B}^\sigma_{p,r}\cap L^\infty$ is an algebra and we have*

$$\|fg\|_{\dot{B}^\sigma_{p,r}}\lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^\sigma_{p,r}}+\|g\|_{L^\infty}\|f\|_{\dot{B}^\sigma_{p,r}}.$$

Let σ_1,σ_2,p_1 and p_2 fulfill

$$\sigma_1+\sigma_2>0,\ \ \sigma_1\le n/p_1,\ \ \sigma_2\le n/p_2,\ \ \sigma_1\ge\sigma_2,\ \ \frac{1}{p_1}+\frac{1}{p_2}\le 1.$$

Then we have

$$\|fg\|_{\dot{B}^{\sigma_2}_{q,1}}\lesssim \|f\|_{\dot{B}^{\sigma_1}_{p_1,1}}\|f\|_{\dot{B}^{\sigma_2}_{p_2,1}} \quad \text{with}\ \ \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma_1}{n}. \tag{2.1}$$

Finally, let $\sigma>0,\ 1\le p_1,p_2,q\le\infty$ and

$$\frac{n}{p_1}+\frac{n}{p_2}-n\le\sigma\le\min(\frac{n}{p_1},\frac{n}{p_2}) \quad \text{and}\ \ \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\sigma}{n}.$$

We have

$$\|fg\|_{\dot{B}^{-\sigma}_{q,\infty}}\lesssim \|f\|_{\dot{B}^{\sigma}_{p_1,1}}\|g\|_{\dot{B}^{-\sigma}_{p_2,\infty}}. \tag{2.2}$$

Lemma 2.5 ([7])**.** *There exists a universal integer N_0 such that for any $2\le p\le 4$, and $\sigma>0$, we have*

$$\|fg^h\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}\lesssim \big(\|f\|_{\dot{B}^\sigma_{p,1}}+\|\dot{S}_{k_0+N_0}f\|_{L^{p^*}}\big)\|g^h\|_{\dot{B}^{-\sigma}_{p,\infty}}, \tag{2.3}$$

$$\|f^hg\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}\lesssim \big(\|f^h\|_{\dot{B}^\sigma_{p,1}}+\|\dot{S}_{k_0+N_0}f^h\|_{L^{p^*}}\big)\|g\|_{\dot{B}^{-\sigma}_{p,\infty}}, \tag{2.4}$$

with $s_0=n\big(\frac{2}{p}-\frac{1}{2}\big)$ and $\frac{1}{p^*}=\frac{1}{2}-\frac{1}{p}$.

Lemma 2.6 ([7]). *Let $1 \le p, p_1 \le \infty$ and*

$$-\min\left(\frac{n}{p_1}, \frac{n}{p'}\right) < \sigma \le 1 + \min\left(\frac{n}{p}, \frac{n}{p_1}\right).$$

There exists a constant $C > 0$, depending only on σ such that for all $j \in \mathbb{Z}$, we have

$$\|[v \cdot \nabla, \nabla \dot{\Delta}_j] z\|_{L^p} \le C c_j 2^{-j(\sigma-1)} \|\nabla v\|_{\dot{B}^{n/p_1}_{p_1,1}} \|\nabla z\|_{\dot{B}^{\sigma-1}_{p,1}}, \tag{2.5}$$

where $(c_j)_{j\in\mathbb{Z}}$ denotes a sequence such that $\|(c_j)\|_{\ell^1} \le 1$.

3. **Reformulation of System** (1.1). In this section, we present intrinsic properties of compressible viscoelastic flows, which have been explored in [30].

Proposition 3.1. *The density $\rho = \rho(t,x)$ and the deformation gradient $F = F(t,x)$ of* (1.1) *satisfy the following relations:*

$$\nabla \cdot (\rho F^T) = 0 \quad \text{and} \quad F^{lk}\partial_l F^{ij} - F^{lj}\partial_l F^{ik} = 0, \tag{3.1}$$

for any $t > 0$, if the initial data (ρ_0, F_0) satisfies

$$\nabla \cdot (\rho_0 F_0^T) = 0 \quad \text{and} \quad F_0^{lk}\partial_l F_0^{ij} - F_0^{lj}\partial_l F_0^{ik} = 0. \tag{3.2}$$

By Proposition 3.1, the i-th component of the vector $\mathrm{div}(\rho F F^T)$ can be written as

$$\begin{aligned} \partial_j(\rho F^{ik}F^{jk}) &= \rho F^{jk}\partial_j F^{ik} + F^{ik}\partial_j(\rho F^{jk}) \\ &= \rho F^{jk}\partial_j F^{ik}, \end{aligned} \tag{3.3}$$

where we used the first equality in (3.1).

Setting $\chi_0 = (P'(1))^{-1/2}$. We define

$$a(t,x) = \rho(\chi_0^2 t, \chi_0 x) - 1, \quad v(t,x) = \chi_0 u(\chi_0^2 t, \chi_0 x), \quad O(t,x) = F(\chi_0^2 t, \chi_0 x) - I.$$

Noticing (3.3), it is easy to check that

$$\begin{cases} \partial_t a + v \cdot \nabla a + \nabla \cdot v = -a \nabla \cdot v, \\ \partial_t v + v \cdot \nabla v - \mathcal{A} v + \nabla a - \beta \nabla \cdot O = \beta O^{jk}\partial_j O^{\bullet k} - I(a)\mathcal{A}v - K(a)\nabla a \\ \qquad\qquad\qquad\qquad + \dfrac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\mathrm{div} v \mathrm{Id}\big), \\ \partial_t O + v \cdot \nabla O - \nabla v = \nabla v O, \end{cases} \tag{3.4}$$

where

$$I(a) \triangleq \frac{a}{1+a}, \; K(a) \triangleq \frac{P'(1+a)}{(1+a)P'(1)} - 1, \; \mathcal{A} = \mu(1)\Delta + (\lambda(1) + \mu(1))\nabla \mathrm{div},$$

and

$$\beta = \frac{\alpha}{P'(1)}, \; \tilde{\mu}(a) = \mu(1+a) - \mu(1), \; \tilde{\lambda}(a) = \lambda(1+a) - \lambda(1).$$

Here, $O^{jk}\partial_j O^{\bullet k}$ is a vector function whose components are given by $(O^{jk}\partial_j O^{ik})_{i=1}^n$. For simplicity, we set $\lambda(1) = \lambda_0, \mu(1) = \mu_0$. Furthermore, we normalize $\beta = 1$ and $\nu(1) := \lambda(1) + 2\mu(1) = 1$ without loss of generality.

For $s \in \mathbb{R}$, we denote

$$\Lambda^s f \triangleq \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f)),$$

and introduce two variables as in [30]:

$$d = \Lambda^{-1}\mathrm{div} v, \quad e^{ij} = \Lambda^{-1}\partial_j v^i. \tag{3.5}$$

Using the second equality in (3.1), we have

$$\Lambda^{-1}(\partial_j\partial_k O^{ik}) = -\Lambda O^{ij} - \Lambda^{-1}\partial_k(O^{lj}\partial_l O^{ik} - O^{lk}\partial_l O^{ij}). \tag{3.6}$$

Hence, with aid of (3.6), the system (3.4) can be reformulated as follows

$$\begin{cases} \partial_t a + \Lambda d = G_1, \\ \partial_t e^{ij} - \mu_0\Delta e^{ij} - (\lambda_0+\mu_0)\partial_i\partial_j d + \Lambda^{-1}\partial_i\partial_j a + \Lambda O^{ij} = G_4^{ij} \\ \partial_t O^{ij} - \Lambda e^{ij} = G_3^{ij}, \\ d = -\Lambda^{-2}\partial_i\partial_j e^{ij}, v^i = -\Lambda^{-1}\partial_j e^{ij}, \end{cases} \tag{3.7}$$

where $G_1 = -a\nabla\cdot v - v\cdot\nabla a$, $G_3^{ij} = \partial_k v^i O^{kj} - v\cdot\nabla O^{ij}$ and

$$\begin{aligned} G_4^{ij} &= -\Lambda^{-1}\partial_j\Big(v\cdot\nabla v^i - O^{lk}\partial_l O^{ik} + I(a)(\mathcal{A}v)^i + K(a)\partial_i a\Big) \\ &\quad -\Lambda^{-1}\partial_k(O^{lj}\partial_l O^{ik} - O^{lk}\partial_l O^{ij}) \\ &\quad +\Lambda^{-1}\partial_j\Big(\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)\Big)^i. \end{aligned}$$

Additionally, we need the auxiliary equation in subsequent estimates

$$\partial_i O^{ij} = -\partial_j a - G_0^j, \qquad G_0^j = \partial_i(aO^{ij}), \tag{3.8}$$

which can be deduced from the first equality in (3.1).

4. **Proof of Theorem 1.2.** Inspired by [5], we may extend those results in [18, 30] such that they hold true in the L^p critical framework. First of all, it is convenient to give the following interpolation inequalities

$$\begin{aligned} \|f\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}} &\lesssim \|f\|^{1/2}_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\|f\|^{1/2}_{L^1_T\dot{B}^{n/2+1,n/p}_{2,p}}, \\ \|f\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}} &\lesssim \|f\|^{1/2}_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|f\|^{1/2}_{L^1_T\dot{B}^{n/2+1,n/p+1}_{2,p}}. \end{aligned} \tag{4.1}$$

The proof Theorem 1.2 is divided into several parts. The first one is to establish two a priori estimates.

4.1. **Two a priori estimates.** Let $T>0$. We denote by $\mathcal{E}^{n/p}_T$ the functional space

$$\begin{aligned} \mathcal{E}^{n/p}_T \quad &\triangleq \quad \big\{(a,O;v)\in\big(\tilde{L}^\infty(0,T;\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p})\cap L^1(0,T;\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p})\big)^{1+n^2} \\ &\qquad\times\big(\tilde{L}^\infty(0,T;\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p})\cap L^1(0,T;\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p})\big)^n\big\} \end{aligned}$$

with the norm

$$\begin{aligned} \|(a,O;v)\|_{\mathcal{E}^{n/p}_T} \quad &\triangleq \quad \|(a,O)\|_{\tilde{L}^\infty_T B^{n/2-1,n/p}_{2,p}\cap L^1_T\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}} \\ &\qquad +\|v\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}\cap L^1_T\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}. \end{aligned} \tag{4.2}$$

Proposition 4.1. *Let $2\le p\le\min(4,\frac{2n}{n-2})$ and $p<2n$. Assume that $(a,O;v)$ is a strong solution of system (3.4) on $[0,T]$ with*

$$\|a\|_{L^\infty([0,T]\times\mathbb{R}^n)}\le\frac{1}{2}.$$

Then we have

$$\begin{aligned} \|(a,O;v)\|_{\mathcal{E}^{n/p}_T} &\le C\Big\{\|(a_0,O_0;v_0)\|_{\mathcal{E}^{n/p}_0} \\ &\qquad +\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}\big(1+\|(a,O;v)\|_{\mathcal{E}^{n/p}_T}\big)^{n+3}\Big\}, \end{aligned} \tag{4.3}$$

where $\|(a_0,O_0;v_0)\|_{\mathcal{E}_0^{n/p}} \triangleq \|(a_0,O_0)\|_{\dot{\mathcal{B}}_{2,p}^{n/2-1,n/p}} + \|v_0\|_{\dot{\mathcal{B}}_{2,p}^{n/2-1,n/p-1}}$.

In addition, we introduce another functional space $E_T^{n/2}$, which is defined by

$$E_T^{n/2} \triangleq \Big\{(a,O;v) \in \big(\tilde{L}^\infty(0,T;\dot{\mathcal{B}}_{2,2}^{n/2-1,n/2}) \cap L^1(0,T;\dot{\mathcal{B}}_{2,2}^{n/2+1,n/2})\big)^{1+n^2} \times\big(\tilde{L}^\infty(0,T;\dot{B}_{2,1}^{n/2-1}) \cap L^1(0,T;\dot{B}_{2,1}^{n/2+1})\big)^n\Big\}$$

with the norm

$$\|(a,O;v)\|_{E_T^{n/2}} \triangleq \|(a,O)\|_{\tilde{L}_T^\infty \dot{\mathcal{B}}_{2,2}^{n/2-1,n/2} \cap L_T^1 \dot{\mathcal{B}}_{2,2}^{n/2+1,n/2}} + \|v\|_{\tilde{L}_T^\infty \dot{B}_{2,1}^{n/2-1} \cap L_T^1 \dot{B}_{2,1}^{n/2+1}}.$$

Proposition 4.2. *Under the assumption of Proposition 4.1, we have*

$$\|(a,O;v)\|_{E_T^{n/2}} \le C\Big\{\|(a_0,O_0;v_0)\|_{E_0^{n/2}} + \|(a,O;v)\|_{E_T^{n/2}}\|(a,O;v)\|_{\mathcal{E}_T^{n/p}}\big(1+\|(a,O;v)\|_{\mathcal{E}_T^{n/p}}\big)^{n+3}\Big\}, \tag{4.4}$$

where $\|(a_0,O_0;v_0)\|_{E_0^{n/2}} \triangleq \|(a_0,O_0)\|_{\dot{\mathcal{B}}_{2,2}^{n/2-1,n/2}} + \|v_0\|_{\dot{B}_{2,1}^{n/2-1}}$.

The proofs of Propositions 4.1-4.2 lie in the pure energy method in terms of low-frequency and high-frequency decompositions.

Step1. Low-frequency estimates ($2^k \le R_0$).

Denote $a_k = \dot{\Delta}_k a, O_k = \dot{\Delta}_k O$ and $d_k = \dot{\Delta}_k d, e_k = \dot{\Delta}_k e$ for simplicity. By applying $\dot{\Delta}_k$ to (3.7), we have

$$\begin{cases} \partial_t a_k + \Lambda d_k = \dot{\Delta}_k G_1, \\ \partial_t e_k^{ij} - \mu_0 \Delta e_k^{ij} - (\lambda_0+\mu_0)\partial_i\partial_j d_k + \Lambda^{-1}\partial_i\partial_j a_k + \Lambda O_k^{ij} = \dot{\Delta}_k G_4^{ij} \\ \partial_t O_k^{ij} - \Lambda e_k^{ij} = \dot{\Delta}_k G_3^{ij}, \\ d_k = -\Lambda^{-2}\partial_i\partial_j e_k^{ij}. \end{cases} \tag{4.5}$$

Taking L^2 inner product of $(4.5)_2$ with e_k^{ij}, and then summing up the resulting equation with respect to indices i,j, we arrive at

$$\begin{aligned} \frac{1}{2}\|e_k\|_{L^2}^2 \quad &+ \quad \mu_0\|\Lambda e_k\|_{L^2}^2 + (\lambda_0+\mu_0)\|\Lambda d_k\|_{L^2}^2 - (a_k|\Lambda d_k) + (\Lambda O_k|e_k) \\ &= \quad (\dot{\Delta}_k G_4|e_k), \end{aligned} \tag{4.6}$$

where we have used the fact $d_k = -\Lambda^{-2}\partial_i\partial_j e_k^{ij}$.

Taking L^2 inner product of $(4.5)_1$ and $(4.5)_3$ with a_k and O_k, respectively, and then adding the resulting equations to (4.6) together, we obtain

$$\begin{aligned} &\frac{1}{2}\Big(\|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2\Big) + \mu_0\|\Lambda e_k\|_{L^2}^2 + (\lambda_0+\mu_0)\|\Lambda d_k\|_{L^2}^2 \\ = \quad &(\dot{\Delta}_k G_1|a_k) + (\dot{\Delta}_k G_4|e_k) + (\dot{\Delta}_k G_3|O_k). \end{aligned} \tag{4.7}$$

To capture the dissipation with respect to (a,O), we next apply the operator Λ to $(4.5)_1$ and take the L^2 inner product of the resulting equation with $-d_k$. Also, we take the L^2 inner product of $(4.5)_2$ with $\Lambda^{-1}\partial_i\partial_j a_k$. Therefore, we add those resulting equations together and get

$$-\frac{d}{dt}(\Lambda a_k|d_k) + \|\Lambda a_k\|_{L^2}^2 - \|\Lambda d_k\|_{L^2}^2 - (\Lambda^2 d_k|\Lambda a_k) + (O_k^{ij}|\partial_i\partial_j a_k)$$

$$= \quad -(\Lambda\dot{\Delta}_k G_1|d_k) + (\dot{\Delta}_k G_4^{ij}|\Lambda^{-1}\partial_i\partial_j a_k). \tag{4.8}$$

On the other hand, we apply Λ to $(4.5)_3$ and then take the L^2 inner product of the resulting equation with e_k^{ij}. We also take the L^2 inner product of $(4.5)_2$ with ΛO_k^{ij}. By summing up those resulting equations, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\Lambda O_k|e_k) + \|\Lambda O_k\|_{L^2}^2 - \|\Lambda e_k\|_{L^2}^2 \\
&-(\lambda_0+\mu_0)(\Lambda O_k^{ij}|\partial_i\partial_j d_k) + \mu_0(\Lambda^2 e_k|\Lambda O_k) + (\partial_i\partial_j a_k|O_k^{ij}) \\
= \quad & (\Lambda\dot{\Delta}_k G_3|e_k) + (\dot{\Delta}_k G_4|\Lambda O_k).
\end{aligned} \tag{4.9}$$

Now, we multiply a small constant $\nu_1 > 0$ (to be determined) to (4.8) and (4.9), respectively, and then add the resulting equations with (4.7) together. Consequently, we are led to the following inequality

$$\begin{aligned}
&\frac{1}{2}\Big(\|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2 + 2\nu_1(\Lambda O_k|e_k) - 2\nu_1(\Lambda a_k|d_k)\Big) \\
&+(\mu_0-\nu_1)\|\Lambda e_k\|_{L^2}^2 + (\lambda_0+\mu_0-\nu_1)\|\Lambda d_k\|_{L^2}^2 + \nu_1(\|\Lambda a_k\|_{L^2}^2 + \|\Lambda O_k\|_{L^2}^2) \\
&+\nu_1\mu_0(\Lambda^2 e_k|\Lambda O_k) - \nu_1(\lambda_0+\mu_0)(\Lambda O_k^{ij}|\partial_i\partial_j d_k) - \nu_1(\Lambda^2 d_k|\Lambda a_k) + 2\nu_1(\partial_i\partial_j a_k|O_k^{ij}) \\
&= (\dot{\Delta}_k G_1|a_k) + (\dot{\Delta}_k G_4|e_k) + (\dot{\Delta}_k G_3|O_k) - \nu_1(\Lambda\dot{\Delta}_k G_1|d_k) \\
&+\nu_1(\dot{\Delta}_k G_4^{ij}|\Lambda^{-1}\partial_i\partial_j a_k) + \nu_1(\Lambda\dot{\Delta}_k G_3|e_k) + \nu_1(\dot{\Delta}_k G_4|\Lambda O_k).
\end{aligned} \tag{4.10}$$

It follows from (3.8) that

$$\begin{aligned}
(\partial_i\partial_j a_k|O_k^{ij}) &= (a_k|\partial_i\partial_j O_k^{ij}) \\
&= \big((-\Delta a_k - \partial_j\dot{\Delta}_k G_0^j)|a_k\big) \\
&= \|\Lambda a_k\|_{L^2}^2 - (a_k|\partial_j\dot{\Delta}_k G_0^j).
\end{aligned} \tag{4.11}$$

Inserting (4.11) into (4.10), we have

$$\begin{aligned}
&\frac{d}{dt}f_{\ell,k}^2 + \tilde{f}_{\ell,k}^2 \\
= \quad & (\dot{\Delta}_k G_1|a_k) + (\dot{\Delta}_k G_4|e_k) + (\dot{\Delta}_k G_3|O_k) \\
& -\nu_1(\Lambda\dot{\Delta}_k G_1|d_k) + \nu_1(\dot{\Delta}_k G_4^{ij}|\Lambda^{-1}\partial_i\partial_j a_k) + \nu_1(\Lambda\dot{\Delta}_k G_3|e_k) \\
& +\nu_1(\dot{\Delta}_k G_4|\Lambda O_k) + 2\nu_1(a_k|\partial_j\dot{\Delta}_k G_0^j),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
f_{\ell,k}^2 \quad &\triangleq \quad \|a_k\|_{L^2}^2 + \|O_k\|_{L^2}^2 + \|e_k\|_{L^2}^2 + 2\nu_1(\Lambda O_k|e_k) - 2\nu_1(\Lambda a_k|d_k), \\
\tilde{f}_{\ell,k}^2 \quad &\triangleq \quad (\mu_0-\nu_1)\|\Lambda e_k\|_{L^2}^2 + (\lambda_0+\mu_0-\nu_1)\|\Lambda d_k\|_{L^2}^2 + 3\nu_1\|\Lambda a_k\|_{L^2}^2 \\
&\quad +\nu_1\|\Lambda O_k\|_{L^2}^2 + \nu_1\mu_0(\Lambda^2 e_k|\Lambda O_k) - \nu_1(\lambda_0+\mu_0)(\Lambda O_k^{ij}|\partial_i\partial_j d_k) \\
&\quad -\nu_1(\Lambda^2 d_k|\Lambda a_k).
\end{aligned}$$

For any fixed R_0, we choose $\nu_1 \sim \nu_1(\lambda_0, \mu_0, R_0)$ sufficiently small such that

$$\begin{aligned}
&f_{\ell,k}^2 \sim \|a_k\|_{L^2}^2 + \|e_k\|_{L^2}^2 + \|O_k\|_{L^2}^2, \\
&\tilde{f}_{\ell,k}^2 \sim 2^{2k}\big(\|a_k\|_{L^2}^2 + \|e_k\|_{L^2}^2 + \|O_k\|_{L^2}^2\big).
\end{aligned} \tag{4.13}$$

By using Cauchy-Schwarz inequality in (4.12), we get the following equality owing to $2^k \le R_0$,

$$\frac{d}{dt}f_{\ell,k} + 2^{2k}f_{\ell,k} \lesssim \sum_{i=0,1,3,4}\|\dot{\Delta}_k G_i\|_{L^2}, \tag{4.14}$$

which indicates that

$$\begin{aligned}&\|(a,O;e)\|^{\ell}_{\tilde{L}^{\infty}_T\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|(a,O;e)\|^{\ell}_{L^1_T\dot{B}^{\frac{n}{2}+1}_{2,1}}\\ \lesssim\ &\|(a_0,O_0;e_0)\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\sum_{i=0,1,3,4}\|G_i\|^{\ell}_{L^1_T\dot{B}^{\frac{n}{2}-1}_{2,1}}.\end{aligned}\tag{4.15}$$

Next we begin to bound those nonlinear terms arising in $G_i(i=0,1,3,4)$. Since the quadratic terms containing a and v have already been done in [5], it suffices to deal with different terms involving in O as well as those cubic terms due to density-dependent viscosities. More precisely, we need to estimate the following terms according to those definitions of G_i,

$$\begin{aligned}&G_0^j:=\partial_i(aO^{ij}),\ G_3^{ij}:=\partial_k v^iO^{kj}-v\cdot\nabla O^{ij},\\ &\Lambda^{-1}\partial_j(O^{lk}\partial_lO^{ik}),\ \Lambda^{-1}\partial_k(O^{lj}\partial_lO^{ik}),\ \Lambda^{-1}\partial_k(O^{lk}\partial_lO^{ij})\text{ in }G_4^{ij},\end{aligned}\tag{4.16}$$

and

$$\Lambda^{-1}\partial_j\Big(\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v)+\tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)\Big)^i\text{ in }G_4^{ij}.\tag{4.17}$$

We write $G_0^j=\partial_iaO^{ij}+a\partial_iO^{ij}$. Regarding ∂_iaO^{ij}, by taking $\gamma=-1,r_1=\infty,r_2=1,r_3=r_4=2,s_1=s_2=n/2-1,t_1=t_2=n/2$ in (A.2) and using (4.1), we arrive at

$$\begin{aligned}&\sum_{2^k\le R_0}2^{k(n/2-1)}\|\dot{\Delta}_k\big(\partial_iaO^{ij}\big)\|_{L^1_TL^2}\\ \lesssim\ &\|O\|_{\tilde{L}^{\infty}_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|\nabla a\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2,n/p-1}_{2,p}}+\|\nabla a\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\ \lesssim\ &\|O\|_{\tilde{L}^{\infty}_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\|a\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}}+\|a\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\ \lesssim\ &\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}.\end{aligned}\tag{4.18}$$

The terms $a\partial_iO^{ij}$, $v\cdot\nabla O^{ij}$ in G_3^{ij} and (4.16) may be treated along the same lines as ∂_iaO^{ij}, so we omit the details for brevity. In order to bound $\partial_kv^iO^{kj}$ in G_3^{ij}, by taking $\gamma=0,r_1=\infty,r_2=1,r_3=r_4=2,s_1=s_2=n/2-1,t_1=t_2=n/2$ in (A.2) and using (4.1), we have

$$\begin{aligned}&\sum_{2^k\le R_0}2^{k(n/2-1)}\|\dot{\Delta}_k(\partial_kv^iO^{kj})\|_{L^1_TL^2}\\ \lesssim\ &\|O\|_{\tilde{L}^{\infty}_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|\nabla v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}+\|\nabla v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\ \lesssim\ &\|O\|_{\tilde{L}^{\infty}_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\|v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}+\|v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\ \lesssim\ &\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}.\end{aligned}\tag{4.19}$$

Next we bound the cubic term (4.17) in G_4^{ij}. Denote

$$\begin{aligned}I:=&\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v)\big)\\ =&\frac{1}{1+a}\tilde{\mu}(a)\nabla^2v+\frac{1}{1+a}\nabla\tilde{\mu}(a)\nabla v\\ :=&I_1+I_2.\end{aligned}$$

To bound I_1, we have

$$\sum_{2^k\le R_0}2^{k(n/2-1)}\|\dot{\Delta}_k(\frac{1}{1+a}\tilde{\mu}(a)\nabla^2v)\|_{L^1_TL^2}$$

$$
\begin{aligned}
&\lesssim \sum_{2^k\le R_0} 2^{k(n/2-1)}\Big(\|\dot\Delta_k(I(a)\tilde\mu(a)\nabla^2 v)\|_{L^1_T L^2}+\|\dot\Delta_k(\tilde\mu(a)\nabla^2 v)\|_{L^1_T L^2}\Big)\\
&\lesssim \|I(a)\|_{\tilde L^\infty_T \dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|\tilde\mu(a)\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2,n/p-1}_{2,p}}\\
&\quad+\|\tilde\mu(a)\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2,n/p}_{2,p}}+\|\tilde\mu(a)\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\\
&\lesssim \Big(1+\|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}}\Big)\|\tilde\mu(a)\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}, \qquad (4.20)
\end{aligned}
$$

where we have chosen $s_1=s_2=n/2-1, t_1=t_2=n/2, r_1=r_4=\infty, r_2=r_3=1, \gamma=-1$ in (A.2) of Proposition A.1 to deal with the term $I(a)\tilde\mu(a)\nabla^2 v$. Now we begin to bound $\|\tilde\mu(a)\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}$. From (A.2) and (A.1), we have

$$
\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot\Delta_k(\tilde\mu(a)\nabla^2 v)\|_{L^1_T L^2}\\
&\lesssim \|\tilde\mu(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2,n/p-1}_{2,p}}\\
&\qquad+\|\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|\tilde\mu(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2,n/p}_{2,p}}\\
&\lesssim \|\tilde\mu(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2+1,n/p+1}_{2,p}}. \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{2^k> R_0} 2^{k(n/p-1)}\|\dot\Delta_k(\tilde\mu(a)\nabla^2 v)\|_{L^1_T L^p}\\
&\lesssim \|\tilde\mu(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2,n/p}_{2,p}}\|\nabla^2 v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}. \qquad (4.22)
\end{aligned}
$$

Inserting (4.21) and (4.22) into (4.20), with aid of Proposition A.2, we can get

$$
\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot\Delta_k(\frac{1}{1+a}\tilde\mu(a)\nabla^2 v)\|_{L^1_T L^2}\\
&\lesssim (1+\|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}})\|\tilde\mu(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2+1,n/p+1}_{2,p}}.\\
&\lesssim (1+\|a\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/p,n/p}_{2,p}})^{n+3}\|a\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}}\|v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2+1,n/p+1}_{2,p}}\\
&\lesssim (1+\|(a,O;v)\|_{\mathcal E^{n/p}_T})^{n+3}\|(a,O;v)\|^2_{\mathcal E^{n/p}_T}. \qquad (4.23)
\end{aligned}
$$

To bound I_2, we have

$$
\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot\Delta_k(\frac{1}{1+a}\nabla\tilde\mu(a)\nabla v)\|_{L^1_T L^2}\\
&\lesssim \sum_{2^k\le R_0} 2^{k(n/2-1)}\Big(\|\dot\Delta_k(I(a)\nabla\tilde\mu(a)\nabla v)\|_{L^1_T L^2}+\|\dot\Delta_k(\nabla\tilde\mu(a)\nabla v)\|_{L^1_T L^2}\Big)\\
&\lesssim \|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|\nabla\tilde\mu(a)\nabla v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2,n/p-1}_{2,p}}\\
&\quad+\|\nabla\tilde\mu(a)\nabla v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2,n/p}_{2,p}}+\|\nabla\tilde\mu(a)\nabla v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\\
&\lesssim \Big(1+\|I(a)\|_{\tilde L^\infty_T\dot{\mathcal B}^{n/2-1,n/p}_{2,p}}\Big)\|\nabla\tilde\mu(a)\nabla v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}, \qquad (4.24)
\end{aligned}
$$

From (A.2) and (A.1), bounding $\|\nabla\tilde\mu(a)\nabla v\|_{\tilde L^1_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}$ is as follows

$$
\begin{aligned}
&\sum_{2^k\le R_0} 2^{k(n/2-1)}\|\dot\Delta_k(\nabla\tilde\mu(a)\nabla v)\|_{L^1_T L^2}\\
&\lesssim \|\nabla v\|_{\tilde L^2_T\dot{\mathcal B}^{n/2-1,n/p-1}_{2,p}}\|\nabla\tilde\mu(a)\|_{\tilde L^2_T\dot{\mathcal B}^{n/2,n/p-1}_{2,p}}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla\tilde{\mu}(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\|\nabla v\|_{L^1_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\
\lesssim\ &\|v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\tilde{\mu}(a)\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\
&+\|\tilde{\mu}(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\|v\|_{L^1_T\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}. \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{2^k>R_0}2^{k(n/p-1)}\|\dot{\Delta}_k(\nabla\tilde{\mu}(a)\nabla v)\|_{L^1_TL^p}\\
\lesssim\ &\|\nabla\tilde{\mu}(a)\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\nabla v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\\
\lesssim\ &\|v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\tilde{\mu}(a)\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}} \qquad (4.26)
\end{aligned}
$$

Inserting (4.25) and (4.26) into (4.24), by Proposition A.2 and (4.1), we can get

$$
\begin{aligned}
&\sum_{2^k\le R_0}2^{k(n/2-1)}\|\dot{\Delta}_k(\frac{1}{1+a}\nabla\tilde{\mu}(a)\nabla v)\|_{L^1_TL^2}\\
\lesssim\ &(1+\|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}})\Big(\|\tilde{\mu}(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\|v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}.\\
&\quad+\|v\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\tilde{\mu}(a)\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\Big)\\
\lesssim\ &(1+\|(a,O;v)\|_{\mathcal{E}^{n/p}_T})^{n+3}\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}. \qquad (4.27)
\end{aligned}
$$

bounding the cubic term $\frac{1}{1+a}\mathrm{div}\big(\tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)$ is same as I, we feel free to omit details. Summing up all the estimates and remembering (4.15), we conclude that

$$
\begin{aligned}
&\|(a,O;e)\|^\ell_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}}+\|(a,O;e)\|^\ell_{L^1_T\dot{B}^{n/2+1}_{2,1}}\\
\lesssim\ &\|(a_0,O_0;e_0)\|^\ell_{\dot{B}^{n/2-1}_{2,1}}+(1+\|(a,O;v)\|_{\mathcal{E}^{n/p}_T})^{n+3}\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}. \qquad (4.28)
\end{aligned}
$$

Step 2. High-frequency estimates ($2^k>R_0$).

Inspired by [15, 16], we perform basic energy approaches in terms of *effective velocities* rather than the Lagrangian change as in [3, 5]. Denote by $\tilde{d}=-\nabla(-\Delta)^{-1}\mathrm{div}v$ the compressible part of v. It is easy to see that $\|\tilde{d}\|_{\tilde{L}^q_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}}\approx\|d\|_{\tilde{L}^q_T\dot{\mathcal{B}}^{s,\sigma}_{2,p}}$. It follows from the first equality in (3.1) that

$$
\begin{aligned}
&-\nabla(-\Delta)^{-1}\mathrm{div}(\nabla\cdot O)\\
=\ &-\nabla(-\Delta)^{-1}\Big(\partial_i\partial_j[(1+a)(\delta^{ij}+O^{ij})]\Big)+\nabla(-\Delta)^{-1}\mathrm{divdiv}(aI+aO)\\
=\ &\nabla(-\Delta)^{-1}\mathrm{divdiv}(aI+aO)\\
=\ &-\nabla a+\nabla(-\Delta)^{-1}\mathrm{divdiv}(aO). \qquad (4.29)
\end{aligned}
$$

Note that (4.29), we get the following equation for the compressible part of v

$$\partial_t\tilde{d}-\Delta\tilde{d}+2\nabla a=G_2, \qquad (4.30)$$

where

$$
\begin{aligned}
G_2=-\nabla(-\Delta)^{-1}\mathrm{div}\Big(&-v\cdot\nabla v+O^{jk}\partial_jO^{\bullet k}-I(a)\mathcal{A}v\\
&-K(a)\nabla a-\mathrm{div}(aO)+\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v)+\tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)\Big). \qquad (4.31)
\end{aligned}
$$

Here, we consider more complicated hyperbolic-parabolic coupled system

$$\begin{cases} \partial_t a + v\cdot\nabla a + \mathrm{div} v = \tilde{G}_1, \\ \partial_t \tilde{d} - \Delta\tilde{d} + 2\nabla a = G_2, \\ \partial_t O^{ij} + v\cdot\nabla O^{ij} - \Lambda e^{ij} = \tilde{G}_3^{ij}, \\ \partial_t e^{ij} - \mu_0\Delta e^{ij} + \Lambda O^{ij} = \tilde{G}_4^{ij} \end{cases} \tag{4.32}$$

where

$$\tilde{G}_1 = -a\nabla\cdot v, \quad \tilde{G}_3^{ij} = \partial_k v^i O^{kj}, \quad \tilde{G}_4^{ij} = G_4^{ij} + (\lambda_0+\mu_0)\partial_i\partial_j d - \Lambda^{-1}\partial_i\partial_j a. \tag{4.33}$$

Introduce two *effective velocities* as follows

$$w = \tilde{d} + 2\nabla(-\Delta)^{-1}a = \nabla(-\Delta)^{-1}(2a - \mathrm{div} v),\ \Omega^{ij} = e^{ij} + \frac{1}{\mu_0}\Lambda(-\Delta)^{-1}O^{ij}.$$

Note that the definition of w is almost same as that in [15, 16]. The subtle difference lies in the coefficient of unknown a, which comes from the coupling of deformation gradient F, see (4.29). The new *effective velocity* Ω^{ij} is used to cancel the coupling between e^{ij} and O^{ij} in the high-frequency estimate.

Firstly, we present those estimates for effective velocities. It follows from (4.32) that

$$\begin{cases} \partial_t w - \Delta w = G_2 + 2\nabla(-\Delta)^{-1}\tilde{G}_1 + 2w - 4\nabla(-\Delta)^{-1}a, \\ \partial_t\Omega^{ij} - \mu_0\Delta\Omega^{ij} = \tilde{G}_4^{ij} + \frac{1}{\mu_0}\Lambda^{-1}\tilde{G}_3^{ij} + \frac{1}{\mu_0}\Omega^{ij} - \frac{1}{\mu_0^2}\Lambda^{-1}O^{ij}. \end{cases} \tag{4.34}$$

Applying (A.6) to the above equations implies that

$$\begin{aligned} \|w\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} \lesssim{}& \|w_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + \|w\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}} + \|a\|^h_{L^1_T\dot{B}^{n/p-2}_{p,1}} \\ &+ \|\tilde{G}_1\|^h_{L^1(\dot{B}^{n/p-2}_{p,1})} + \|G_2\|^h_{L^1(\dot{B}^{n/p-1}_{p,1})}, \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \|\Omega^{ij}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} \lesssim{}& \|\Omega^{ij}_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + \|\Omega^{ij}\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}} + \|O^{ij}\|^h_{L^1_T\dot{B}^{n/p-2}_{p,1}} \\ &+ \|\tilde{G}_3^{ij}\|^h_{L^1(\dot{B}^{n/p-2}_{p,1})} + \|\tilde{G}_4^{ij}\|^h_{L^1(\dot{B}^{n/p-1}_{p,1})}. \end{aligned} \tag{4.36}$$

Owing to the high frequency cut-off $2^k > R_0$, we have

$$\|w\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}} \lesssim R_0^{-2}\|w\|^h_{L^1_T\dot{B}^{n/p+1}_{p,1}}, \quad \|a\|^h_{L^1_T\dot{B}^{n/p-2}_{p,1}} \lesssim R_0^{-2}\|a\|^h_{L^1_T\dot{B}^{n/p}_{p,1}},$$

and

$$\|\Omega^{ij}\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}} \lesssim R_0^{-2}\|w^{ij}\|^h_{L^1_T\dot{B}^{n/p+1}_{p,1}}, \quad \|O^{ij}\|^h_{L^1_T\dot{B}^{n/p-2}_{p,1}} \lesssim R_0^{-2}\|O^{ij}\|^h_{L^1_T\dot{B}^{n/p}_{p,1}}.$$

Choosing $R_0 > 0$ sufficient large, the terms $\|w\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}}$ and $\|\Omega^{ij}\|^h_{L^1_T\dot{B}^{n/p-1}_{p,1}}$ on the right-side of (4.35) and (4.36) can be absorbed by the corresponding parts in the left-hand side. Consequently, we conclude that

$$\begin{aligned} \|w\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} \lesssim{}& \|w_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + R_0^{-2}\|a\|^h_{L^1_T\dot{B}^{n/p}_{p,1}} \\ &+ \|\tilde{G}_1\|^h_{L^1(\dot{B}^{n/p-2}_{p,1})} + \|G_2\|^h_{L^1(\dot{B}^{n/p-1}_{p,1})}, \end{aligned} \tag{4.37}$$

and

$$\|\Omega^{ij}\|^h_{\tilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} \lesssim \|\Omega^{ij}_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + R_0^{-2}\|O^{ij}\|^h_{L^1_T\dot{B}^{n/p}_{p,1}} + \|\tilde{G}^{ij}_3\|^h_{L^1(\dot{B}^{n/p-2}_{p,1})} + \|\tilde{G}^{ij}_4\|^h_{L^1(\dot{B}^{n/p-1}_{p,1})}. \tag{4.38}$$

Secondly, we see that (a, O^{ij}) satisfies the following damped equations in terms of effective velocities

$$\begin{cases} \partial_t a + v\cdot\nabla a + 2a = \tilde{G}_1 - \nabla\cdot w, \\ \partial_t O^{ij} + v\cdot\nabla O^{ij} + \dfrac{1}{\mu_0}O^{ij} = \tilde{G}^{ij}_3 + \Lambda\Omega^{ij}. \end{cases}$$

Applying $\dot{\Delta}_k$ to the above equations, we obtain

$$\begin{cases} \partial_t \dot{\Delta}_k a + v\cdot\nabla\dot{\Delta}_k a + 2\dot{\Delta}_k a = \dot{\Delta}_k\tilde{G}_1 - \dot{\Delta}_k\nabla\cdot w + R^1_k, \\ \partial_t \dot{\Delta}_k O^{ij} + v\cdot\nabla\dot{\Delta}_k O^{ij} + \dfrac{1}{\mu_0}\dot{\Delta}_k O^{ij} = \dot{\Delta}_k\tilde{G}^{ij}_3 + \dot{\Delta}_k\Lambda\Omega^{ij} + R^2_k, \end{cases} \tag{4.39}$$

where $R^1_k := [v\cdot\nabla, \dot{\Delta}_k]a$ and $R^2_k := [v\cdot\nabla, \dot{\Delta}_k]O^{ij}$. Multiplying $(4.39)_1$ by $\dot{\Delta}_k a|\dot{\Delta}_j a|^{p-2}$ and $(4.39)_2$ by $\dot{\Delta}_k O^{ij}|\dot{\Delta}_k O^{ij}|^{p-2}$, and then integrating over $\mathbb{R}^n\times[0,t]$, we can obtain

$$\|\dot{\Delta}_k a(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_k a\|_{L^p}d\tau \lesssim \|\dot{\Delta}_k a_0\|_{L^p} + \int_0^t \|\nabla v\|_{L^\infty}\|\dot{\Delta}_k a\|_{L^p}d\tau + \int_0^t \|\dot{\Delta}_k(\tilde{G}_1 - \Lambda w)\|_{L^p}d\tau + \int_0^t \|R^1_k\|_{L^p}d\tau \tag{4.40}$$

and

$$\|\dot{\Delta}_k O^{ij}(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_k O^{ij}\|_{L^p}d\tau \lesssim \|\dot{\Delta}_k O^{ij}_0\|_{L^p} + \int_0^t \|\nabla v\|_{L^\infty}\|\dot{\Delta}_k O^{ij}\|_{L^p}d\tau + \int_0^t \|\dot{\Delta}_k(\tilde{G}^{ij}_3 + \Lambda\Omega^{ij})\|_{L^p}d\tau + \int_0^t \|R^2_k\|_{L^p}d\tau. \tag{4.41}$$

It follows from commutator estimates in [2] that

$$\sum_{j\in\mathbb{Z}} 2^{js}\|(R^1_j, R^2_j)\|_{L^p} \lesssim \|\nabla v\|_{\dot{B}^{n/p}_{p,1}}\|(a, O^{ij})\|_{\dot{B}^s_{p,1}}.$$

Now multiplying (4.40) and (4.41) by $2^{k\frac{n}{p}}$, respectively, and then summing over the index k satisfying $2^k > R_0$, we are led to

$$\begin{aligned} &\|a\|^h_{\tilde{L}^\infty_T \dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} \\ \lesssim\ & \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\nabla v\|_{L^1_T\dot{B}^{n/p}_{p,1}}\|a\|_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}} + \|\tilde{G}_1\|^h_{L^1_T\dot{B}^{n/p}_{p,1}} + \|w\|^h_{L^1_T(\dot{B}^{n/p+1}_{p,1})} \\ \lesssim\ & \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|w\|^h_{L^1_T(\dot{B}^{n/p+1}_{p,1})} + \|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}_1\|^h_{L^1_T\dot{B}^{n/p}_{p,1}} \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} &\|O^{ij}\|^h_{\tilde{L}^\infty_T \dot{B}^s_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} \\ \lesssim\ & \|O^{ij}_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\nabla v\|_{L^1_T\dot{B}^{n/p}_{p,1}}\|O^{ij}\|_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}} + \|\tilde{G}^{ij}_3\|^h_{L^1_T\dot{B}^{n/p}_{p,1}} + \|\Omega^{ij}\|^h_{L^1_T(\dot{B}^{n/p+1}_{p,1})} \\ \lesssim\ & \|O^{ij}_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\Omega^{ij}\|^h_{L^1_T(\dot{B}^{n/p+1}_{p,1})} + \|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}^{ij}_3\|^h_{L^1_T\dot{B}^{n/p}_{p,1}}. \end{aligned} \tag{4.43}$$

Multiply (4.42) and (4.43) by $\delta > 0$ respectively, and then add two resulting inequalities to (4.37) and (4.38) together. By choosing R_0 sufficiently large, we can get

$$\begin{aligned}\|a\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} + \|w\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} &\lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|w_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ &+\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}_1\|^h_{L^1_T(\dot{B}^{n/p}_{p,1})} + \|G_2\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})},\end{aligned}$$

and

$$\begin{aligned}\|O^{ij}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} + \|\Omega^{ij}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} &\lesssim \|O^{ij}_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\Omega^{ij}_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ &+\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}^{ij}_3\|^h_{L^1_T(\dot{B}^{n/p}_{p,1})} + \|\tilde{G}^{ij}_4\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})}.\end{aligned}$$

Keep in mind that $w = \tilde{d} + 2\nabla(-\Delta)^{-1}a,\ \Omega^{ij} = e^{ij} + \frac{1}{\mu_0}(-\Delta)^{-1}\Lambda O^{ij}$, we arrive at

$$\begin{aligned}\|a\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} + \|\tilde{d}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} &\lesssim \|a_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|\tilde{d}_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ &+\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}_1\|^h_{L^1_T(\dot{B}^{n/p}_{p,1})} + \|G_2\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})}\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\|O^{ij}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} + \|e^{ij}\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} &\lesssim \|O^{ij}_0\|^h_{\dot{B}^{n/p}_{p,1}} + \|e^{ij}_0\|^h_{\dot{B}^{n/p-1}_{p,1}} \\ &+\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|\tilde{G}^{ij}_3\|^h_{L^1_T(\dot{B}^{n/p}_{p,1})} + \|\tilde{G}^{ij}_4\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})}.\end{aligned} \tag{4.45}$$

In addition, remembering (4.33), we have

$$\|\tilde{G}^{ij}_4\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \lesssim \|G^{ij}_4\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})} + \|\tilde{d}\|^h_{L^1_T(\dot{B}^{n/p+1}_{p,1})} + \|a\|^h_{L^1(\dot{B}^{n/p}_{p,1})} \tag{4.46}$$

Hence, together with (4.44)-(4.46), we deduce that

$$\begin{aligned}&\|(a,O)\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}} + \|e\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}} \\ \lesssim\ &\|(a_0,O_0)\|^h_{\dot{B}^{n/p}_{p,1}} + \|e_0\|^h_{\dot{B}^{n/p-1}_{p,1}} + \|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T} + \|(\tilde{G}_1,\tilde{G}_3)\|^h_{L^1_T(\dot{B}^{n/p}_{p,1})} \\ &+\|(G_2,G_4)\|^h_{L^1_T(\dot{B}^{n/p-1}_{p,1})}.\end{aligned} \tag{4.47}$$

Likely, we need to bound those different terms in $\tilde{G}_i(i=1,3)$ and $G_i(i=2,4)$ compared to [5], for example,

$$\tilde{G}^{ij}_3 := \partial_k v^i O^{kj},$$

$$O\nabla O,\ \mathrm{div}(aO) \text{ in } G_2 \text{ and } G_4,$$

and

$$\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big) \text{ in } G_2 \text{ and } G_4. \tag{4.48}$$

In order to bound $\partial_k v^i O^{kj}$, from (A.1) of Proposition A.1 with $,r_1 = 1, r_2 = \infty, \sigma = \tau = n/p$, we have

$$\begin{aligned}&\sum_{2^k > R_0} 2^{k(n/p)}\|\dot{\Delta}_k(\partial_k v^i O^{kj})\|_{L^1_T L^p} \\ \lesssim\ &\|\nabla v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|O\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}} \lesssim \|v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}}\|O\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}} \\ \lesssim\ &\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}.\end{aligned}$$

For $O\nabla O$, from (A.1) of Proposition A.1 with $r_1=r_2=2, \sigma=n/p, \tau=n/p-1$ and by applying interpolation (4.1), we have

$$\begin{aligned}
&\sum_{2^k>R_0} 2^{k(n/p-1)}\|\dot{\Delta}_k\big(O\nabla O\big)\|_{L^1_T L^p}\\
\lesssim\ & \|O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\nabla O\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}} \lesssim \|O\|^2_{\tilde{L}^2_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\\
\lesssim\ & \|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}. \qquad (4.49)
\end{aligned}$$

Bounding $\operatorname{div}(aO)=a\nabla\cdot O+\nabla aO$ may be handled with at the same away as $O\nabla O$. Next, we handle the cubic term (4.48) in G^{ij}_4. Following from the the same notation, we have

$$\begin{aligned}
I &= \frac{1}{1+a}\operatorname{div}\big(2\tilde{\mu}(a)D(v)\big)\\
&= \frac{1}{1+a}\tilde{\mu}(a)\nabla^2 v+\frac{1}{1+a}\nabla\tilde{\mu}(a)\nabla v\\
&\triangleq I_1+I_2.
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
&\sum_{2^k>R_0} 2^{k(n/p-1)}\|\dot{\Delta}_k(\frac{1}{1+a}\tilde{\mu}(a)\nabla^2 v)\|_{L^1_T L^p}\\
\lesssim\ & \sum_{2^k>R_0} 2^{k(n/p-1)}\Big(\|\dot{\Delta}_k(I(a)\tilde{\mu}(a)\nabla^2 v)\|_{L^1_T L^p}+\|\dot{\Delta}_k(\tilde{\mu}(a)\nabla^2 v)\|_{L^1_T L^p}\Big)\\
\lesssim\ & \|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\tilde{\mu}(a)\nabla^2 v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}+\|\tilde{\mu}(a)\nabla^2 v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\\
\lesssim\ & \Big(1+\|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\Big)\|\tilde{\mu}(a)\nabla^2 v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\\
\lesssim\ & (1+\|(a,O;v)\|_{\mathcal{E}^{n/p}_T})^{n+3}\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T},
\end{aligned}$$

where the third line is followed by taking $\sigma=n/p, \tau=n/p-1, r_1=\infty, r_2=1$ in (A.1).

On the other hand, regarding I_2, we deduce that

$$\begin{aligned}
&\sum_{2^k>R_0} 2^{k(n/p-1)}\|\dot{\Delta}_k(\frac{1}{1+a}\nabla\tilde{\mu}(a)\nabla v)\|_{L^1_T L^p}\\
\lesssim\ & \sum_{2^k>R_0} 2^{k(n/p-1)}\Big(\|\dot{\Delta}_k(I(a)\nabla\tilde{\mu}(a)\nabla v)\|_{L^1_T L^p}+\|\dot{\Delta}_k(\nabla\tilde{\mu}(a)\nabla v)\|_{L^1_T L^p}\Big)\\
\lesssim\ & \|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2,n/p}_{2,p}}\|\nabla\tilde{\mu}(a)\nabla v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}+\|\nabla\tilde{\mu}(a)\nabla v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\\
\lesssim\ & \Big(1+\|I(a)\|_{\tilde{L}^\infty_T\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\Big)\|\nabla\tilde{\mu}(a)\nabla v\|_{\tilde{L}^1_T\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\\
\lesssim\ & (1+\|(a,O;v)\|_{\mathcal{E}^{n/p}_T})^{n+3}\|(a,O;v)\|^2_{\mathcal{E}^{n/p}_T}.
\end{aligned}$$

The computation for $\frac{1}{1+a}\operatorname{div}\big(\tilde{\lambda}(a)\operatorname{div}v\mathrm{Id}\big)$ totally follows from the same procedure as I, so we omit details. By putting above estimates together, remembering (4.47), we achieve that

$$\begin{aligned}
&\|(a,O)\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}}+\|e\|^h_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}}\\
\lesssim\ & \|(a_0,O_0)\|^h_{\dot{B}^{n/p}_{p,1}}+\|e_0\|^h_{\dot{B}^{n/p-1}_{p,1}}
\end{aligned}$$

$$+(1+\|(a,O;v)\|_{\mathcal{E}_T^{n/p}})^{n+3}\|(a,O;v)\|^2_{\mathcal{E}_T^{n/p}}. \tag{4.50}$$

Step 3. Combination of two-step analysis.

The inequality (4.3) is the consequence of (4.28) and (4.50), so the proof of Proposition 4.1 is finished. By using (A.3) in Proposition A.1, we can infer that

$$\begin{aligned}&\|(a,O;e)\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{n/2-1}_{2,1}}+\|(a,O;e)\|^{\ell}_{L^1_T\dot{B}^{n/2+1}_{2,1}}\\ \lesssim\;&\|(a_0,O_0;e_0)\|^{\ell}_{\dot{B}^{n/2-1}_{2,1}}\\ &+(1+\|(a,O;v)\|_{\mathcal{E}_T^{n/p}})^{n+3}\|(a,O;v)\|_{\mathcal{E}_T^{n/p}}\|(a,O;v)\|_{E_T^{n/p}}\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}&\|(a,O)\|^{h}_{\tilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap L^1_T\dot{B}^{n/p}_{p,1}}+\|e\|^{h}_{\tilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap L^1_T\dot{B}^{n/p+1}_{p,1}}\\ \lesssim\;&\|(a_0,O_0)\|^{h}_{\dot{B}^{n/p}_{p,1}}+\|e_0\|^{h}_{\dot{B}^{n/p-1}_{p,1}}\\ &+(1+\|(a,O;v)\|_{\mathcal{E}_T^{n/p}})^{n+3}\|(a,O;v)\|_{\mathcal{E}_T^{n/p}}\|(a,O;v)\|_{E_T^{n/p}}.\end{aligned} \tag{4.52}$$

The inequality (4.4) is followed by (4.51) and (4.52). Therefore, the proof of Proposition 4.2 is complete.

4.2. **Approximate solutions and uniform estimates.** The construction of approximate solutions is based on the following local-in-time existence.

Theorem 4.1 ([30])**.** *Assume $(\rho_0-1,F_0-I)\in\big(\dot{B}^{n/2}_{2,1}\big)^{1+n^2}$ and $u_0\in\big(\dot{B}^{n/2-1}_{2,1}\big)^n$ with ρ_0 bounded away from 0. There exists a positive time T such that system (1.1) has a unique solution $(\rho,F;u)$ with ρ bounded away from 0 and*

$$(\rho-1,F-I)\in\Big(C([0,T];\dot{B}^{n/2}_{2,1})\Big)^{1+n^2},u\in\Big(C([0,T];\dot{B}^{n/2-1}_{2,1})\cap L^1([0,T];\dot{B}^{n/2+1}_{2,1})\Big)^n.$$

Additionally, if $(\rho_0-1,F_0-I)\in\big(\dot{B}^{n/2-1}_{2,1}\big)^{1+n^2}$, we have

$$(\rho-1,F-I)\in\big(C([0,T];\dot{B}^{n/2-1}_{2,1})\big)^{1+n^2}.$$

In order to apply Theorem 4.1, we need a lemma, which can be shown by the proof of Lemma 4.2 in [1].

Lemma 4.1. *Let $p\geq 2$. For any*

$$(\rho_0-1,F_0-I;u_0)\in\Big(\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}\Big)^{1+n^2}\times\Big(\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}\Big)^n$$

satisfying $\rho_0\geq c_0>0$, then there exists a sequence $\{(\rho_{0,k},F_{0,k};u_{0,k})\}_{k\in\mathbb{N}}$ with $\{(\rho_{0,k}-1,F_{0,k}-I;u_{0,k})\}\in\Big(\dot{\mathcal{B}}^{n/2-1,n/2}_{2,2}\Big)^{1+n^2}\times\Big(\dot{B}^{n/2-1}_{2,1}\Big)^n$ such that

$$\|(\rho_{0,k}-\rho_0,F_{0,k}-F_0)\|_{\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}\longrightarrow 0,\quad \|u_{0,k}-u_0\|_{\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}}\longrightarrow 0 \tag{4.53}$$

when $k\to 0$. we also have $\rho_{0,k}\geq\frac{c_0}{2}$ for any $k\in\mathbb{N}$.

Let $(\rho_{0,k},F_{0,k};u_{0,k})$ be the sequence for initial data stated in Lemma 4.1. Then Theorem 4.1 indicates that there exists a maximal existence time $T_k>0$ such that System (1.1) with initial data $(\rho_{0,k},F_{0,k};u_{0,k})$ admits a unique solution $(\rho_k,F_k;u_k)$ with ρ_k bounded away from zero satisfying

$$(\rho_k-1,F_k-I)\in\Big(C([0,T_k];\dot{B}^{n/2}_{2,1}\cap\dot{B}^{n/2-1}_{2,1})\Big)^{1+n^2},$$

$$u_k \in \Big(C([0,T_k);\dot{B}^{n/2-1}_{2,1})\cap L^1(\dot{B}^{n/2+1}_{2,1})\Big)^n.$$

Then using the definition of Hybird Besov spaces and Bernstein inequality in Lemma 2.1, we have

$$(\rho_k-1, F_k-I)\in \Big(C([0,T_k);\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p})\Big)^{1+n^2},$$
$$u_k \in \Big(C([0,T_k);\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p})\cap L^1([0,T_k);\dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p})\Big)^n.$$

Set

$$a_k(t,x)=\rho_k(\chi_0^2 t,\chi_0 x)-1, v_k(t,x)=\chi_0 u_k(\chi_0^2 t,\chi_0 x), O_k(t,x)=F_k(\chi_0^2 t,\chi_0 x)-I.$$

From (1.3) and (4.53), we

$$\|(a_{0,k},O_{0,k};v_{0,k})\|_{\mathcal{E}^{n/p}_0}\le C_0\eta,$$

for some constant $C_0>0$. Let M be a constant (to be determined later). We define

$$T_k^*\triangleq \sup\{t\in[0,T_k)\big|\|(a_k,O_k;v_k)\|_{\mathcal{E}^{n/p}_t}\le M\eta\}.$$

First we claim that

$$T_k^*=T_k \qquad \forall k\in\mathbb{N}.$$

With the help of the continuity argument, it suffices to show for all $k\in\mathbb{N}$,

$$\|(a_k,O_k;v_k)\|_{\mathcal{E}^{n/p}_{T_k^*}}\le \frac{1}{2}M\eta. \tag{4.54}$$

Indeed, noting that $\|a_k\|_{L^\infty([0,T_k^*)\times\mathbb{R}^n)}\le C_1\|a^k\|_{L^\infty_{T_k^*}\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}}$, we can choose η sufficiently small such that

$$M\eta\le\frac{1}{2C_1}.$$

Then

$$\|a_k\|_{L^\infty([0,T_k^*)\times\mathbb{R}^n)}\le\frac{1}{2}.$$

By applying Proposition 4.1, we obtain

$$\|(a_k,O_k;v_k)\|_{\mathcal{E}^{n/p}_{T_k^*}}\le C\{C_0\eta+(M\eta)^2(1+M\eta)^{n+3}\}. \tag{4.55}$$

By choosing $M=3CC_0$ and η sufficient small enough such that

$$C(M\eta)(1+M\eta)^{n+3}\le\frac{1}{6},$$

so (4.54) is followed by (4.55) directly.

Therefore, we obtain a sequence of approximate solutions $(\rho_k,F_k;u_k)$ to the system (1.1) on $[0,T_k)$ satisfying

$$\|(a_k,O_k;v_k)\|_{\mathcal{E}^{n/p}_{T_k}}\le M\eta, \tag{4.56}$$

for any $k\in\mathbb{N}$. From (4.4) and (4.56), we have

$$\|(a_k,O_k;v_k)\|_{E^{n/2}_{T_k}}\le C\Big\{\|(a_{0,k},O_{0,k};v_{0,k})\|_{E^{n/2}_0} + \|(a_k,O_k;v_k)\|_{E^{n/2}_{T_k}}(M\eta)\big(1+M\eta\big)^{n+3}\Big\},$$

which implies

$$\|(a_k,O_k;v_k)\|_{E^{n/2}_{T_k}}\le C\|(a_{0,k},O_{0,k};v_{0,k})\|_{E^{n/2}_0}, \tag{4.57}$$

provided η is sufficiently small. Consequently, based on Proposition 4.2, the continuity argument ensures that $T_k = +\infty$ for any $k \in \mathbb{N}$.

4.3. **Passing to the limit and existence.** Next, the existence of the solution will be proved by the compact argument. We show that, up to an extraction, the sequence $(a_k, O_k; v_k)$ converges in the distributional sense to some function $(a, O; v)$ such that

$$(a, O; v) \in \Big(\tilde{L}^\infty \dot{\mathcal{B}}^{n/2-1,n/p}_{2,p} \cap L^1 \dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}\Big)^{1+n^2} \times \Big(\tilde{L}^\infty \dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p} \cap L^1 \dot{\mathcal{B}}^{n/2+1,n/p+1}_{2,p}\Big)^n. \tag{4.58}$$

Indeed, it follows from (4.56) that (a_k, O_k) is uniformly bounded in $\tilde{L}^\infty(0,\infty; \dot{B}^{n/p}_{p,1})$ and v_k is uniformly bounded in $\tilde{L}^\infty(0,\infty;\dot{B}^{n/p-1}_{p,1}) \cap L^1(0,\infty;\dot{B}^{n/p+1}_{p,1})$. By interpolation, we also deduce that v_k is uniformly bounded in $\tilde{L}^{\frac{2}{2-\varepsilon}}(0,\infty;\dot{B}^{n/p+1-\varepsilon}_{p,1})$ for any $\varepsilon \in [0,2]$. We claim that $(a_k, O_k; v_k)$ is uniformly bounded in

$$\Big(C^{1/2}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\Big)^{1+n^2} \times \Big(C^{\frac{2-\zeta}{2}}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1-\zeta}_{p,1})\Big)^n \tag{4.59}$$

with $\zeta = \min\{\frac{2n}{p} - 1, 1\}$, which is a direct consequence of

$$(\partial_t a_k, \partial_t O_k; \partial_t v_k) \in \Big(\tilde{L}^2_{loc}\dot{B}^{n/p-1}_{p,1}\Big)^{1+n^2} \times \Big(\tilde{L}^{\frac{2}{2-\zeta}}_{loc}\dot{B}^{n/p-1-\zeta}_{p,1}\Big)^n. \tag{4.60}$$

Recalling (3.4), we have

$$\partial_t a_k = -v_k \cdot \nabla a_k - \nabla \cdot v_k - a_k \nabla \cdot v_k$$

and

$$\partial_t O_k = -v_k \cdot \nabla O_k + \nabla v_k + \nabla v_k O_k.$$

By interpolation and Lemma 2.3, it follows from (4.56) that

$$(\partial_t a_k, \partial_t O_k) \in \Big(\tilde{L}^2_{loc}\dot{B}^{n/p-1}_{p,1}\Big)^{1+n^2},$$

which implies that (a_k, O_k) is uniformly bounded in $\Big(C^{1/2}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\Big)^{1+n^2}$. On the other hand,

$$\partial_t v_k = -v_k \cdot \nabla v_k + \mathcal{A} v_k - \nabla a_k + \nabla \cdot O_k + O^{jl}_k \partial_j O^{\bullet l}_k$$
$$-I(a_k)\mathcal{A}v_k - K(a_k)\nabla a_k + \frac{1}{1+a_k}\mathrm{div}\big(2\tilde{\mu}(a_k)D(v_k) + \tilde{\lambda}(a_k)\mathrm{div} v_k \mathrm{Id}\big).$$

It's easy to see that

$$\|\mathcal{A}v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}} \lesssim \|v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p+1-\zeta}_{p,1}}. \tag{4.61}$$

Thanks to Lemma 2.3 and Proposition A.2, we have

$$\big\|\big(v_k \cdot \nabla v_k, I(a_k)\mathcal{A}v_k\big)\big\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}$$
$$\lesssim \|v_k\|_{\tilde{L}^2\dot{B}^{n/p}_{p,1}}\|\nabla v_k\|_{\tilde{L}^{\frac{2}{1-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}} + \|a_k\|_{\tilde{L}^\infty\dot{B}^{n/p}_{p,1}}\|\nabla^2 v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}. \tag{4.62}$$

Also, due to the embedding $\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p} \hookrightarrow \dot{B}^{n/p-\zeta}_{p,1}$ and Proposition A.2, we arrive at

$$\|(\nabla a_k, \nabla O_k)\|_{\tilde{L}^\infty\dot{B}^{n/p-1-\zeta}_{p,1}} + \|(K(a_k)\nabla a_k, O_k\nabla O_k)\|_{\tilde{L}^\infty\dot{B}^{n/p-1-\zeta}_{p,1}}$$

$$\lesssim \quad \big(1+\|(a_k,O_k)\|_{\tilde{L}^\infty \dot{B}^{n/p}_{p,1}}\big)\|(a_k,O_k)\|_{\tilde{L}^\infty \dot{B}^{n/p-\zeta}_{p,1}}. \tag{4.63}$$

As above, we write

$$\frac{1}{1+a_k}\mathrm{div}\big(2\tilde{\mu}(a_k)D(v_k)\big)=\frac{1}{1+a_k}\tilde{\mu}(a_k)\nabla^2 v_k+\frac{1}{1+a_k}\nabla\tilde{\mu}(a_k)\nabla v_k.$$

Then by applying Lemma 2.3 and Proposition A.2, we get

$$\begin{aligned}
&\|\frac{1}{1+a_k}\mathrm{div}\big(2\tilde{\mu}(a_k)D(v_k)\big)\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}\\
\lesssim\ &\|\frac{1}{1+a_k}\tilde{\mu}(a_k)\nabla^2 v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}+\|\frac{1}{1+a_k}\nabla\tilde{\mu}(a_k)\nabla v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}\\
\lesssim\ &(1+\|I(a_k)\|_{\tilde{L}^\infty\dot{B}^{n/p}_{p,1}})\Big(\|\tilde{\mu}(a_k)\|_{\tilde{L}^\infty\dot{B}^{n/p}_{p,1}}\|\nabla^2 v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-1-\zeta}_{p,1}}\\
&\qquad+\|\nabla\tilde{\mu}(a_k)\|_{\tilde{L}^\infty\dot{B}^{n/p-1}_{p,1}}\|\nabla v_k\|_{\tilde{L}^{\frac{2}{2-\zeta}}\dot{B}^{n/p-\zeta}_{p,1}}\Big)
\end{aligned}\tag{4.64}$$

and $\frac{1}{1+a_k}\mathrm{div}\big(\tilde{\lambda}(a_k)\mathrm{div}v_k\mathrm{Id}\big)$ may be treated along the same way. Consequently, combining (4.61) – (4.64), we conclude that

$$\partial_t v_k\in\Big(L^{\frac{2}{2-\zeta}}_{loc}\dot{B}^{n/p-1-\zeta}_{p,1}\Big)^n,$$

which implies that v_k is uniformly bounded in $C^{\frac{2-\zeta}{2}}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1-\zeta}_{p,1})$. Therefore the claim (4.59) is proved. Furthermore, we see that $(a_k,O_k;v_k)$ is equicontinuous on $\mathbb{R}_+$ valued in $\big(\dot{B}^{n/p-1}_{p,1}\big)^{1+n^2}\times\big(\dot{B}^{n/p-1-\zeta}_{p,1}\big)^n$. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be a sequence of smooth functions supported in the ball $B(0,j+1)$ and equal to 1 in $B(0,j)$. It follows from (4.59) that $(\phi_j a_k,\phi_j O_k;\phi_j v_k)$ is uniformly bounded in

$$\Big(C^{1/2}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\Big)^{1+n^2}\times\Big(C^{\frac{2-\zeta}{2}}_{loc}(\mathbb{R}_+;\dot{B}^{n/p-1-\zeta}_{p,1})\Big)^n.$$

Observe that the map $(a_k,O_k;v_k)\mapsto(\phi_j a_k,\phi_j O_k;\phi_j v_k)$ is compact from

$$\big(\dot{B}^{n/p-1}_{p,1}\cap\dot{B}^{n/p}_{p,1}\big)^{1+n^2}\times\big(\dot{B}^{n/p-1-\zeta}_{p,1}\cap\dot{B}^{n/p-1}_{p,1}\big)^n$$

into

$$\big(\dot{B}^{n/p-1}_{p,1}\big)^{1+n^2}\times\big(\dot{B}^{n/p-1-\zeta}_{p,1}\big)^n.$$

By applying Ascoli's theorem and Cantor's diagonal process, there exist a $(a,O;v)$ such that for any smooth function $\phi\in C^\infty_0(\mathbb{R}^n)$,

$$\begin{aligned}
&(\phi a_k,\phi O_k)\to(\phi a,\phi O)\quad\text{in}\quad\big(L^\infty(\mathbb{R}_+;\dot{B}^{n/p-1}_{p,1})\big)^{1+n^2},\\
&\phi v_k\to\phi v\quad\text{in}\quad\big(L^\infty(\mathbb{R}_+;\dot{B}^{n/p-1-\zeta}_{p,1})\big)^n,
\end{aligned}\tag{4.65}$$

when $k\to+\infty$ (up to an extraction). Actually, by interpolation, we also have

$$\begin{aligned}
&(\phi a_k,\phi O_k)\to(\phi a,\phi O)\quad\text{in}\quad\big(L^\infty(\mathbb{R}_+;\dot{B}^{n/p-s}_{p,1})\big)^{1+n^2}\quad\forall\ 0<s\le 1,\\
&\phi v_k\to\phi v\quad\text{in}\quad\big(L^1(\mathbb{R}_+;\dot{B}^{n/p+s}_{p,1})\big)^n\quad\forall -1\le s<1.
\end{aligned}\tag{4.66}$$

Then, using the so-called Fatou property in Besov spaces and the uniform bound in (4.56), we conclude that (4.58) is fulfilled. It is a routine process to verify that $(a,O;v)$ satisfies the system (3.4) in the sense of distributions. Below is to check the desired regularity of solutions. Noticing that

$$\partial_t a+v\cdot\nabla a=-\nabla\cdot v-a\nabla\cdot v\in L^1_{loc}(\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p})\cap L^1(\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}),$$

$$\partial_t O + v\cdot\nabla O = \nabla v + \nabla v O \in L^1_{loc}(\dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}) \cap L^1(\dot{\mathcal{B}}^{n/2+1,n/p}_{2,p}).$$

since $(a_0, O_0) \in \dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}$, the classical result for transport equations indicates that

$$(a, O) \in C(\mathbb{R}_+; \dot{\mathcal{B}}^{n/2-1,n/p}_{2,p}).$$

On the other hand,

$$\begin{aligned}\partial_t v - \mathcal{A}v &= -v\cdot\nabla v - \nabla a + \nabla\cdot O + O^{jk}\partial_j O^{\bullet k} - I(a)\mathcal{A}v - K(a)\nabla a,\\ &+\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v) + \tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big),\\ &\in \tilde{L}^1_{loc}(\dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p}),\end{aligned}$$

So the maximal regularity of heat equation enables us to get $v \in C(\mathbb{R}_+; \dot{\mathcal{B}}^{n/2-1,n/p-1}_{2,p})$.

4.4. **Uniqueness.** Due to technical reasons, allow us to deal with the case $2 \le p \le n$ in the present paper only. We shall work on the remaining interval with respect to p in near future. Here, the proof of uniqueness depends on a logarithmic inequality, which is given it by a lemma.

Lemma 4.2 ([10])**.** *Let $s \in \mathbb{R}$. Then for any $1 \le p, r \le +\infty$ and $0 < \varepsilon \le 1$, we have*

$$\|f\|_{\tilde{L}^r_t\dot{B}^s_{p,1}} \le C\frac{\|f\|_{\tilde{L}^r_t\dot{B}^s_{p,\infty}}}{\varepsilon}\log\Big(e + \frac{\|f\|_{\tilde{L}^r_t\dot{B}^{s-\varepsilon}_{p,\infty}} + \|f\|_{\tilde{L}^r_t\dot{B}^{s+\varepsilon}_{p,\infty}}}{\|f\|_{\tilde{L}^r_t\dot{B}^s_{p,\infty}}}\Big).$$

Assume that $(\rho_i, F_i; u_i)(i = 1, 2)$ are two solution to the system (1.1) with the same initial data. Without loss of generality, we may assume that

$$\|(\rho_i - 1, F_i - I; u_i)\|_{\mathcal{E}^{n/p}} \le M\eta. \quad \text{for} \quad i = 1, 2. \tag{4.67}$$

Using embedding and (4.67), we have

$$\|\rho_i - 1\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^n)} \le C\|\rho_i - 1\|_{\mathcal{E}^{n/p}} \le CM\eta \le \frac{1}{2}, \quad \text{for} \quad i = 1.2$$

for $\eta > 0$ sufficiently small. Set

$$\begin{aligned}a_i(t,x) &= \rho_i(\chi_0^2 t, \chi_0 x) - 1,\\ O_i(t,x) &= F_i(\chi_0^2 t, \chi_0 x) - I,\\ v_i(t,x) &= \chi_0 u_i(\chi_0^2 t, \chi_0 x),\end{aligned}$$

for $i = 1, 2.$ and

$$\delta a = a_1 - a_2, \delta O = O_1 - O_2; \delta v = v_1 - v_2.$$

Thanks to (3.4), we find that $(\delta a, \delta v, \delta O)$ satisfies

$$\begin{cases}\partial_t\delta a + v_2\cdot\nabla\delta a = \delta F,\\ \partial_t\delta v - \mathcal{A}\delta v = \delta G,\\ \partial_t\delta O + v_2\cdot\nabla\delta O = \delta H,\\ (\delta a, \delta O; \delta v) = (0,0,0),\end{cases} \tag{4.68}$$

with

$$\begin{aligned}\delta F &= -\delta v\cdot\nabla a_1 - \nabla\cdot\delta v - a_1\nabla\cdot\delta v - \delta a\nabla\cdot v_2,\\ \delta H &= \delta v\cdot\nabla O_1 + \nabla\delta v + \nabla\delta v O_1 + \nabla v_2\delta O,\\ \delta G &= -\nabla\delta a + \nabla\cdot\delta O - (v_1\cdot\nabla v_1 - v_2\cdot\nabla v_2) + (O_1^{jk}\partial_j O_1^{\bullet k} - O_2^{jk}\partial_j O_2^{\bullet k})\end{aligned}$$

$$\begin{aligned}&-I(a_1)\mathcal{A}v_1+I(a_2)\mathcal{A}v_2-K(a_1)\nabla a_1+K(a_2)\nabla a_2\\&+\frac{1}{1+a_1}\mathrm{div}\big(2\tilde{\mu}(a_1)D(v_1)+\tilde{\lambda}(a_1)\mathrm{div}v_1\mathrm{Id}\big)\\&-\frac{1}{1+a_2}\mathrm{div}\big(2\tilde{\mu}(a_2)D(v_2)+\tilde{\lambda}(a_2)\mathrm{div}v_2\mathrm{Id}\big).\end{aligned}\tag{4.69}$$

In the following, we denote

$$V_i(t)=\int_0^t\|v_i(\tau)\|_{\dot{B}^{n/p+1}_{p,1}}d\tau\quad\text{for } i=1,2\tag{4.70}$$

and we denote by A_t a constant depending on $\|a_i\|_{\tilde{L}^\infty_t\dot{B}^{n/p}_{p,1}}$ for $i=1,2$. Due to the embedding $\mathcal{E}^{n/p}\subseteq\mathcal{E}^1(p\le n)$, it suffices to prove the uniqueness in $\mathcal{E}^1$. Therefore, we can take $p=n$ in the subsequent process.

By Proposition A.3, we get

$$\|\big(\delta a(t),\delta O(t)\big)\|_{\dot{B}^0_{p,\infty}}\le e^{CV_2(t)}\int_0^t\|\big(\delta F(\tau),\delta H(\tau)\big)\|_{\dot{B}^0_{p,\infty}}d\tau,\tag{4.71}$$

where it follows from Lemma 2.3 that

$$\begin{aligned}&\|\big(\delta F(\tau),\delta H(\tau)\big)\|_{\dot{B}^0_{p,\infty}}\\\lesssim\;&\|v_2\|_{\dot{B}^2_{p,1}}\|(\delta a,\delta O)\|_{\dot{B}^0_{p,\infty}}+\big(1+\|(a_1,O_1)\|_{\dot{B}^1_{p,1}}\big)\|\delta v\|_{\dot{B}^1_{p,1}}.\end{aligned}$$

Hence, by inserting the above inequality into (4.71), we arrive at

$$\|\big(\delta a(t),\delta O(t)\big)\|_{\dot{B}^0_{p,\infty}}\le e^{CV_2(t)}\int_0^t\big(1+\|(a_1,O_1)\|_{\dot{B}^1_{p,1}}\big)\|\delta v\|_{\dot{B}^1_{p,1}}d\tau.\tag{4.72}$$

Using Proposition A.4 to the second equation of (4.68) gives

$$\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}+\|\delta v\|_{\tilde{L}^2_t\dot{B}^0_{p,\infty}}\lesssim\|\delta G(\tau)\|_{\tilde{L}^1_t\dot{B}^{-1}_{p,\infty}}.\tag{4.73}$$

Furthermore, by Lemma 2.3 and Proposition A.2, it is shown that

$$\begin{aligned}\|\delta G(\tau)\|_{\tilde{L}^1_t\dot{B}^{-1}_{p,\infty}}\lesssim\;&\|(v_1,v_2)\|_{\tilde{L}^2_t\dot{B}^1_{p,1}}\|\delta v\|_{\tilde{L}^2_t\dot{B}^0_{p,\infty}}+A_t\|a_1\|_{\tilde{L}^\infty_t\dot{B}^1_{p,1}}\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}\\&+A_t\int_0^t(1+\|v_2\|_{\dot{B}^2_{p,1}})\|(\delta a,\delta O)\|_{\dot{B}^0_{p,\infty}}d\tau.\end{aligned}\tag{4.74}$$

According to a prior estimates, by choosing η small, we have

$$A_t\|a_1\|_{\tilde{L}^\infty_t\dot{B}^1_{p,1}}+\|(v_1,v_2)\|_{\tilde{L}^2_t\dot{B}^1_{p,1}}\lesssim M\eta\ll 1.$$

Consequently, inserting (4.74) into (4.73) to implies that

$$\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}+\|\delta v\|_{\tilde{L}^2_t\dot{B}^0_{p,\infty}}\lesssim A_t\int_0^t(1+\|v_2\|_{\dot{B}^2_{p,1}})\|(\delta a,\delta O)\|_{\dot{B}^0_{p,\infty}}d\tau.\tag{4.75}$$

Combining (4.72) and (4.75), we get

$$\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}\lesssim\int_0^t(1+\|v_2\|_{\dot{B}^2_{p,1}})\|\delta v\|_{\tilde{L}^1_\tau\dot{B}^1_{p,1}}d\tau.\tag{4.76}$$

By applying Lemma 4.2 with $s=r=\varepsilon=1$ and $f=\delta v$, we obtain

$$\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,1}}\le C\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}\log\Big(e+\frac{\|\delta v\|_{\tilde{L}^1_t\dot{B}^0_{p,\infty}}+\|\delta v\|_{\tilde{L}^1_t\dot{B}^2_{p,\infty}}}{\|\delta v\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}}}\Big),$$

which together with (4.75) and (4.72) indicates that

$$\|\delta v(t)\|_{\tilde{L}^1_t\dot{B}^1_{p,\infty}} \le e^{CV_2(t)}A_t\int_0^t(1+\|v_2\|_{\dot{B}^2_{p,1}})\|\delta v\|_{\tilde{L}^1_\tau\dot{B}^1_{p,\infty}} \times \log\left(e+C_\tau\|\delta v\|^{-1}_{\tilde{L}^1_\tau\dot{B}^1_{p,\infty}}\right)d\tau,$$

where $C_\tau=\|\delta v\|_{\tilde{L}^1_\tau\dot{B}^0_{p,\infty}}+\|\delta v\|_{\tilde{L}^1_\tau\dot{B}^2_{p,\infty}}$. Noting $\|v_2\|_{\dot{B}^2_{p,1}}$ is integrable on $[0,\infty]$ and

$$\int_0^1\frac{dr}{r\log(r+C_tr^{-1})}=+\infty, \tag{4.77}$$

the Osgood lemma implies that $(\delta a,\delta O;\delta v)=0$ on $[0,t]$. Hence, a continuity argument ensures that $(a_1,O_1;v_1)=(a_2,O_2;v_2)$ for any $t\in[0,\infty)$.

5. **The Proof of time-decay estimates.** In this section, we aim at proving the time-weighted energy inequality (1.6) taking for granted Theorem 1.2. We will proceed the proof into three subsections, according to the three terms in $\mathcal{G}_p(t)$. Subsection 5.1 is devoted to the low-frequency estimate. In the spirit of [7], we only need to perform nonlinear estimates in terms of deformation tensor. In Subsection 5.2, in order to overcome the technical difficulty that there is loss of one derivative for the density and deformation tensor at high frequencies, we develop "*two effective velocities*" and obtain the upper bound for the second term in $\mathcal{G}_p(t)$. To close the high-frequency estimates, in Subsection 5.3, a crucial observation enables us to establish gain of regularity and decay altogether for the velocity, which strongly depends on Proposition A.4.

For simplicity, we define

$$\mathcal{X}_p(t)\triangleq\|(a,O;v)\|_{\mathcal{E}^{n/p}_t}. \tag{5.1}$$

In what follows, we will use the two key lemmas repeatedly.

Lemma 5.1. *Let $0\le\sigma_1\le\sigma_2$ with $\sigma_2>1$. It holds that*

$$\int_0^t\langle t-\tau\rangle^{-\sigma_1}\langle\tau\rangle^{-\sigma_2}d\tau\lesssim\langle t\rangle^{-\sigma_1} \tag{5.2}$$

and

$$\int_0^t\langle t-\tau\rangle^{-\sigma_1}\tau^{-\theta}\langle\tau\rangle^{\theta-\sigma_2}d\tau\lesssim\langle t\rangle^{-\sigma_1}\quad \text{if}\quad 0<\theta<1. \tag{5.3}$$

Lemma 5.2 ([7]). *Let $X:[0,T]\to\mathbb{R}_+$ be a continuous function such that X^p is a differentiable for some $p\ge1$ and satisfies*

$$\frac{1}{p}\frac{d}{dt}X^p+BX^p\le AX^{p-1}$$

for some constant $B\ge0$ and measurable function $A:[0,T]\to\mathbb{R}_+$. Define $X_\delta=(X^p+\delta^p)^{\frac{1}{p}}$ for $\delta>0$. Then it holds that

$$\frac{d}{dt}X_\delta+BX_\delta\le A+B\delta. \tag{5.4}$$

For convenience, we denote by $\|\cdot\|_{\delta,L^p}:=(\|\cdot\|^p_{L^p}+\delta^p)^{1/p}$ for $1\le p<\infty$.

5.1. **Bounds for the low frequencies.** From (4.12) and (4.13), we have

$$\frac{d}{dt}\big(\|(a_k,O_k;v_k)\|_{L^2}^2\big)+2^{2k}\|(a_k,O_k;v_k)\|_{L^2}^2\lesssim\Big(\sum_{i=0,1,3,4}\|\dot\Delta_kG_i\|_{L^2}\Big)\|(a_k,O_k;v_k)\|_{L^2}.$$

It follows from Lemma 5.2 that

$$\frac{d}{dt}\|(a_k,O_k;v_k)\|_{\delta,L^2}+2^{2k}\|(a_k,O_k;v_k)\|_{\delta,L^2}\lesssim\sum_{i=0,1,3,4}\|\dot\Delta_kG_i\|_{L^2}+2^{2k}\delta.$$

Then integrate the above inequality in time and let $\delta\to0$. There exists a constant $c_0>0$ such that

$$\begin{aligned}\|(a_k,O_k;v_k)\|_{L^2}\ &\lesssim\ e^{-c_02^{2k}t}\|(\dot\Delta_ka_0,\dot\Delta_kO_0;\dot\Delta_kv_0)\|_{L^2}\\&\qquad+\int_0^te^{c_02^{2k}(\tau-t)}\sum_{i=0,1,3,4}\|\dot\Delta_kG_i\|_{L^2}d\tau.\end{aligned}\tag{5.5}$$

Regarding the first term in (5.5), we multiply the factor $\langle t\rangle^{\frac{s+s_0}{2}}2^{ks}$ and sum up on $2^k\le R_0$ to get

$$\begin{aligned}&\langle t\rangle^{\frac{s+s_0}{2}}\sum_{2^k\le R_0}2^{ks}e^{-c_02^{2k}t}\|(\dot\Delta_ka_0,\dot\Delta_kO_0;\dot\Delta_kv_0)\|_{L^2}\\\lesssim\ &\|(a_0,O_0;v_0)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\sum_{2^k\le R_0}(2^k\langle t\rangle^{\frac12})^{s+s_0}e^{-c_0(2^k\sqrt t)^2}.\\\lesssim\ &\|(a_0,O_0;v_0)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}\Big(\sum_{2^k\le R_0}(2^k\sqrt t)^{s+s_0}e^{-c_0(2^k\sqrt t)^2}+2^{k_0(s+s_0)}\Big)\\\lesssim\ &\|(a_0,O_0;v_0)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}},\end{aligned}\tag{5.6}$$

where we have used the fact $\sum\limits_{2^k\le R_0}(2^k\sqrt t)^{s+s_0}e^{-c_0(2^k\sqrt t)^2}\le C$ when $s+s_0>0$. So we have

$$\sum_{2^k\le R_0}2^{ks}e^{-c_02^{2k}t}\|(\dot\Delta_ka_0,\dot\Delta_kO_0;\dot\Delta_kv_0)\|_{L^2}\lesssim\langle t\rangle^{-\frac{s+s_0}{2}}\|(a_0,O_0;v_0)\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}.\tag{5.7}$$

Furthermore, the corresponding nonlinear term in (5.5) can be estimated as

$$\begin{aligned}&\sum_{2^k\le R_0}2^{ks}\int_0^te^{c_02^{2k}(\tau-t)}\sum_{i=0,1,3,4}\|\dot\Delta_kG_i\|_{L^2}d\tau\\\lesssim\ &\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\sum_{i=0,1,3,4}\|G_i\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}d\tau.\end{aligned}\tag{5.8}$$

We claim that if p fulfills the assumption as in Theorem 1.3, then we have for all $t\ge0$,

$$\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\sum_{i=0,1,2,3}\|G_i\|^{\ell}_{\dot B^{-s_0}_{2,\infty}}d\tau\lesssim\langle t\rangle^{-\frac{s+s_0}{2}}\big(\mathcal G_p^2(t)+\mathcal X_p^2(t)\big),\tag{5.9}$$

where $\mathcal G_p(t)$ and $\mathcal X_p(t)$ are defined by (1.7) and (5.1).

Since those quadratic terms containing a and v in $G_i(i=0,1,3,4)$ have already been done in [7], it suffices to give suitable decay estimates for some terms involving in O. Precisely, we need to hand the following integral

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}} \|\big(\partial_i(aO^{ij}), \partial_k v^i O^{kj}, v\cdot\nabla O^{ij}, \\ O^{jk}\partial_j O^{ik}, O^{lj}\partial_l O^{ik}, O^{lk}\partial_l O^{ij}\big)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau.$$

As far as we know, the regularity level remains the same between the density and deformation tensor. Hence, these terms $\partial_i aO^{ij}$, $a\partial_i O^{ij}$, $O^{jk}\partial_j O^{ik}$, $O^{lj}\partial_l O^{ik}$, $O^{lk}\partial_l O^{ij}$ can be treated along the same line. In principle, the above integral can be reduced to

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}} \|\big(\nabla a\cdot O, \nabla v O, v\cdot\nabla O\big)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau. \tag{5.10}$$

We decompose (5.10) as follows

$$(5.10) \triangleq I^{\ell} + I^{h},$$

where

$$I^{\ell} = \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}} \|\big(O\cdot\nabla a^{\ell}, O\nabla v^{\ell}, v\cdot\nabla O^{\ell}\big)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau,$$

and

$$I^{h} = \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}} \|\big(O\cdot\nabla a^{h}, O\nabla v^{h}, v\cdot\nabla O^{h}\big)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau.$$

In order to handle I^{ℓ} in terms of with a^{ℓ}, O^{ℓ} and v^{ℓ}, we use the following Lemma.

Lemma 5.3. *Let $s_0 = n(2/p-1/2)$ and p satisfy the assumption in Theorem 1.3. It holds that*

$$\|fg\|_{\dot{B}^{-s_0}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{1-n/p}_{p,1}} \|g\|_{\dot{B}^{n/2-1}_{2,1}}, \tag{5.11}$$

and

$$\|fg\|_{\dot{B}^{-n/p}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{n/p-1}_{p,1}} \|g\|_{\dot{B}^{1-n/p}_{2,1}}. \tag{5.12}$$

The reader is referred to [7] for the detailed proof. Owing to the embedding theorem and the definition of $\mathcal{G}_p(t)$, we shall often use the following inequalities

$$\|(a,O;v)^{\ell}(\tau)\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}} \lesssim \|(a,O;v)^{\ell}(\tau)\|_{\dot{B}^{1-s_0}_{2,1}} \lesssim \langle\tau\rangle^{-\frac{1}{2}}\mathcal{G}_p(\tau), \tag{5.13}$$

and

$$\|(a,O)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \langle\tau\rangle^{-\frac{n}{p}}\mathcal{G}_p(\tau). \tag{5.14}$$

Indeed, the above inequality is obvious for the high frequencies since $\alpha \geq \frac{n}{p}$, and we have

$$\|(a,O)^{\ell}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \|(a,O)\|^{\ell}_{\dot{B}^{\frac{n}{2}}_{2,1}} \lesssim \langle\tau\rangle^{-\frac{1}{2}(s_0+n/2)}\mathcal{G}_p(\tau) = \langle\tau\rangle^{-\frac{n}{p}}\mathcal{G}_p(\tau). \tag{5.15}$$

Notice that $1-\frac{n}{p} \leq \frac{n}{p}$ and the definition of $\mathcal{G}_p(t)$, we arrive at

$$\|v^h\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}} \lesssim \|v^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \Big(\|v^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} \|\nabla v^h\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)^{\frac{1}{2}} \lesssim \tau^{-\frac{1}{2}}\langle\tau\rangle^{-\frac{\alpha}{2}}\mathcal{G}_p(\tau). \tag{5.16}$$

In what follows, we estimate those nonlinear terms I^{ℓ} and I^{h}.

Estimates for I^{ℓ}

Taking advantage of (5.11), (5.13), (5.14) and (5.16), we get

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}} \|(v\cdot\nabla O^{\ell})\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} d\tau$$

$$\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|v\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\|\nabla O^\ell\|_{\dot B^{\frac{n}{2}-1}_{2,1}}d\tau$$
$$\lesssim \mathcal{G}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\Big(\langle\tau\rangle^{-\frac12}+\tau^{-\frac12}\langle\tau\rangle^{-\frac{\alpha}{2}}\Big)\langle\tau\rangle^{-\frac{n}{p}}d\tau.$$

Due to the fact that $\frac{n}{p}+\frac12>1$ and $\frac{s+s_0}{2}\le\frac{n}{p}+\frac12$ for all $s\le 1+\frac{n}{2}$, Lemma (5.1) implies that

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|(v\cdot\nabla O^\ell)\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau \lesssim \langle t\rangle^{-\frac{s+s_0}{2}}\mathcal{G}_p^2(t). \tag{5.17}$$

The terms $O\cdot\nabla a^\ell$ and $O\nabla v^\ell$ can be treated along with the same lines, so we omit details.

Estimates for I^h

For the term I^h containing a^h, O^h and v^h, as in [7], we proceed differently depending on whether $p>n$ and $p\le n$. Let's first consider the case $2\le p\le n$. Applying (2.4) with $\sigma=\frac{n}{p}-1$ yields

$$\|fg^h\|_{\dot B^{-s_0}_{2,\infty}}\lesssim\|f\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\Big(\|\dot S_{k_0+N_0}g^h\|_{L^{p^*}}+\|g^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\Big)\lesssim\|f\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\|g^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}, \tag{5.18}$$

where we have used the Berstein inequality ($p^*=\frac{2p}{p-2}\ge p$) and the fact that only finite middle frequencies of g are involving in $\dot S_{k_0+N_0}g^h$.[1]

Taking $f=v$ and $g=\nabla O$ in (5.18), we get

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|v\cdot\nabla O^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau\\ \lesssim&\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|v\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\|\nabla O^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}d\tau.\end{aligned} \tag{5.19}$$

It follows from (5.13) and (5.16) that

$$\|v\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\lesssim\big(\langle\tau\rangle^{-\frac12}+\tau^{-\frac12}\langle\tau\rangle^{-\frac{\alpha}{2}}\big)\mathcal{G}_p(\tau). \tag{5.20}$$

The definition of $\mathcal{G}_p(t)$ implies that

$$\|\nabla O^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\lesssim\langle\tau\rangle^{-\alpha}\mathcal{G}_p(\tau)\quad\text{with}\quad \alpha=\frac{n}{p}+\frac12-\varepsilon. \tag{5.21}$$

Inserting (5.20) and (5.21) into (5.19), we conclude that for $-s_0\le s\le\frac{d}{2}+1$,

$$\begin{aligned}&\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|v\cdot\nabla O^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau\\ \lesssim&\ \mathcal{G}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\langle\tau\rangle^{-\alpha}\big(\langle\tau\rangle^{-\frac12}+\tau^{-\frac12}\langle\tau\rangle^{-\frac{\alpha}{2}}\big)d\tau\\ \lesssim&\ \langle t\rangle^{-\frac{s+s_0}{2}}\mathcal{G}_p^2(t).\end{aligned} \tag{5.22}$$

Handling with the term $O\cdot\nabla a^h$ is similar. With aid of (5.18), we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\cdot\nabla a^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau \quad\lesssim\quad \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\|_{\dot B^{1-\frac{n}{p}}_{p,1}}\|\nabla a^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}d\tau$$

[1]The limit case $p=n$ follows from $\|fg^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}\lesssim\|fg^h\|_{L^{\frac{n}{2}}}\lesssim\|f\|_{L^n}\|g^h\|_{L^n}\lesssim\|f\|_{\dot B^0_{n,1}}\|g^h\|_{\dot B^0_{n,1}}$.

$$\lesssim \mathcal{G}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\langle\tau\rangle^{-\frac{1}{2}-\alpha}d\tau$$
$$\lesssim \langle t\rangle^{-\frac{s+s_0}{2}}\mathcal{G}_p^2(t). \tag{5.23}$$

Regarding the term $O\nabla v^h$, combining the embedding $L^{\frac{p}{2}} \hookrightarrow \dot{B}^{-s_0}_{2,\infty}$ and Hölder inequality, we obtain

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\nabla v^h\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}d\tau \lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O(\tau)\|_{L^p}\|\nabla v^h(\tau)\|_{L^p}d\tau. \tag{5.24}$$

By embedding, the definition of $\mathcal{G}_p(t)$ and the fact that $\alpha \geq \frac{n}{2p}$ for sufficiently small $\varepsilon > 0$, we have

$$\|O\|_{L^p} \lesssim \|O^\ell\|_{L^p} + \|O^h\|_{L^p} \lesssim \|O\|^{\ell}_{\dot{B}^{\frac{n}{2}-\frac{n}{p}}_{2,1}} + \|O\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$$
$$\lesssim (\langle\tau\rangle^{-\frac{n}{2p}} + \langle\tau\rangle^{-\alpha})\mathcal{G}_p(t) \lesssim \langle\tau\rangle^{-\frac{n}{2p}}\mathcal{G}_p(t). \tag{5.25}$$

Arguing as for proving (5.16), it is easy to get for $2 \leq p \leq n$,

$$\|\nabla v^h(\tau)\|_{L^p} \lesssim \|v^h(\tau)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \tau^{-\frac{1}{2}}\langle\tau\rangle^{-\frac{\alpha}{2}}\mathcal{G}_p(\tau). \tag{5.26}$$

Furthermore, together with (5.25)-(5.26), we have

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\nabla v^h\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}d\tau$$
$$\lesssim \mathcal{G}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\tau^{-\frac{1}{2}}\langle\tau\rangle^{-(\frac{\alpha}{2}+\frac{n}{2p})}d\tau \lesssim \mathcal{G}_p^2(t)\langle t\rangle^{-\frac{s+s_0}{2}}. \tag{5.27}$$

Let's end that step by considering I^h involving a^h, O^h and v^h in the case of $p > n$. Applying Inequality (2.3) with $\sigma = 1-\frac{n}{p}$ and the embedding $\dot{B}^{\frac{n}{p}}_{2,1} \hookrightarrow L^{p*}$ give that

$$\|fg^h\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}} \lesssim \big(\|f\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}} + \|\dot{S}_{k_0+N_0}f\|_{L^{p*}}\big)\|g^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$$
$$\lesssim (\|f^\ell\|_{\dot{B}^{\frac{n}{p}}_{2,1}} + \|f\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}})\|g^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}, \tag{5.28}$$

where $\frac{1}{p*} = \frac{1}{2} - \frac{1}{p}$. Taking $f = v$ and $g = \nabla O$, and then using (5.13), (5.16) as well as the definition of $\mathcal{G}_p(t)$, we arrive at

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|(v\cdot\nabla O^h)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}d\tau$$
$$\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}(\|v^\ell\|_{\dot{B}^{\frac{n}{p}}_{2,1}} + \|v\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}})\|\nabla O^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau$$
$$\lesssim \mathcal{G}_p^2(t)\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\big(\langle\tau\rangle^{-(\frac{3n}{2p}-\frac{n}{4})} + \langle\tau\rangle^{-\frac{1}{2}} + \tau^{-\frac{1}{2}}\langle\tau\rangle^{-\frac{\alpha}{2}}\big)\langle\tau\rangle^{-\alpha}d\tau$$
$$\lesssim \langle t\rangle^{-\frac{s+s_0}{2}}\mathcal{G}_p^2(t). \tag{5.29}$$

Next, by taking $f = O$ and $g = \nabla a$ in (5.28), we obtain

$$\int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\cdot\nabla a^h\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}d\tau$$
$$\lesssim \int_0^t \langle t-\tau\rangle^{-\frac{s+s_0}{2}}(\|O^\ell\|_{\dot{B}^{\frac{n}{p}}_{2,1}} + \|O\|_{\dot{B}^{1-\frac{n}{p}}_{p,1}})\|\nabla a^h\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau. \tag{5.30}$$

It follows from (5.13) and (5.14) that

$$\|O^\ell\|_{\dot B^{\frac{n}{p}}_{2,1}}+\|O\|_{\dot B^{1-\frac{n}{p}}_{p,1}} \lesssim (\langle\tau\rangle^{-(\frac{3n}{2p}-\frac{n}{4})}+\langle\tau\rangle^{-\frac12}+\langle\tau\rangle^{-\alpha})\mathcal{G}_p(\tau). \tag{5.31}$$

Consequently, we deduce that

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\nabla a^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau\\ \lesssim\ & \mathcal{G}_p^2(t)\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}(\langle\tau\rangle^{-(\frac{3n}{2p}-\frac{n}{4})}+\langle\tau\rangle^{-\frac12}+\langle\tau\rangle^{-\alpha})\langle\tau\rangle^{-\alpha}d\tau\\ \lesssim\ & \mathcal{G}_p^2(t)\langle t\rangle^{-\frac{s+s_0}{2}}.\end{aligned} \tag{5.32}$$

To bound the last term $O\nabla v^h$, we need to take $f=O$ and $g=\nabla v$ in (5.28) and get

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\nabla v^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau\\ \lesssim\ & \int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}(\|O^\ell\|_{\dot B^{\frac{n}{p}}_{2,1}}+\|O\|_{\dot B^{1-\frac{n}{p}}_{p,1}})\|\nabla v^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}d\tau.\end{aligned} \tag{5.33}$$

By interpolation, for all $\tau\geq 0$,

$$\|\nabla v^h\|_{\dot B^{\frac{n}{p}-1}_{p,1}}\lesssim\Big(\|v\|^h_{\dot B^{\frac{n}{p}-1}_{p,1}}\|\nabla v\|^h_{\dot B^{\frac{n}{p}}_{p,1}}\Big)^{\frac12}\lesssim\tau^{-\frac12}\langle\tau\rangle^{-\frac{\alpha}{2}}\mathcal{G}_p(\tau).$$

Therefore, we are led to

$$\begin{aligned}&\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\|O\nabla v^h\|^\ell_{\dot B^{-s_0}_{2,\infty}}d\tau\\ \lesssim\ & \mathcal{G}_p^2(t)\int_0^t\langle t-\tau\rangle^{-\frac{s+s_0}{2}}\langle\tau\rangle^{-\min(\frac12,\frac{3n}{2p}-\frac{n}{4},\alpha)}\tau^{-\frac12}\langle\tau\rangle^{-\frac{\alpha}{2}}d\tau\\ \lesssim\ & \mathcal{G}_p^2(t)\langle t\rangle^{-\frac{s+s_0}{2}}.\end{aligned} \tag{5.34}$$

Putting together all the above estimates for those terms involving in O and those computations with respect to a and v (see [7]), we can finish the proof of (5.9). Then by combining (5.7) and (5.9), we deduce that

$$\langle t\rangle^{-\frac{s+s_0}{2}}\|(a,O;v)(t)\|^\ell_{\dot B^s_{2,1}}\lesssim\mathcal{G}_{p,0}+\mathcal{G}_p^2(t)+\mathcal{X}_p^2(t). \tag{5.35}$$

5.2. **Decay estimates for the high frequencies of $(\nabla a,\nabla O;v)$.** This part is devoted to bounding the second term in $\mathcal{G}_p(t)$. The usual Duhamel principle is no longer true, since there is a loss of one derivative for the density and deformation tensor at high frequencies. To eliminate the technical difficulty, we need to perform a suitable quasi-diagonalization (say, *effective velocities*), to handle high frequencies. Let $\dot\Delta_k w=w_k$ and $\dot\Delta_k\Omega^{ij}=\Omega^{ij}_k$. From (4.34) and (4.39), by employing the energy methods of L^p type, we obtain

$$\frac{d}{dt}\|w_k\|_p^p+c_p2^{2k}\|w_k\|_p^p\lesssim\{2^{-k}(\|a_k\|_p+\|\dot\Delta_k\tilde G_1\|_p)+\|\dot\Delta_kG_2\|_p\}\|w_k\|_p^{p-1}, \tag{5.36}$$

$$\frac{d}{dt}\|\Omega^{ij}_k\|_p^p+c_p2^{2k}\|\Omega^{ij}_k\|_p^p\lesssim\{2^{-k}(\|O^{ij}_k\|_p+\|\dot\Delta_k\tilde G^{ij}_3\|_p)+\|\dot\Delta_k\tilde G^{ij}_4\|_p\}\|\Omega^{ij}_k\|_p^{p-1}, \tag{5.37}$$

$$\frac{d}{dt}\|\Lambda a_k\|_p^p+c_p\|\Lambda a_k\|_p^p$$

$$\lesssim \quad \big\{\|\Lambda\dot{\Delta}_k\tilde{G}_1\|_p+2^{2k}\|w\|_p+\|\mathrm{div}v\|_\infty\|\Lambda a_k\|_p+\|\mathcal{R}_{1,k}\|_p\big\}\|\Lambda a_k\|_p^{p-1} \quad (5.38)$$

and

$$\begin{aligned}&\frac{d}{dt}\|\Lambda O_k^{ij}\|_p^p+c_p\|\Lambda O_k^{ij}\|_p^p\\ &\lesssim\big\{\|\Lambda\dot{\Delta}_k\tilde{G}_3^{ij}\|_p+2^{2k}\|\Omega^{ij}\|_p+\|\mathrm{div}v\|_\infty\|\Lambda O_k^{ij}\|_p+\|\mathcal{R}_{2,k}\|_p\big\}\|\Lambda O_k^{ij}\|_p^{p-1} \quad (5.39)\end{aligned}$$

with $\mathcal{R}_{1,k}\triangleq[v\cdot\nabla,\Lambda\dot{\Delta}_k]a$ and $\mathcal{R}_{2,k}\triangleq[v\cdot\nabla,\Lambda\dot{\Delta}_k]O^{ij}$, where we chosen R_0 sufficiently large such that $2^k>R_0$. Furthermore, with aid of Lemma 5.2, it is shown that there exists a constant $c_p>0$ such that

$$\begin{aligned}&\frac{d}{dt}\big(\|\Lambda a_k\|_{\delta,L^p}+\|d_k\|_{\delta,L^p}\big)+c_p\big(\|\Lambda a_k\|_{\delta,L^p}+2^{2k}\|d_k\|_{\delta,L^p}\big)\\ \lesssim\quad&\|\Lambda\dot{\Delta}_k\tilde{G}_1\|_p+\|\dot{\Delta}_kG_2\|_p+\|\mathrm{div}v\|_\infty\|\Lambda a_k\|_p+\|\mathcal{R}_{1,k}\|_p+2^{2k}\delta, \quad (5.40)\end{aligned}$$

where we used the effective velocity in terms of a and d. Similar estimates for O^{ij} and e^{ij} stems from (5.37) and (5.39):

$$\begin{aligned}&\frac{d}{dt}\big(\|\Lambda O_k\|_{\delta,L^p}+\|e_k^{ij}\|_{\delta,L^p}\big)+\tilde{c}_p\big(\|\Lambda O_k^{ij}\|_{\delta,L^p}+2^{2k}\|e_k^{ij}\|_{\delta,L^p}\big)\\ \lesssim\quad&\|\Lambda\dot{\Delta}_k\tilde{G}_3^{ij}\|_p+\|\dot{\Delta}_k\tilde{G}_4^{ij}\|_p+\|\mathrm{div}v\|_\infty\|\Lambda O_k^{ij}\|_p+\|\mathcal{R}_{2,k}\|_p+2^{2k}\delta \quad (5.41)\end{aligned}$$

for some constant $\tilde{c}_p>0$. It's easy to see that

$$\begin{aligned}\|\dot{\Delta}_k\tilde{G}_4^{ij}\|_p&\lesssim\|\dot{\Delta}_kG_4^{ij}\|_p+2^{2k}\|d_k\|_p+\|\Lambda a_k\|_p\\ &\lesssim\|\dot{\Delta}_kG_4^{ij}\|_p+2^{2k}\|d_k\|_{\delta,L^p}+\|\Lambda a_k\|_{\delta,L^p}. \quad (5.42)\end{aligned}$$

Therefore, it follows from (5.40), (5.41) and (5.42) that

$$\begin{aligned}&\frac{d}{dt}\big(\|(\Lambda a_k,\Lambda O_k;v_k)\|_{\delta,L^p}+c_0\|(\Lambda a_k,\Lambda O_k;v_k)\|_{\delta,L^p}\\ \lesssim\quad&\|\mathrm{div}v\|_\infty\|(\Lambda a_k,\Lambda O_k)\|_p+\|\mathcal{R}_{1,k}\|_p+\|\mathcal{R}_{2,k}\|_p\\ &+\|\Lambda\dot{\Delta}_k(\tilde{G}_1,\tilde{G}_3)\|_p+\|\dot{\Delta}_k(G_2,G_4)\|_p+2^{2k}\delta \quad (5.43)\end{aligned}$$

for $c_0>0$. Integrating in time on both sides and letting $\delta\to0$, we eventually get

$$\|(\Lambda a_k,\Lambda O_k;v_k)(t)\|_p\lesssim e^{-c_0t}\|(\Lambda a_k(0),\Lambda O_k(0),v_k(0))\|_p+\int_0^te^{c_0(\tau-t)}g_k(\tau)d\tau, \quad (5.44)$$

where

$$\begin{aligned}g_k\quad\triangleq\quad&\underbrace{\|\mathrm{div}v\|_{L^\infty}\big(\|\Lambda a_k\|_p+\|\Lambda O_k\|_p\big)}_{g_k^1}\\ &+\underbrace{\|\dot{\Delta}_k(G_2,G_4,\Lambda\tilde{G}_1,\Lambda\tilde{G}_3)\|_p}_{g_k^2}\\ &+\underbrace{\|\big(\mathcal{R}_{1,k},\mathcal{R}_{2,k}\big)\|_p}_{g_k^3}.\end{aligned}$$

Multiplying (5.44) by $\langle t\rangle^\alpha2^{k(\frac{n}{p}-1)}$, taking the supremum on $[0,T]$ and summing up over k satisfying $2^k>R_0$ yields

$$\begin{aligned}&\|\langle t\rangle^\alpha(\Lambda a,\Lambda O;v)\|^h_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}-1}_{p,1}}\\ \lesssim\quad&\|(\Lambda a_0,\Lambda O_0;v_0)\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\end{aligned}$$

$$+\sum_{2^k>R_0}\sup_{0\le t\le T}\left(\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{n}{p}-1)}g_k(\tau)d\tau\right). \tag{5.45}$$

Without loss of generality, we assume that $T\ge 2$ and first bound the the supremum for $0\le t\le 2$. Notice that

$$\sum_{2^k>R_0}\sup_{0\le t\le 2}\left(\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{n}{p}-1)}g_k(\tau)d\tau\right)\lesssim\int_0^2\sum_{2^k>R_0}2^{k(\frac{n}{p}-1)}g_k(\tau)d\tau. \tag{5.46}$$

Furthermore, it will be shown that the right side of (5.46) can be bounded by $\mathcal{X}_p^2(2)$. Indeed, using Lemma 2.6 and the representation of $\tilde{G}_i(i=1,3)$ and $G_i(i=2,4)$, we get

$$\begin{aligned}
&\int_0^2\sum_{2^k>R_0}2^{k(\frac{n}{p}-1)}g_k(\tau)d\tau\\
&\le\int_0^2\Big\{\underbrace{\|\mathrm{div}v\|_{L^\infty}\big(\|\Lambda O\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\Lambda a\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}\big)}_{\text{coming from } g_k^1}\\
&+\underbrace{\|(\mathrm{div}(aO),\Lambda(\partial_k v^iO^{kj}),O^{lk}\partial_lO^{ik},O^{lj}\partial_lO^{ik},O^{lk}\partial_lO^{ij},O^{jk}\partial_jO^{\bullet k})\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}}_{\text{coming from } g_k^2}\\
&+\underbrace{\Big\|\Big(a\cdot\nabla v,v\cdot\nabla v,I(a)\mathcal{A}v,K(a)\nabla a,\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v)+\tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)\Big)\Big\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}}_{\text{coming from } g_k^2}\\
&+\underbrace{\|\nabla v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\big(\|O\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+(\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\big)}_{\text{coming from } g_k^3}\Big\}d\tau.
\end{aligned}\tag{5.47}$$

In contrast with [7], we pay attention to those terms involving O only. For instance, we have

$$\int_0^2\Big(\|\mathrm{div}v\|_{L^\infty}\|\Lambda O\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\nabla v\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|O\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\Big)d\tau\lesssim\|O\|_{L^\infty_t\dot{B}^{\frac{n}{p}}_{p,1}}\int_0^2\|v\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}d\tau\lesssim\mathcal{X}_p^2(2).$$

Owing to Lemma 2.3 and interpolation inequalities, we obtain

$$\begin{aligned}
&\int_0^2\|(\mathrm{div}(aO),\Lambda(\partial_kv^iO^{kj}),O^{lk}\partial_lO^{ik},O^{lj}\partial_lO^{ik},O^{lk}\partial_lO^{ij},O^{jk}\partial_jO^{\bullet k}\|^h_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\
\lesssim&\int_0^2\|a\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|O\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|v\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|O\|_{\dot{B}^{\frac{n}{p}}_{p,1}}+\|O\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|\nabla O\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}d\tau\\
\lesssim&\Big(\|a\|_{L^2\dot{B}^{\frac{n}{p}}_{p,1}}+\|O\|_{L^2\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\|O\|_{L^2\dot{B}^{\frac{n}{p}}_{p,1}}+\|v\|_{L^1\dot{B}^{\frac{n}{p}+1}_{p,1}}\|O\|_{L^\infty\dot{B}^{\frac{n}{p}}_{p,1}}.\\
\lesssim&\mathcal{X}_p^2(2).
\end{aligned}$$

Hence, we infer that

$$\sum_{2^k>R_0}\sup_{0\le t\le 2}\langle t\rangle^\alpha\int_0^t e^{c_0(\tau-t)}2^{k(\frac{n}{p}-1)}g_k(\tau)d\tau\lesssim\mathcal{X}_p^2(2). \tag{5.48}$$

Let us now bound the supremum for $2 \le t \le T$ in the last term of (5.45). For that end, we split the integral on $[0,t]$ into integrals on $[0,1]$ and $[1,t]$. The integral on $[0,1]$ is easy to handle: because $e^{c_0(\tau-t)} \le e^{-\frac{c_0}{2}t}$ for $2 \le t \le T$ and $0 \le \tau \le 1$, so one can write

$$\begin{aligned}&\sum_{2^k>R_0} \sup_{2\le t\le T} \langle t\rangle^\alpha \int_0^1 e^{c_0(\tau-t)} 2^{k(\frac{n}{p}-1)} g_k(\tau)d\tau\\ \lesssim&\sum_{2^k>R_0} \sup_{2\le t\le T} \langle t\rangle^\alpha e^{-\frac{c_0}{2}t} \int_0^1 2^{k(\frac{n}{p}-1)} g_k(\tau)d\tau \lesssim \sum_{2^k>R_0} \int_0^1 2^{k(\frac{n}{p}-1)} g_k(\tau)d\tau. \end{aligned}\tag{5.49}$$

Therefore, following the procedure leading to (5.48), we end up with

$$\sum_{2^k>R_0} \sup_{2\le t\le T} \langle t\rangle^\alpha \int_0^1 e^{c_0(\tau-t)} 2^{k(\frac{n}{p}-1)} g_k(\tau)d\tau \lesssim \mathcal{X}_p^2(1). \tag{5.50}$$

In order to bound the integral on $[1,t]$ for $2 \le t \le T$, we notice that

$$\sum_{2^k>R_0} \sup_{2\le t\le T} \left(\langle t\rangle^\alpha \int_1^t e^{c_0(\tau-t)} 2^{k(\frac{n}{p}-1)} g_k(\tau)d\tau \right) \lesssim \sum_{2^k>R_0} 2^{k(\frac{n}{p}-1)} \sup_{1\le t\le T} (t^\alpha g_k(t)). \tag{5.51}$$

In nonlinear sources g_k^1, g_k^2 and g_k^3, the calculations for those terms with respect to O are totally similar, so we only bound $\mathrm{div}(aO)$ and $\Lambda(\partial_k v^i O^{kj})$ for brevity. We write

$$\mathrm{div}(aO) = a\nabla\cdot O + \nabla a \cdot O.$$

Due to the same regularity level, it suffices to estimate the term $a\nabla\cdot O$. By using Lemma 2.3, we deduce that

$$\|t^\alpha(a\nabla\cdot O^h)\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}-1}_{p,1}} \lesssim \|a\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}} \|t^\alpha \nabla O\|^h_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}-1}_{p,1}} \le \mathcal{X}_p(T)\mathcal{G}_p(T), \tag{5.52}$$

and

$$\begin{aligned} &\|t^\alpha(a\nabla\cdot O^\ell)\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}-1}_{p,1}}\\ \lesssim\ & \|t^{\frac{\alpha}{2}} a\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}} \|t^{\frac{\alpha}{2}} O\|^\ell_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}}\\ \lesssim\ & \left(\|t^{\frac{\alpha}{2}} a\|^\ell_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{2}}_{2,1}} + \|t^{\frac{\alpha}{2}} a\|^h_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}} \right) \|t^{\frac{\alpha}{2}} O\|^\ell_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{2}}_{2,1}} \le \mathcal{G}_p^2(T), \end{aligned}\tag{5.53}$$

since the fact $\frac{\alpha}{2} \le \frac{s_0}{2} + \frac{n}{4} - \frac{\varepsilon}{2}$ indicates that $\|t^{\frac{\alpha}{2}} z\|^\ell_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{2}}_{2,1}} \lesssim \|t^{\frac{\alpha}{2}} z\|^\ell_{L^\infty_T \dot{B}^{\frac{n}{2}-\varepsilon}_{2,1}} \lesssim \mathcal{G}_p(T)$ for $z = a, O, v$. Combining (5.52) and (5.53), we get

$$\|t^\alpha(a\nabla\cdot O)\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}-1}_{p,1}} \lesssim \mathcal{X}_p(T)\mathcal{G}_p(T) + \mathcal{G}_p^2(T). \tag{5.54}$$

In addition, it follows from (1.7) that

$$\|\tau\nabla v\|_{\tilde{L}^\infty_t \dot{B}^{\frac{n}{p}}_{p,1}} \lesssim \mathcal{G}_p(t). \tag{5.55}$$

By Lemmas 2.1 and 2.3, we have

$$\begin{aligned} &\|t^\alpha \Lambda(\partial_k v^i O^{kj})\|^h_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}-1}_{p,1}}\\ \lesssim\ & \|t^{\alpha-1} O\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}} \|t\nabla v\|_{\tilde{L}^\infty_T \dot{B}^{\frac{n}{p}}_{p,1}} \end{aligned}$$

$$\lesssim \Big(\|t^{\alpha-1}O\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}}_{p,1}}+\|t^{\alpha-1}O\|^{h}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}}_{p,1}}\Big)\|t\nabla v\|_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{5.56}$$

It is obvious that $\|t^{\alpha-1}O\|^{h}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}}_{p,1}}\le \mathcal{G}_p(T)$ according to the definition of $\mathcal{G}_p(T)$. On the other hand, we have the following estimates for $z=a,O,v$,

$$\|t^{\alpha-1}z\|^{\ell}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim \|t^{\alpha-1}z\|^{\ell}_{L^\infty_T\dot{B}^{\frac{n}{2}-1-2\varepsilon}_{2,1}}\le \mathcal{G}_p(T), \tag{5.57}$$

as $\alpha-1=\frac{1}{2}(s_0+n/2-1-2\varepsilon)$ with enough small ε. Consequently, we arrive at

$$\|t^{\alpha}\Lambda(\partial_k v^i O^{kj})\|^{h}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim \mathcal{G}^2_p(T). \tag{5.58}$$

In a conclusion, by combining those estimates involving a and v in [7], we can conclude that

$$\|\langle t\rangle^{\alpha}(\Lambda a,\Lambda O;v)\|^{h}_{\tilde{L}^\infty_T\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim \|(\Lambda a_0,\Lambda O_0;v_0)\|^{h}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\mathcal{G}^2_p(T)+\mathcal{X}^2_p(T). \tag{5.59}$$

5.3. **Decay and gain of regularity for the high frequencies of v.** In order to bound the last term in $\mathcal{G}_p(t)$, it is convenient to rewrite the velocity equation in the following way. First, it follows from (3.4) that

$$\begin{aligned}\partial_t v-\mathcal{A}v &= F\\ &\triangleq -(1+K(a))\nabla a-v\cdot\nabla v+\nabla\cdot O+O^{jk}\partial_j O^{\bullet k}-I(a)\mathcal{A}v\\ &\quad+\frac{1}{1+a}\mathrm{div}\big(2\tilde{\mu}(a)D(v)+\tilde{\lambda}(a)\mathrm{div}v\mathrm{Id}\big)\end{aligned} \tag{5.60}$$

Hence, we have

$$\partial_t(t\mathcal{A}v)-\mathcal{A}(t\mathcal{A}v)=\mathcal{A}v+t\mathcal{A}F. \tag{5.61}$$

It follows from Proposition A.4 and the subsequent remark that

$$\begin{aligned}\|\tau\nabla^2 v\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}} &\lesssim \|\mathcal{A}v\|^{h}_{\tilde{L}^1_t\dot{B}^{\frac{n}{p}-1}_{p,1}}+\|\tau\mathcal{A}F\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-3}_{p,1}}\\ &\lesssim \|v\|^{h}_{\tilde{L}^1_t\dot{B}^{\frac{n}{p}+1}_{p,1}}+\|\tau F\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}}\\ &\lesssim \mathcal{X}_p(t)+\|\tau F\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}},\end{aligned} \tag{5.62}$$

where we used Theorem 1.2. Secondly, we turn to bound the norm $\|\tau F\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}}$. Because $\alpha\ge 1$, we have

$$\|\tau(\nabla a,\nabla\cdot O)\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim \|\langle\tau\rangle^{\alpha}(a,O)\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{5.63}$$

Product and composition estimates indicate that

$$\|\tau(K(a)\nabla a,O^{jk}\partial_j O^{\bullet k})\|_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}-1}_{p,1}}\lesssim \|\tau^{\frac{1}{2}}(a,O)\|^2_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}}_{p,1}}\lesssim \mathcal{G}^2_p(t). \tag{5.64}$$

Together with those estimates for other nonlinear terms (see [7]), we can conclude that

$$\|\tau\nabla v\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}}_{p,1}}\lesssim \mathcal{X}^2_p(t)+\mathcal{G}^2_p(t)+\|\langle\tau\rangle^{\alpha}(a,O)\|^{h}_{\tilde{L}^\infty_t\dot{B}^{\frac{n}{p}}_{p,1}}, \tag{5.65}$$

where the last term on the right-side of (5.65) can be bounded by (5.59). Finally, adding up (5.35), (5.59) and (5.65) yields for all $t\ge 0$

$$\mathcal{G}_p(t)\lesssim \mathcal{G}_{p,0}+\|(a_0,O_0;v_0)\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}}+\|(\nabla a_0,\nabla O_0;v_0)\|^{h}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}+\mathcal{G}^2_p(t)+\mathcal{X}^2_p(t)$$

$$\lesssim \quad \mathcal{G}_{p,0} + \mathcal{E}_0^{n/p} + \mathcal{G}_p^2(t) + \mathcal{X}_p^2(t). \tag{5.66}$$

It follows from Theorem 1.2 that $\mathcal{X}_p(t) \leq M\mathcal{E}_0^{n/p} \leq M\eta \ll 1$. On the other hand, it is easy to check that $\|(a_0, O_0; v_0)\|^{\ell}_{\dot{B}^{\frac{n}{2}-1}_{2,1}} \lesssim \|(a_0, O_0; v_0)\|^{\ell}_{\dot{B}^{-s_0}_{2,\infty}}$, so we conclude that (1.6) is fulfilled for all time if $\mathcal{G}_{p,0}$ and $\|(\nabla a_0, \nabla O_0; v_0)\|^{h}_{\dot{B}^{\frac{n}{p}-1}_{p,1}}$ are small enough. This finishes the proof of Theorem 1.3 eventually. □

Appendix: Some estimates in the hybrid Besov space.

Proposition A.1 ([5]). *Let $s_1, s_2, t_1, t_2, \sigma, \tau \in \mathbb{R}$, $2 \leq p \leq 4$ and $1 \leq r, r_1, r_2, r_3, r_4 \leq \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}$. Then we have the following:*

- *If $\sigma, \tau \leq n/p$ and $\sigma + \tau > 0$, then*

$$\sum_{2^k > R_0} 2^{k(\sigma+\tau-n/p)} \|\dot{\Delta}_k(fg)\|_{L^r_T L^p} \leq C\|f\|_{\tilde{L}^{r_1}_T \dot{\mathcal{B}}^{n/2-n/p+\sigma,\sigma}_{2,p}} \|g\|_{\tilde{L}^{r_2}_T \dot{\mathcal{B}}^{n/2-n/p+\tau,\tau}_{2,p}}. \tag{A.1}$$

- *If $s_1, s_2 \leq n/p$ and $s_1 + t_1 > n - \frac{2n}{p}$ with $s_1 + t_1 = s_2 + t_2$ and $\gamma \in \mathbb{R}$, then*

$$\begin{aligned} &\sum_{2^k \leq R_0} 2^{k(s_1+t_1-n/2)} \|\dot{\Delta}_k(fg)\|_{L^r_T L^2} \\ \leq\ & C\big(\|f\|_{\tilde{L}^{r_1}_T \dot{\mathcal{B}}^{s_1, s_1-n/2+n/p}_{2,p}} \|g\|_{\tilde{L}^{r_2}_T \dot{\mathcal{B}}^{t_1, t_1-n/2+n/p+\gamma}_{2,p}} \\ &\quad + \|g\|_{\tilde{L}^{r_3}_T \dot{\mathcal{B}}^{s_2, s_2-n/2+n/p}_{2,p}} \|f\|_{\tilde{L}^{r_4}_T \dot{\mathcal{B}}^{t_2, t_2-n/2+n/p}_{2,p}}\big). \end{aligned} \tag{A.2}$$

- *If $s_1, s_2 \leq n/2$ and $s_1 + t_1 > \frac{n}{2} - \frac{n}{p}$ with $s_1 + t_1 = s_2 + t_2$, then*

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} 2^{k(s_1+t_1-n/2)} \|\dot{\Delta}_k(fg)\|_{L^r_T L^2} \\ \leq\ & C\big(\|f\|_{\tilde{L}^{r_1}_T \dot{\mathcal{B}}^{s_1, s_1-n/2+n/p}_{2,p}} \|g\|_{\tilde{L}^{r_2}_T \dot{B}^{t_1}_{2,1}} + \|g\|_{\tilde{L}^{r_3}_T \dot{\mathcal{B}}^{s_2, s_2-n/2+n/p}_{2,p}} \|f\|_{\tilde{L}^{r_4}_T \dot{B}^{t_2}_{2,1}}\big). \end{aligned} \tag{A.3}$$

Proposition A.2 ([5]). *Let $2 \leq p \leq 4$, $s, \sigma > 0$, and $s \geq \sigma - n/2 + n/p$, $r \geq 1$. Assume that $F \in W^{[s]+2,\infty}_{loc} \cap W^{[\sigma]+2,\infty}_{loc}$ with $F(0) = 0$. Then there haves*

$$\|F(f)\|_{\tilde{L}^r_T \dot{\mathcal{B}}^{s,\sigma}_{2,p}} \leq C(1 + \|f\|_{\tilde{L}^\infty_T \dot{\mathcal{B}}^{n/p,n/p}_{2,p}})^{\max([s],[\sigma])+1} \|f\|_{\tilde{L}^r_T \dot{\mathcal{B}}^{s,\sigma}_{2,p}}. \tag{A.4}$$

For any $s > 0$ and $p \geq 1$, there haves

$$\|F(f)\|_{\tilde{L}^r_T \dot{B}^s_{p,1}} \leq C(1 + \|f\|_{L^\infty_T L^\infty})^{[s]+1} \|f\|_{\tilde{L}^r_T \dot{B}^s_{p,1}}. \tag{A.5}$$

Proposition A.3 ([11]). *Let $s \in (-n\min(1/p, 1/p'), 1 + n/p)$ and $1 \leq p, q \leq \infty$. Let v be a vector field such that $\nabla v \in L^1_T \dot{B}^{n/p}_{p,1}$. Assume that $f_0 \in \dot{B}^s_{p,q}$, $g \in L^1_T \dot{B}^s_{p,q}$, and f is a solution of the transport equation*

$$\partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0.$$

Then for $t \in [0, T]$, there holds

$$\|f\|_{\tilde{L}_t \dot{B}^s_{p,q}} \leq \exp\big(C\int_0^t \|\nabla v(\tau)\|_{\dot{B}^{n/p}_{p,1}} d\tau\big)\big(\|f_0\|_{\dot{B}^s_{p,q}} + \int_0^t \|g(\tau)\|_{\dot{B}^s_{p,q}} d\tau\big).$$

For the heat equation, one has the following parabolic regularity estimate.

Proposition A.4. *Let $p, r \in [1,\infty]$, $s \in \mathbb{R}$, and $1 \le \rho_2 \le \rho_1 \le \infty$ Assume that $u_0 \in \dot{B}^{s-1}_{p,r}$, $f \in \tilde{L}^{\rho_2}_T \dot{B}^{s-3+\frac{2}{\rho_2}}_{p,r}$. Let u be a solution of the equation*

$$\partial_t u - \mu \Delta u = f, \quad u|_{t=0} = u_0.$$

Then for $t \in [0,T]$, there holds

$$\mu^{\frac{1}{\rho_1}} \|u\|_{\tilde{L}^{\rho_1}_T \dot{B}^{s-1+2/\rho_1}_{p,r}} \le C\big(\|u_0\|_{\dot{B}^{s-1}_{p,r}} + \mu^{1/\rho_2-1}\|f\|_{\tilde{L}^{\rho_2}_T \dot{B}^{s-3+\frac{2}{\rho_2}}_{p,r}}\big). \tag{A.6}$$

Remark A.1. The estimate (A.6) is still hold for the following equation

$$\partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = f, \quad u|_{t=0} = u_0, \tag{A.7}$$

where λ and μ are constants such that $\mu > 0$ and $\lambda + \mu > 0$(up to the different dependence on the viscous coefficients). Indeed, both $\mathcal{P}u$ and $\mathcal{P}^\perp u$ satisfy the heat equation. We can apply $\mathcal{P}$ and $\mathcal{P}^\perp$ to (A.7) to get the heat estimate (A.6).

Acknowledgments. The authors would like to thank the anonymous referee for his (her) valuable suggestions. X. Pan is partially supported by the Research Foundation of Nanjing University of Aeronautics and Astronautics (No.1008-YAH17070), Natural Science Foundation of Jiangsu Province (SBK2018041027) and National Natural Science Foundation of China (11801268). J. Xu is partially supported by the National Natural Science Foundation of China (11471158, 11871274) and the Fundamental Research Funds for the Central Universities (NE2015005).

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Received March 2018; revised August 2018.

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