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Global existence of solutions to 1-d Euler equations with time-dependent damping

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ABSTRACT

In this paper, we study the 1-d isentropic Euler equations with time-dependent damping. Our damping decays at a speed of order −1 with respect to time which is a little weaker than the linear one. Under our assumption, we will prove the global existence of solutions to the Euler system and obtain L_2 and L_∞ estimates for the solutions. Our approach is based on some detailed energy estimates.

1. Introduction

We study the isentropic Euler equations with time-dependent damping in 1 dimension:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\dfrac{\mu}{1+t}\rho u, \\ \rho|_{t=0} = 1 + \varepsilon\rho_0(x), \qquad u|_{t=0} = \varepsilon u_0(x), \end{cases} \tag{1.1}$$

where $\rho_0(x), u_0(x) \in C_0^\infty(\mathbb{R})$, supported in $|x| \le R$ and $\varepsilon > 0$ is sufficient small. Here $\rho(x), u(x)$ and $P(x)$ represent the density, fluid velocity and pressure respectively and $\mu > 0$ is a positive number to describe the scale of the damping. We assume the fluid is a polytropic gas which means we assume $P(\rho) = \frac{1}{\gamma}\rho^\gamma$, $1 < \gamma < 3$. We denote $c^2 = P'(\rho)$.

As is well known, for the small data, when the damping is vanishing, the smooth solution of compressible Euler flow will blow up in finite time. While 1-d Euler equations with constant damping, read as

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = -\kappa\rho u, \end{cases} \tag{1.2}$$

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where κ is a constant, have a global smooth solution. Many authors [2,7,9] have proved the global existence and given a convergence rate of solutions to system (1.2). See also [3,5] etc. and the references therein.

It is natural to ask whether the global solution exists when the damping is decayed and what is the critical decayed rate to separate the global existence and the finite-time blow up of solutions with small data. Our article discusses the 1-dimension Euler equations with damping of time-decayed order -1. More precisely, for the damping decayed as $\frac{\mu}{1+t}$, we believe that $\mu = 2$ is the critical value. That is, when $\mu > 2$, system (1.1) has a global smooth solution. While $0 \le \mu \le 2$, the C^1 solution of (1.1) will blow up in finite time.

The main purpose of this paper is to study the global existence of solutions to (1.1) when $\mu > 2$. For the blow up of the C^1 solution, readers can see the second part of [13] for details.

Our system can be transformed into the nonlinear damped wave equation (see Eq. (2.3)) and developed to the problem of "diffusion phenomenon" of the damped wave equation (see [8,6,1,14] etc.). That is, the solution to the Cauchy problem for

$$w_{tt} - \Delta w + b(t) w_t = 0$$

behaves, as $t \to \infty$, like the solution of the corresponding parabolic equation

$$-\Delta \phi + b(t)\phi_t = 0$$

with suitable conditions on $b(t)$. Simply saying, when $b(t) = (1+t)^{-\beta}(-1 < \beta < 1)$, the diffusion phenomenon holds, while if $b(t) = (1+t)^{-\beta}(\beta > 1)$, then the solution behaves like the corresponding wave equation [12,11]. The case $b(t) = \frac{\mu}{1+t}$ corresponds to the critical one. And the constant $\mu = 2$ is also critical in the following sense. It holds that, when the equation is linear and $\mu > 2$, the energy $E(t)$ decays with order t^{-1} and decays slower for $\mu < 2$ [10]. Also, $\mu = 2$ separates the global existence and finite-time blow up of solutions to system (1.1).

By the Darcy's law or the diffusion phenomenon, the solution (ρ, u) is expected to behave, as $t \to \infty$, like the solution to the corresponding parabolic system

$$\begin{cases} \partial_t \bar\rho + \partial_x(\bar\rho \bar u) = 0, \\ \partial_x P(\bar\rho) = -\dfrac{\mu}{1+t}\bar\rho \bar u, \end{cases}$$

when $\mu > 2$. This equations can be transformed into

$$\begin{cases} \dfrac{\mu}{1+t}\partial_t \bar c - \bar c^2 \partial_x^2 \bar c - \dfrac{\gamma+1}{\gamma-1}\bar c(\partial_x \bar c)^2 = 0, \\ \bar u = -\dfrac{1+t}{\mu(\gamma-1)}\partial_x(\bar c^2), \end{cases}$$

where $\bar c = \sqrt{P'(\bar\rho)} = (\bar\rho)^{\frac{\gamma-1}{2}}$. In our next paper, we will prove that $(\bar c, \bar u)$ is an approximate solution to the solution (c, u) of system (1.1) and show that the decay of $(c - \bar c, u - \bar u)$ is faster than $(1+t)^{-1}$.

Throughout this paper we denote a generic constant by C. $H^m(R)$ denotes the usual Sobolev space with its norm

$$\|f\|_{H^m} \triangleq \sum_{k=0}^{m} \|\partial_x^k f\|_{L_p}.$$

For convenience, we use $\|\cdot\|$ to denote $\|\cdot\|_{L_2}$ and $\|\cdot\|_m$ for $\|\cdot\|_{H^m}$.

Theorem 1.1. *Suppose $(\rho_0, u_0) \in H^m(R), m \ge 3$ and $\mu > 2$. Then there exists a unique global classical solution $(\rho(x,t), u(x,t))$ of (1.1) satisfies*

$$\begin{aligned} &\|(1+t)\partial_t\rho(t)\|_{m-1}^2 + \|(1+t)\partial_x\rho(t)\|_{m-1}^2 + \|(1+t)\partial_x u(t)\|_{m-1}^2 + \|(\rho(t)-1)\|^2 + \|u(t)\|^2 \\ &\quad + \int_0^t (1+\tau)(\|\partial_\tau \rho(\tau)\|_{m-1}^2 + \|\partial_x\rho(\tau)\|_{m-1} + \|\partial_x u(\tau)\|_{m-1}) d\tau \le C\varepsilon^2(\|\rho_0\|_m^2 + \|u_0\|_m^2). \end{aligned}$$

Remark 1.1. By Sobolev inequality, we have the pointwise estimates

$$\sum_{\alpha\le m-1}\sup\left\{(1+t)^{\frac{1+\min\{\alpha,1\}}{2}}(|\partial_x^\alpha v(x,t)|+|\partial_x^\alpha u(x,t)|)\right\}\le C\varepsilon.$$

Remark 1.2. Actually when $0\le\mu\le 2$, we can prove the blow up of C^1 solutions to system (1.1). See [13].

In Section 2, we will be devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by energy methods. Remember $c=\sqrt{P'(\rho)}=\rho^{\frac{\gamma-1}{2}}$. First we transform (1.1) into

$$\begin{cases}\dfrac{2}{\gamma-1}\partial_t c+c\partial_x u+\dfrac{2}{\gamma-1}u\partial_x c=0,\\ \partial_t u+u\partial_x u+\dfrac{2}{\gamma-1}c\partial_x c+\dfrac{\mu}{1+t}u=0,\\ c|_{t=0}=1+\varepsilon c_0(x),\qquad u|_{t=0}=\varepsilon u_0(x),\end{cases}\tag{2.1}$$

where $c_0(x)\in C_0^\infty(\mathbb{R})$, supported in $|x|\le R$.

Let $v=\frac{2}{\gamma-1}(c-1)$, then v,u satisfy

$$\begin{cases}\partial_t v+\partial_x u=-u\partial_x v-\dfrac{\gamma-1}{2}v\partial_x u,\\ \partial_t u+\partial_x v+\dfrac{\mu}{1+t}u=-u\partial_x u-\dfrac{\gamma-1}{2}v\partial_x v,\\ v|_{t=0}=\varepsilon v_0(x),\qquad u|_{t=0}=\varepsilon u_0(x),\end{cases}\tag{2.2}$$

where $v_0(x)=\frac{2}{\gamma-1}c_0(x)$.

From (2.2), we have

$$\partial_{tt}v-\partial_{xx}v+\frac{\mu}{1+t}\partial_t v=Q(v,u),\tag{2.3}$$

where

$$Q(v,u)=\frac{\mu}{1+t}\left(-u\partial_x v-\frac{\gamma-1}{2}v\partial_x u\right)-\partial_t(u\partial_x v)-\frac{\gamma-1}{2}\partial_t(v\partial_x u)+\partial_x(u\partial_x u)+\frac{\gamma-1}{2}\partial_x(v\partial_x v).$$

In the following, for $m\ge 3$, we will estimate (v,u) under the a priori assumption

$$\begin{aligned}E_m(T)=:&\sup_{0<t<T}\left\{\|(1+t)\partial_x v(t)\|_{m-1}^2+\|(1+t)\partial_x u(t)\|_{m-1}^2+\|v(t)\|^2+\|u(t)\|^2\right\}^{\frac12}\\ \le& M\varepsilon\end{aligned}\tag{2.4}$$

where M, independent of ε, will be determined later. By choosing M big and ε small suitably, we will prove

$$E_m(T)\le\frac12 M\varepsilon.\tag{2.5}$$

By Sobolev inequality, we know that

$$\sum_{|\alpha|\le 1}\sup\left\{(1+t)^{\frac{\alpha+1}{2}}(|\partial_x^\alpha v(x,t)|+|\partial_x^\alpha u(x,t)|)\right\}\le CE_2(t)\le CM\varepsilon.$$

Since (2.2) implies

$$|(1+t)\partial_t v|+|(1+t)\partial_t u|\le C\left\{|u|+(1+t)|\partial_x v|+(1+t)|\partial_x u|\right\},$$

we also have

$$\sum_{\alpha+\beta\le 1} \sup\left\{(1+t)^{\frac{\alpha+\beta+1}{2}}(|\partial_x^\alpha\partial_t^\beta v(x,t)|+|\partial_x^\alpha\partial_t^\beta u(x,t)|)\right\}\le CM\varepsilon. \tag{2.6}$$

In the following, we will first obtain some elementary estimates for the 1-order derivatives of the solution. Then the higher derivatives will be handled in a similar way.

2.1. Estimate 1

For some constant λ, to be determined later, multiplying (2.3) by $\lambda(1+t)^2\partial_t v$ yields

$$\frac{\lambda}{2}\partial_t[(1+t)^2(\partial_t v)^2]+\lambda(\mu-1)(1+t)(\partial_t v)^2-\lambda(1+t)^2\partial_t v\partial_{xx}v=\lambda(1+t)^2\partial_t vQ(v,u). \tag{2.7}$$

Integrating it over $\mathbb{R}\times[0,t]$ and using integration by parts give

$$\begin{aligned}&\frac{\lambda}{2}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx+\lambda(\mu-1)\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau\\&\quad+\frac{\lambda}{2}\int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx-\lambda\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau\\&=\frac{\lambda}{2}\int_{\mathbb{R}}(\partial_t v)^2|_{t=0}dx+\frac{\lambda}{2}\int_{\mathbb{R}}(\partial_x v)^2|_{t=0}dx+\int_0^t\int_{\mathbb{R}}\lambda(1+\tau)^2\partial_\tau vQ(v,u)dxd\tau\\&\le C\varepsilon^2(\|v_0\|_1^2+\|u_0\|_1^2)+\int_0^t\int_{\mathbb{R}}\lambda(1+\tau)^2\partial_\tau vQ(v,u)dxd\tau.\end{aligned} \tag{2.8}$$

Also multiplying (2.3) by $(1+t)v$, we get

$$\partial_t[(1+t)v\partial_t v]+\frac{\mu-1}{2}\partial_t(v^2)-(1+t)v\partial_{xx}v-(1+t)(\partial_t v)^2=(1+t)vQ(v,u). \tag{2.9}$$

Then integrating (2.9) over $\mathbb{R}\times[0,t]$ and using integrating by parts give

$$\begin{aligned}&\int_{\mathbb{R}}(1+t)v\partial_t vdx+\frac{\mu-1}{2}\int_{\mathbb{R}}v^2dx+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau-\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau\\&=\int_{\mathbb{R}}(v\partial_t v)|_{t=0}dx+\frac{\mu-1}{2}\int_{\mathbb{R}}v^2|_{t=0}dx+\int_0^t\int_{\mathbb{R}}(1+\tau)vQ(v,u)dxd\tau\\&\le C\varepsilon^2(\|v_0\|_1^2+\|u_0\|_1^2)+\int_0^t\int_{\mathbb{R}}(1+\tau)vQ(v,u)dxd\tau.\end{aligned} \tag{2.10}$$

Adding (2.8) and (2.10), we have

$$\begin{aligned}&\frac{\lambda}{2}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx+\frac{\lambda}{2}\int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx+\frac{\mu-1}{2}\int_{\mathbb{R}}v^2dx\\&\quad+[\lambda(\mu-1)-1]\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau+(1-\lambda)\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau+\int_{\mathbb{R}}(1+t)v\partial_t vdx\\&\le C\varepsilon^2(\|v_0\|_1^2+\|u_0\|_1^2)+\int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v+(1+\tau)v]Q(v,u)dxd\tau.\end{aligned} \tag{2.11}$$

If $\mu=2+4\delta$ for some $\delta>0$, using Cauchy–Schwarz inequality, we have

$$\int_{\mathbb{R}}(1+t)v\partial_t vdx\ge-\frac{1+4\delta}{2(1+\delta)}\int_{\mathbb{R}}v^2dx-\frac{1+\delta}{2(1+4\delta)}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx. \tag{2.12}$$

From (2.11) and (2.12) by choosing $\lambda = \frac{1+2\delta}{1+4\delta}$, we have

$$\begin{aligned}
&\frac{\delta}{2(1+4\delta)}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx + \frac{1+2\delta}{2(1+4\delta)}\int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx + \frac{\delta(1+4\delta)}{2(1+\delta)}\int_{\mathbb{R}}v^2dx \\
&\quad + 2\delta\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau + \frac{2\delta}{1+4\delta}\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau \\
&\leq CE_1^2(0) + \int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v + (1+\tau)v]Q(v,u)dxd\tau. \qquad (2.13)
\end{aligned}$$

So we have

$$\begin{aligned}
&\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx + \int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx + \int_{\mathbb{R}}v^2dx \\
&\quad + \int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau + \int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau \leq CE_1^2(0) + CI \qquad (2.14)
\end{aligned}$$

where C depends on μ and

$$\begin{aligned}
I &= \int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v + (1+\tau)v]Q(v,u)dxd\tau \\
&= \int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v + (1+\tau)v]\frac{\mu}{1+\tau}\left(-u\partial_x v - \frac{\gamma-1}{2}v\partial_x u\right)dxd\tau \\
&\quad - \int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v + (1+\tau)v]\left[\partial_\tau(u\partial_x v) + \frac{\gamma-1}{2}\partial_\tau(v\partial_x u)\right]dxd\tau \\
&\quad + \int_0^t\int_{\mathbb{R}}[\lambda(1+\tau)^2\partial_\tau v + (1+\tau)v]\left[\partial_x(u\partial_x u) + \frac{\gamma-1}{2}\partial_x(v\partial_x v)\right]dxd\tau \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now we estimate I_1, I_2 and I_3.

$$I_1 = \lambda\mu\int_0^t\int_{\mathbb{R}}(1+\tau)\partial_\tau v\left(-u\partial_x v - \frac{\gamma-1}{2}v\partial_x u\right)dxd\tau + \mu\int_0^t\int_{\mathbb{R}}v\left(-u\partial_x v - \frac{\gamma-1}{2}v\partial_x u\right)dxd\tau.$$

From (2.6), using Cauchy–Schwarz inequality and integration by parts, we have

$$\begin{aligned}
I_1 &\leq CM\varepsilon\int_0^t\int_{\mathbb{R}}\left[(1+\tau)(|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_x u|^2) + \frac{u^2}{1+\tau}\right]dxd\tau + C\int_0^t\int_{\mathbb{R}}uv\partial_x v dxd\tau \\
&\leq CM\varepsilon\int_0^t\int_{\mathbb{R}}\left[(1+\tau)(|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_x u|^2) + \frac{u^2}{1+\tau}\right]dxd\tau. \qquad (2.15)
\end{aligned}$$

Now we focus on the estimate of I_2, then I_3 will be essentially the same with I_2. Dealing I_2 the same with I_1, we have

$$\begin{aligned}
I_2 &\leq CM\varepsilon\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau u|^2 + |\partial_x v|^2 + |\partial_\tau v|^2)dxd\tau \\
&\quad - \int_0^t\int_{\mathbb{R}}(1+\tau)^2u\partial_\tau v\partial^2_{x\tau}v dxd\tau - \int_0^t\int_{\mathbb{R}}(1+\tau)vu\partial^2_{x\tau}v dxd\tau \\
&\quad - \frac{\gamma-1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2v\partial_\tau v\partial^2_{x\tau}u dxd\tau - \frac{\gamma-1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)v^2\partial^2_{x\tau}u dxd\tau \\
&= CM\varepsilon\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau u|^2 + |\partial_x v|^2 + |\partial_\tau v|^2)dxd\tau
\end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x u(\partial_\tau v)^2dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)\partial_x(vu)\partial_\tau vdxd\tau$$
$$-\frac{\gamma-1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2v\partial_\tau v\partial_{x\tau}^2udxd\tau+\frac{\gamma-1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)\partial_x v^2\partial_\tau udxd\tau.$$

Using (2.6) again, we have

$$I_2\le CM\varepsilon\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau u|^2+|\partial_x v|^2+|\partial_\tau v|^2)dxd\tau-\frac{\gamma-1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2v\partial_\tau v\partial_{x\tau}^2udxd\tau. \tag{2.16}$$

From (2.2)

$$\partial_x u=\frac{-\partial_t v-u\partial_x v}{1+\frac{\gamma-1}{2}v}.$$

Then

$$\partial_{xt}^2u=-\frac{\partial_t^2v+\partial_t u\partial_x v+u\partial_{xt}^2v}{1+\frac{\gamma-1}{2}v}+\frac{\frac{\gamma-1}{2}\partial_t v(\partial_t v+u\partial_x v)}{\left(1+\frac{\gamma-1}{2}v\right)^2}. \tag{2.17}$$

From (2.6), we have

$$\frac{1}{1+\frac{\gamma-1}{2}v}\le\frac{1}{1-CM\varepsilon}. \tag{2.18}$$

Inserting (2.17) and (2.18) into (2.16), we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}(1+\tau)^2v\partial_\tau v\partial_{x\tau}^2udxd\tau\\&=\int_0^t\int_{\mathbb{R}}(1+\tau)^2v\partial_\tau v\left[-\frac{\partial_\tau^2v+\partial_\tau u\partial_x v+u\partial_{x\tau}^2v}{1+\frac{\gamma-1}{2}v}+\frac{\frac{\gamma-1}{2}\partial_\tau v(\partial_\tau v+u\partial_x v)}{\left(1+\frac{\gamma-1}{2}v\right)^2}\right]dxd\tau\\&\le-\int_0^t\int_{\mathbb{R}}(1+\tau)^2\frac{v}{1+\frac{\gamma-1}{2}v}\partial_\tau v(\partial_\tau^2v+u\partial_{x\tau}^2v)dxd\tau\\&\quad+\frac{CM^2\varepsilon^2}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau v|^2+|\partial_x v|^2)dxd\tau.\end{aligned} \tag{2.19}$$

Combining (2.16) and (2.19), we have

$$\begin{aligned}I_2&\le C\left[M\varepsilon+\frac{M^2\varepsilon^2}{(1-CM\varepsilon)^2}\right]\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau v|^2+|\partial_x v|^2+|\partial_\tau u|^2)dxd\tau\\&\quad-\int_0^t\int_{\mathbb{R}}\left[(1+\tau)^2\frac{v}{1+\frac{\gamma-1}{2}v}\partial_\tau v\partial_\tau^2v+(1+\tau)^2\frac{vu}{1+\frac{\gamma-1}{2}v}\partial_\tau v\partial_{x\tau}^2v\right]dxd\tau\\&=C\left[M\varepsilon+\frac{M^2\varepsilon^2}{(1-CM\varepsilon)^2}\right]\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau v|^2+|\partial_x v|^2+|\partial_\tau u|^2)dxd\tau+I_{2,1}+I_{2,2}.\end{aligned} \tag{2.20}$$

Using integration by parts with respect to τ, we have

$$\begin{aligned}I_{2,1}&=-\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2\frac{v}{1+\frac{\gamma-1}{2}v}\partial_\tau(\partial_\tau v)^2dxd\tau\\&=-\frac{1}{2}\int_{\mathbb{R}}(1+t)^2\frac{v}{1+\frac{\gamma-1}{2}v}(\partial_t v)^2dx+\frac{1}{2}\int_{\mathbb{R}}\frac{\varepsilon v_0}{1+\frac{\gamma-1}{2}\varepsilon v_0}(\partial_\tau v(\cdot,0))^2dx\end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\partial_\tau\left[(1+\tau)^2\frac{v}{1+\frac{\gamma-1}{2}v}\right](\partial_\tau v)^2dxd\tau$$

$$\leq\frac{C\varepsilon}{1-C\varepsilon}E_1^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau. \quad (2.21)$$

And using integration by parts, we have

$$\begin{aligned}I_{2,2}&=-\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2\frac{vu}{1+\frac{\gamma-1}{2}v}\partial_x(\partial_\tau v)^2dxd\tau\\&=\frac{1}{2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x\left[\frac{vu}{1+\frac{\gamma-1}{2}v}\right](\partial_\tau v)^2dxd\tau\\&\leq\frac{CM^2\varepsilon^2}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}(1+\tau)^2(\partial_\tau v)^2dxd\tau. \quad (2.22)\end{aligned}$$

Inserting (2.21) and (2.22) into (2.20), we have

$$\begin{aligned}I_2\leq&\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau v|^2+|\partial_x v|^2+|\partial_\tau u|^2)dxd\tau\\&+\frac{C\varepsilon}{1-C\varepsilon}E_1^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx. \quad (2.23)\end{aligned}$$

We can deal I_3 almost the same with I_2. Then we can get

$$\begin{aligned}I_3\leq&\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}(1+\tau)(|\partial_\tau v|^2+|\partial_x v|^2+|\partial_x u|^2)dxd\tau\\&+\frac{C\varepsilon}{1-C\varepsilon}E_1(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx. \quad (2.24)\end{aligned}$$

Remember that in (2.2), we have

$$|\partial_t u|\leq C\left(|\partial_x v|+|\partial_x u|+\frac{|u|}{1+t}\right).$$

Inserting the estimates of I_1 (2.15), I_2 (2.23) and I_3 (2.24) into (2.14), we have

$$\begin{aligned}&\int_{\mathbb{R}}(1+t)^2(\partial_t v)^2dx+\int_{\mathbb{R}}(1+t)^2(\partial_x v)^2dx+\int_{\mathbb{R}}v^2dx\\&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_\tau v)^2dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau\\&\leq C\frac{1+\varepsilon}{1-C\varepsilon}E_1^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2\left[(\partial_t v)^2+(\partial_x v)^2\right]dx\\&\quad+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_\tau v)^2+(\partial_x v)^2+(\partial_x u)^2\right]+\frac{u^2}{1+\tau}\right\}dxd\tau. \quad (2.25)\end{aligned}$$

2.2. Estimate 2

Multiplying (2.2) $_2$ by u and integrating on $\mathbb{R}\times[0,t]$ yield

$$\int_0^t\int_{\mathbb{R}}\left(\frac{1}{2}\partial_\tau u^2+u\partial_x v+\frac{\mu}{1+\tau}u^2\right)dxd\tau=\int_0^t\int_{\mathbb{R}}\left(-u\partial_x u-\frac{\gamma-1}{2}v\partial_x v\right)udxd\tau.$$

Then using integration by parts, we have

$$\frac{1}{2}\int_{\mathbb{R}} u^2dx - \frac{\varepsilon^2}{2}\int_{\mathbb{R}} u_0^2dx - \int_0^t\int_{\mathbb{R}} v\partial_x udxd\tau + \int_0^t\int_{\mathbb{R}} \frac{\mu}{1+\tau}u^2dxd\tau$$
$$= \int_0^t\int_{\mathbb{R}} \left(-u\partial_x u - \frac{\gamma-1}{2}v\partial_x v\right)udxd\tau.$$

Using (2.2) and (2.6), we have

$$\frac{1}{2}\int_{\mathbb{R}} u^2dx + \int_0^t\int_{\mathbb{R}} v(\partial_\tau v + u\partial_x v + \frac{\gamma-1}{2}v\partial_x u)dxd\tau + \int_0^t\int_{\mathbb{R}} \frac{\mu}{1+\tau}u^2dxd\tau$$
$$\leq CE_1^2(0) + CM\varepsilon\int_0^t\int_{\mathbb{R}}\left\{\frac{1}{1+\tau}u^2 + (1+\tau)\left[(\partial_x v)^2 + (\partial_x u)^2\right]\right\}dxd\tau.$$

Then we can have

$$\int_{\mathbb{R}}(u^2+v^2)dx + \int_0^t\int_{\mathbb{R}} \frac{\mu}{1+\tau}u^2dxd\tau$$
$$\leq CE_1^2(0) + CM\varepsilon\int_0^t\int_{\mathbb{R}}\left\{\frac{1}{1+\tau}u^2 + (1+\tau)\left[(\partial_x v)^2 + (\partial_x u)^2\right]\right\}dxd\tau. \tag{2.26}$$

2.3. Estimate 3

By differentiating (2.2) $_2$ with respect to x and integrating its product with $(1+\tau)^2\partial_x u$ on $\mathbb{R}\times[0,t]$, we have

$$\int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x u\left(\partial_{x\tau}^2 u + \partial_x^2 v + \frac{\mu}{1+\tau}\partial_x u\right)dxd\tau = \int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x u\partial_x\left(-u\partial_x u - \frac{\gamma-1}{2}v\partial_x v\right)dxd\tau.$$

Then

$$\frac{1}{2}\int_{\mathbb{R}}(1+t)^2(\partial_x u)^2dx + (\mu-1)\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x u)^2dxd\tau$$
$$+\int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x\left(\partial_\tau v + u\partial_x v + \frac{\gamma-1}{2}v\partial_x u\right)\partial_x vdxd\tau$$
$$\leq CE_1^2(0) + \int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x u\partial_x\left(-u\partial_x u - \frac{\gamma-1}{2}v\partial_x v\right)dxd\tau.$$

So

$$\frac{1}{2}\int_{\mathbb{R}}(1+t)^2\left[(\partial_x u)^2 + (\partial_x v)^2\right]dx + (\mu-1)\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x u)^2dxd\tau - \int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau$$
$$\leq -\int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x(u\partial_x v + \frac{\gamma-1}{2}v\partial_x u)\partial_x vdxd\tau$$
$$+CE_1^2(0) + \int_0^t\int_{\mathbb{R}}(1+\tau)^2\partial_x u\partial_x\left(-u\partial_x u - \frac{\gamma-1}{2}v\partial_x v\right)dxd\tau.$$

We can deal with the right terms almost the same with I_2 and I_3. Then we can get the estimate

$$\int_{\mathbb{R}}(1+t)^2\left[(\partial_x u)^2 + (\partial_x v)^2\right]dx + \int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x u)^2dxd\tau - \int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x v)^2dxd\tau$$
$$\leq C\frac{1+\varepsilon}{1-C\varepsilon}E_1^2(0) + \frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2\left[(\partial_t v)^2 + (\partial_x v)^2\right]dx$$
$$+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_\tau v)^2 + (\partial_x v)^2 + (\partial_x u)^2\right] + \frac{u^2}{1+\tau}\right\}dxd\tau. \tag{2.27}$$

Let (2.25) + (2.26) + $\frac{1}{2}$ × (2.27), then we get

$$\begin{aligned}&\int_{\mathbb{R}}\left\{(1+t)^2\left[(\partial_t v)^2+(\partial_x v)^2+(\partial_x u)^2\right]+v^2+u^2\right\}dx\\&\quad+\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_\tau v)^2+(\partial_x v)^2+(\partial_x u)^2\right]+\frac{u^2}{1+\tau}\right\}dxd\tau\\&\le C\frac{1+\varepsilon}{1-C\varepsilon}E_1^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2\left[(\partial_t v)^2+(\partial_x v)^2\right]dx\\&\quad+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_\tau v)^2+(\partial_x v)^2+(\partial_x u)^2\right]+\frac{u^2}{1+\tau}\right\}dxd\tau. \qquad (2.28)\end{aligned}$$

2.4. Estimates for higher derivatives

The estimates as (2.25) and (2.27) can also be obtained for higher derivatives. In fact, by multiplying (2.3) by $\partial_x^2[\lambda(1+\tau)^2\partial_\tau v+(1+\tau)v]$ and integrating it on $\mathbb{R}\times[0,t]$, we can get

$$\begin{aligned}&\int_{\mathbb{R}}(1+t)^2(\partial_{xt}^2 v)^2dx+\int_{\mathbb{R}}(1+t)^2(\partial_x^2 v)^2dx+\int_{\mathbb{R}}(\partial_x v)^2dx\\&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_{x\tau}^2 v)^2dxd\tau+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x^2 v)^2dxd\tau\\&\le C\frac{1+\varepsilon}{1-C\varepsilon}E_2^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2\left[(\partial_{xt}^2 v)^2+(\partial_x^2 v)^2\right]dx\\&\quad+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_{x\tau}^2 v)^2+(\partial_x^2 v)^2+(\partial_x^2 u)^2\right]\right\}dxd\tau. \qquad (2.29)\end{aligned}$$

By differentiating (2.2) $_2$ two times with respect to x and integrating its product with $(1+\tau)^2\partial_x^2 u$ on $\mathbb{R}\times[0,t]$, we have

$$\begin{aligned}&\int_{\mathbb{R}}(1+t)^2\left[(\partial_x^2 u)^2+(\partial_x^2 v)^2\right]dx+\int_0^t\int_{\mathbb{R}}(1+\tau)(\partial_x^2 u)^2dxd\tau\\&\le C\frac{1+\varepsilon}{1-C\varepsilon}E_2^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}\int_{\mathbb{R}}(1+t)^2\left[(\partial_{xt}^2 v)^2+(\partial_x^2 v)^2\right]dx\\&\quad+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\int_{\mathbb{R}}\left\{(1+\tau)\left[(\partial_{x\tau}^2 v)^2+(\partial_x^2 v)^2+(\partial_x^2 u)^2\right]\right\}dxd\tau. \qquad (2.30)\end{aligned}$$

Combining (2.28)–(2.30) gives

$$\begin{aligned}&\|(1+t)\partial_t v\|_1^2+\|(1+t)\partial_x v\|_1^2+\|(1+t)\partial_x u\|_1^2+\|v\|^2+\|u\|^2\\&\quad+\int_0^t\left[(1+\tau)(\|\partial_\tau v\|_1^2+\|\partial_x v\|_1^2+\|\partial_x u\|_1^2)+\frac{\|u\|^2}{1+\tau}\right]d\tau\\&\le C\frac{1+\varepsilon}{1-C\varepsilon}E_2^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}(\|(1+t)\partial_t v\|_1^2+\|(1+t)\partial_x v\|_1^2)\\&\quad+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\left[(1+\tau)(\|\partial_\tau v\|_1^2+\|\partial_x v\|_1^2+\|\partial_x u\|_1^2)+\frac{\|u\|^2}{1+\tau}\right]d\tau. \qquad (2.31)\end{aligned}$$

Actually, we can prove for any $m\ge 1$

$$\begin{aligned}&\|(1+t)\partial_t v\|_{m-1}^2+\|(1+t)\partial_x v\|_{m-1}^2+\|(1+t)\partial_x u\|_{m-1}^2+\|v\|^2+\|u\|^2\\&\quad+\int_0^t\left[(1+\tau)(\|\partial_\tau v\|_{m-1}^2+\|\partial_x v\|_{m-1}^2+\|\partial_x u\|_{m-1}^2)+\frac{\|u\|^2}{1+\tau}\right]d\tau\end{aligned}$$

$$\leq C\frac{1+\varepsilon}{1-C\varepsilon}E_m^2(0)+\frac{CM\varepsilon}{1-CM\varepsilon}(\|(1+t)\partial_t v\|_{m-1}^2+\|(1+t)\partial_x v\|_{m-1}^2)$$
$$+\frac{CM\varepsilon}{(1-CM\varepsilon)^2}\int_0^t\left[(1+\tau)(\|\partial_\tau v\|_{m-1}^2+\|\partial_x v\|_{m-1}^2+\|\partial_x u\|_{m-1}^2)+\frac{\|u\|^2}{1+\tau}\right]d\tau. \tag{2.32}$$

When ε is small, for some C_0, we get

$$\|(1+t)\partial_t v\|_{m-1}^2+\|(1+t)\partial_x v\|_{m-1}^2+\|(1+t)\partial_x u\|_{m-1}^2+\|v\|^2+\|u\|^2$$
$$+\int_0^t\left[(1+\tau)(\|\partial_\tau v\|_{m-1}^2+\|\partial_x v\|_1^2+\|\partial_x u\|_{m-1}^2)+\frac{\|u\|^2}{1+\tau}\right]d\tau\leq\frac{(1-CM\varepsilon)^2}{(1-CM\varepsilon)^2-CM\varepsilon}C_0\varepsilon^2.$$

Let $M^2=5C_0$. By using the smallness of ε, we can have

$$E_m^2(t)\leq\frac{1}{4}M^2\varepsilon^2. \tag{2.33}$$

The local existence of symmetrizable hyperbolic equations has been proved by using the fixed point theorem in [4]. In order to get the global existence of the system, we only need a priori estimate. Based on our above estimate (2.33), we yield (2.5). This finishes the proof of our Theorem 1.1.

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References

[1] Takafumi Hosono, Takayoshi Ogawa, Large time behavior and Lp-Lq estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (1) (2004) 82–118.
[2] Ling Hsiao, Tai-Ping Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, Comm. Math. Phys. 143 (3) (1992) 599–605.
[3] Feimin Huang, Pierangelo Marcati, Ronghua Pan, Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum, Arch. Ration. Mech. Anal. 176 (1) (2005) 1–24.
[4] Tosio Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Ration. Mech. Anal. 58 (3) (1975) 181–205.
[5] Pierangelo Marcati, Ming Mei, Convergence to nonlinear diffusion waves for solutions of the initial boundary problem to the hyperbolic conservation laws with damping, Quart. Appl. Math. 58 (4) (2000) 763–784.
[6] Pierangelo Marcati, Kenji Nishihara, The $L_p - L_q$ estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2) (2003) 445–469.
[7] Kenji Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, J. Differential Equations 131 (2) (1996) 171–188.
[8] Kenji Nishihara, $L_p - L_q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (3) (2003) 631–649.
[9] Kenji Nishihara, Weike Wang, Tong Yang, L_p-convergence rate to nonlinear diffusion waves for p-system with damping, J. Differential Equations 161 (1) (2000) 191–218.
[10] Jens Wirth, Asymptotic properties of solutions to wave equations with time-dependent dissipation (Ph.D.-Thesis), TU Bergakademie Freiberg, 2004.
[11] Jens Wirth, Wave equations with time-dependent dissipation. I. Non-effective dissipation, J. Differential Equations 222 (2) (2006) 487–514.
[12] Jens Wirth, Wave equations with time-dependent dissipation. II. Effective dissipation, J. Differential Equations 232 (1) (2007) 74–103.
[13] Pan Xinghong, Remarks on 1-D Euler equations with time-decayed damping, arXiv:1510.08115.
[14] Han Yang, Albert Milani, On the diffusion phenomenon of quasilinear hyperbolic waves, Bull. Sci. Math. 124 (5) (2000) 415–433.