Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 8485–8529

Journal of Differential Equations

www.elsevier.com/locate/jde

Regularity of solutions to axisymmetric Navier–Stokes equations with a slightly supercritical condition

Xinghong Pan

Department of Mathematics and IMS, Nanjing University, Nanjing 210093, China

Received 18 September 2015; revised 25 January 2016

Available online 3 March 2016

Abstract

Consider an axisymmetric suitable weak solution of 3D incompressible Navier–Stokes equations with nontrivial swirl, $v = v_r e_r + v_\theta e_\theta + v_z e_z$. Let z denote the axis of symmetry and r be the distance to the z-axis. If the solution satisfies a slightly supercritical assumption (that is, $|v| \le C\frac{(\ln|\ln r|)^\alpha}{r}$ for $\alpha \in [0, 0.028]$ when r is small), then we prove that v is regular. This extends the results in [6,16,18] where regularities under critical assumptions, such as $|v| \le \frac{C}{r}$, were proven.

As a useful tool in the proof of our main result, an upper-bound estimate to the fundamental solution of the parabolic equation with a critical drift term will be given in the last part of this paper.

MSC: 35Q30; 76N10

Keywords: Axisymmetric Navier–Stokes equations; Slightly supercritical; Moser iteration; Regularity; The fundamental solution

1. Introduction

The incompressible Navier–Stokes equations in *cartesian coordinates* are given by

$$\partial_t v + (v \cdot \nabla)v + \nabla p = \Delta v, \quad \nabla \cdot v = 0,$$

E-mail address: math.scrat@gmail.com.

http://dx.doi.org/10.1016/j.jde.2016.02.026

where v is the velocity field and p is the pressure. We consider the axisymmetric solution of the equations. That means, in *cylindrical coordinates* r, θ, z with $x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$, the solution is of this form

$$v = v_r e_r + v_\theta e_\theta + v_z e_z,$$

where the basis vectors e_r, e_θ, e_z are

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_z = (0, 0, 1),$$

and the components v_r, v_θ, v_z do not depend on θ.

Recall that v_r, v_θ, v_z satisfy

$$\begin{cases} \partial_t v_r + (b\cdot\nabla)v_r - \dfrac{(v_\theta)^2}{r} + \partial_r p = (\Delta - \dfrac{1}{r^2})v_r, \\ \partial_t v_\theta + (b\cdot\nabla)v_\theta + \dfrac{v_\theta v_r}{r} = (\Delta - \dfrac{1}{r^2})v_\theta, \\ \partial_t v_z + (v\cdot\nabla)v_z + \partial_z p = \Delta v_z, \\ b = v_r e_r + v_z e_z, \quad \nabla\cdot b = \partial_r v_r + \dfrac{v_r}{r} + \partial_z v_z = 0. \end{cases} \tag{1.1}$$

In this paper we study the axisymmetric Navier–Stokes equations under a slightly supercritical assumption on the drift term b. To be precise, we consider b such that

$$|b| = \sqrt{v_r^2 + v_z^2} \le \begin{cases} \dfrac{(\ln|\ln\frac{r}{3}|)^\alpha}{r} & \text{if } r \le 1, \\ \dfrac{C}{r} & \text{if } r > 1. \end{cases} \tag{1.2}$$

Here $\alpha \in [0, 0.028]$ is a fixed constant. Later we will see how 0.028 is obtained.

Our main result is the following.

Theorem 1.1. *Let (v, p) be a suitable weak solution of the axisymmetric Navier–Stokes equation (1.1) in $\mathbb{R}^3 \times [-1, 0)$. Assume that b satisfies (1.2) and $\sup_{x\in\mathbb{R}^3} |rv_\theta(\cdot, -1)| < +\infty$. Then we have*

$$\sup_{(x,t)\in\mathbb{R}^3\times[-1,0)} |v| < +\infty.$$

Remark 1.1. Readers can refer to [5] for the definition of suitable weak solutions.

Define the quantity $\Gamma = rv_\theta$, which satisfies

$$\partial_t\Gamma + (b\cdot\nabla)\Gamma - \Delta\Gamma + \frac{2}{r}\partial_r\Gamma = 0. \tag{1.3}$$

It satisfies the following maximum principle:

$$\sup_t \|\Gamma(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \le \|\Gamma(\cdot,-1)\|_{L^\infty(\mathbb{R}^3)}.$$

Global in-time regularity of the solution to the axisymmetric Navier–Stokes equations is still open. Under the no swirl assumption, $v^\theta = 0$, Ladyzhenskaya [17] and Ukhovskii–Iudovich [28] independently proved that weak solutions are regular for all time. When the swirl v^θ is non-trivial, recently, some efforts and progress have been made on the regularity of the axisymmetric solutions. In [6], Chen–Strain–Yau–Tsai proved that the suitable weak solutions are regular if the solution satisfies $r|b| \le C_* < \infty$. Their method is based on Nash [23], Moser [22] and De Giorgi [9]. Also, Koch–Nadirashvili–Seregin–Sverak in [16] proved the same result using a Liouville theorem and scaling-invariant property. Lei–Zhang in [18] proved regularity of the solution under a more general assumption on the drift term b where $b \in L^\infty\big([-1,0), BMO^{-1}\big)$.

The natural scaling of Navier–Stokes equations is: If (v, p) is a solution of equations (1.1), then for any $\lambda > 0$, the scaled pair $v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t)$, $p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ is also a solution.

It seems that their assumptions on b are all critical in the following scaled sense:

Denote $b^\lambda = v_r^\lambda e_r + v_z^\lambda e_z$ and let $x_0 = (r_0\cos\theta_0, r_0\sin\theta_0, z_0)$ be a fixed space point. For the scaled solution, the assumption $r_0|b(x_0,t)| \le C_*$ becomes

$$r_0|b^\lambda(x_0,t)| = r_0|\lambda b(\lambda x_0, \lambda^2 t)| \le C_*.$$

When $\lambda \to 0$, the bound C_* is invariant, independent of λ.

Our assumption (1.2) is supercritical which means:

For the scaled solution, the assumption $r_0|b(x_0,t)| \le C_*$ becomes

$$r_0|b^\lambda(x_0,t)| = r_0|\lambda b(\lambda x_0, \lambda^2 t)| \le C_*(\ln|\ln\lambda\frac{r_0}{3}|)^\alpha.$$

When $\lambda \to 0$, the bound goes to infinity, which means, when one zooms in at a point, the bound on the drift term becomes worse, so the regularity of our solution must be handled more carefully.

Our assumption on b is closely related to a counterexample in [26]. In [26], the authors consider elliptic equation of this form

$$-\Delta u + (b\cdot\nabla)u = 0. \tag{1.4}$$

They construct a counterexample to state that (1.4) does not have Liouville theorem when the divergence-free vector field b satisfies $|b| \le \frac{\ln\ln|x|}{|x|}$ for large $|x|$. Moreover, Hölder continuity, as well as Harnack inequality, to solutions of (1.4) are also not to be expected. So under the assumption of (1.2), we do not expect a Hölder continuity to solutions of (1.3) even if the exponent α is small, which, however, is true under the critical assumption $r|b| \le C_*$.

Therefore, under the current techniques, maybe the “lnln” supercritical assumption on b seems to be the best that one can expect for some continuity results (weaker than the Hölder continuity) to solutions of (1.3) which can be used to prove the regularity of solutions to (1.1).

The idea of proving Theorem 1.1 comes from [6]. We sketch it as follows.

First, we will follow [18], using Nash–Moser type method to prove continuity of Γ at $r=0$. It satisfies a log decay near $r=0$, that is

$$|\Gamma| \le C|\ln\frac{r}{3}|^{-c_0} \quad \text{when } r \le 1, \tag{1.5}$$

for some small positive c_0. See Theorem 1.2. This estimate breaks the scaling-invariant bound of v_θ.

Let $w = \nabla \times v$ be the vorticity of v. In cylindrical coordinates,

$$w(x,t) = w_r e_r + w_\theta e_\theta + w_z e_z,$$

where

$$w_r = -\partial_z v_\theta,\ \ w_\theta = \partial_z v_r - \partial_r v_z,\ \ w_z = (\partial_r + \frac{1}{r})v_\theta.$$

And w_θ satisfies

$$\left[\partial_t + b\cdot\nabla - \Delta - \frac{v_r}{r}\right] w_\theta - \partial_z \frac{(v_\theta)^2}{r} + \frac{w_\theta}{r^2} = 0.$$

Define $\Omega = \frac{w_\theta}{r}$, then Ω satisfies

$$(\partial_t - L)\Omega = r^{-2}\partial_z(v_\theta)^2, \quad L = \Delta + \frac{2}{r}\partial_r - b\cdot\nabla. \tag{1.6}$$

Next, combining (1.5) and (1.6), we can get an estimate of w_θ. Handling of (1.6) involves an estimate to the fundamental solution of $(\partial_t - L)u = 0$ which will be described in Theorem 1.3.

By noting that $\mathrm{div}b = 0$, $\mathrm{curl}b = w_\theta e_\theta$, $-\Delta b = \mathrm{curlcurl}b - \nabla\mathrm{div}b$, then we have

$$-\Delta b = \nabla \times (w_\theta e_\theta).$$

At last, using the regularity theory of elliptic equations, we can get the boundedness of b. This will prove the regularity of our solution v.

Theorem 1.2. *For the divergence-free drift term b, define a zero-dimensional integral norm*

$$E_{R,p}(b) \triangleq \sup_{-R^2\le t\le 0}\left\{\frac{1}{R^{3-p}}\int_{B_R}|b|^p dx\right\}^{\frac{1}{p}}, \tag{1.7}$$

where $\frac{5}{3} < p \le 2$ *and* $R \le 1$, B_R *is the ball of radius* R *centered at* $x=0$. *If*

$$E_{R,p}(b) \le C\left(\ln\left|\ln\frac{R}{3}\right|\right)^{\frac{3p-5}{77p-120}-} \qquad \forall R\in(0,1], \tag{1.8}$$

here $\beta-$ means any number smaller than β. Then the weak solution of (1.3) is continuous at $(0,0)$ and it has a log decay near $r=0$. That is, there exists some positive c_0, such that

$$|\Gamma| \le C\Big|\ln\frac{r}{3}\Big|^{-c_0} \quad \textit{for } r \le 1. \tag{1.9}$$

Remark 1.2. We note that when $|b| \le \frac{(\ln|\ln\frac{r}{3}|)^{\alpha}}{r}$, $\alpha \in [0, 0.028]$, there exists a $p_0 \in (\frac{5}{3}, 2]$ such that the assumption (1.8) is satisfied. See **Claim** in Appendix.

Remark 1.3. For a general linear problem (1.3) with a divergence-free drift term b, we think it is a little hard to deduce (1.9) from (1.2) due to the following facts:

1. In our Navier–Stokes equation, $\Gamma|_{r=0} = 0$.
2. In the cylindrical coordinates, $\Gamma = \Gamma(r, z)$ which is independent of θ.
3. The drift term, expressed as $b = u_r e_r + u_z e_z$, is swirl-free ($u_\theta = 0$) and u_r, u_z is independent of θ.

But we guess that if the above three conditions are fulfilled, we can get some similar result as (1.9) by assuming (1.2).

For convenience, we will simply denote $\mathbf{E_{R,p}(b)}$ with $\mathbf{E_R}$ later on.

The next theorem gives an upper bound estimate to the fundamental solution of equation

$$\partial_t u - \Delta u + b \cdot \nabla u - \frac{2}{r}\partial_r u = 0, \quad \nabla \cdot b = 0 \tag{1.10}$$

under certain bound for b, which will be useful in the proof of Theorem 1.1. Due to the term $\frac{2}{r}\partial_r$, the result is not covered by the standard theory.

Before stating the theorem, we give the definition of fundamental solutions to (1.10).

Definition 1.1. Let $Q = \{(x,t) | x \in \mathbb{R}^3, t > s\}$, we say $0 \le p(x,t;y,s) \in L_{\text{loc}}(Q) \bigcap C^2(\overline{Q} \backslash (y,s))$ is a fundamental solution of (1.10) in Q if it satisfies

1. For any $\psi(x,t) \in C_0^\infty(Q)$ and $\psi|_{r=0} = 0$,

$$\int_s^{+\infty}\int_{\mathbb{R}^3} p(x,t;y,s)(\psi_t + \Delta\psi + b \cdot \nabla\psi - \frac{2}{r}\partial_r\psi)dxdt = 0.$$

2. For any $\phi(x) \in C_0^\infty(\mathbb{R}^3)$,

$$\lim_{t\to s}\int_{\mathbb{R}^3} p(x,t;y,s)\phi(y)dy = \phi(x).$$

3. Let $y = (y_1, y_2, y_3)$ and denote $y' = (y_1, y_2, 0)$, we require

$$p(x,t;y,s)|_{|y'|=0} = 0.$$

This third condition marks an important difference with the standard theory where fundamental solutions are positive everywhere. Our choice of this fundamental solution coincides with some quantities in the axisymmetric Navier–Stokes equations, such as Γ, w_θ.

Remark 1.4. Due to our assumption $p(x,t;y,s) \in C^2(\overline{Q}\backslash(y,s))$, $p(x,t;y,s)$ satisfies (1.10) in classical sense except at point (y,s).

Theorem 1.3. *Let $p(x,t;y,s)$ be a fundamental solution of (1.10) and the divergence-free smooth vector function $b(x,t)$ satisfies $|b| \le C_0 + \frac{1}{r}$. Then we have*

$$p(x,t;y,s) \le C(t-s)^{-3/2} \exp\left\{-C_1 \frac{|x-y|^2}{t-s}\left(1 - C_0\frac{t-s}{|x-y|}\right)_+^2\right\} \tag{1.11}$$

for some positive constants C, C_1. Moreover,

$$\int_{\mathbb{R}^3} p(x,t;y,s)dx \le 1, \quad \int_{\mathbb{R}^3} p(x,t;y,s)dy = 1.$$

The idea of proving Theorem 1.3 is based on Theorem 5 of [7], but due to the term $\frac{2}{r}\partial_r$, the proof will be more complicated. In [7], the authors consider the equation

$$\partial_t u = \Delta u - b \cdot \nabla u. \tag{1.12}$$

In their proof, the Davies-type exponent $r(t)$ can map from $[0,T]$ to $[1,\infty)$ and with the help of the logarithmic Sobolev inequality, a $L^1 \to L^\infty$ estimate to the solution of (1.12) can be obtained. But for (1.10), we must deal with a singular term $\frac{2}{r}\partial_r$ which will create some difficulties when using their method to estimate the solution. However, stimulated by Fabes–Stroock [11] and Davies [8], we use a dual technique from harmonic analysis to overcome this difficulty. We proceed as follows.

First, we choose $r(t) : [0,T] \to [2,\infty)$ to get a $L^2 \to L^\infty$ estimate of the fundamental solution $p(x,t;y,s)$, then the same estimate can be applied to the adjoint $p^*(x,t;y,s)$ of $p(x,t;y,s)$. By duality, we get $L^1 \to L^\infty$ estimate of $p(x,t;y,s)$. This will prove our Theorem 1.3.

Estimates to the kernels of parabolic equations have had a long history especially when the drift b is a divergence-free singular term. Under different assumptions on b, Osada H. [25], Liskevich–Zhang [21], Zhang Qi S. [31] give bounds for the fundamental solution of (1.12). Readers can refer to their papers and their references for more information. Here we add a singular term $\frac{2}{r}\partial_r$ in the equation and give an upper bound to the fundamental solution. We hope our estimate can not only be applied to the axis-symmetric Navier–Stokes equations, but also to other related incompressible fluid fields.

We now recall some regularity results on the axisymmetric Navier–Stokes equations. When the initial data is non-swirl and satisfies some integral conditions, recently Abidi–Zhang in [1] prove the global existence of the solution and give the decay rate of the solution which is exactly the same with the classical Navier–Stokes equations. In the presence of swirl, from the partial regularity theory of [3], any singular points of the axis-symmetric suitable weak solution of (1.1) can only lie on the z axis. In [2], Burke–Zhang give a priori bounds for the vorticity of axially symmetric solutions which indicates that the result of [3] can be applied to a large class of weak solutions. Chan–Vasseur in [29] give a logarithmically improved Serrin criterion for global regularity to solutions of Navier–Stokes equations. See also an extension in Zhou–Lei [30]. Neustupa and Pokorny [24] proved certain regularity of one component (either v^θ or v^r) imply regularity of the other components of the solutions. Chae–Lee [4] proved regularity assuming a zero-dimensional integral norm on w^θ: $w^\theta \in L_t^s L_x^q$ with $3/q + 2/s = 2$. Also regularity results come from the work of Jiu–Xin [15] under the assumption that another zero-dimensional scaled norms $\int_{Q_R} (R^{-1}|w^\theta|^2 + R^{-3}|v^\theta|^2) dz$ is sufficiently small for $R > 0$ is small enough. On the other hand, Lei–Zhang [19] give a structure of singularity of 3D axis-symmetric equation near maximum point. Tian–Xin [27] constructed a family of singular axi-symmetric solutions with singular initial data. Recently, Hou–Li [14] construct a special class of global smooth solutions. See also a recent extension: Hou–Lei–Li [13].

Here and throughout the paper, we will use c and C to denote a generic constant. It may be different from line to line. Also we use $A \lesssim B$ to denote $A \le CB$.

The paper is organized as follows: In section 2, we establish a local maximum estimate using a Moser's iteration. Based on the local maximal estimate, in section 3, we obtain the continuity of Γ and prove Theorem 1.2 by Nash's method. In section 4, we prove Theorem 1.1. The argument is based on [6]. In section 5, we give the proof of Theorem 1.3.

2. Local maximum estimate

In this section, using Moser's iteration, we prove a local maximum estimate of Γ which will be used to obtain continuity of Γ in the next section. The main idea is to exploit the divergence-free property of $b(x,t)$ and a special cut-off function. We learn from Lei–Zhang [18] and [5] where the authors treated the term $\frac{2}{r}\partial_r \Gamma$ and $b \cdot \nabla \Gamma$.

We first derive a parabolic De Giorgi type energy estimates of (1.3). Set $\frac{1}{2} \le \sigma_2 < \sigma_1 \le 1$ and consider a test function $\psi(y,s) = \phi(|y|)\eta(s)$ satisfying

$$\begin{cases} \text{supp}\ \phi \subset B(\sigma_1), \phi = 1 \text{ in } B(\sigma_2), 0 \le \phi \le 1; \\ \text{supp}\ \eta \subset (-(\sigma_1)^2, 0], \eta = 1 \text{ in } (-(\sigma_2)^2, 0], 0 \le \eta \le 1; \\ |\eta'| \lesssim \dfrac{1}{(\sigma_1 - \sigma_2)^2}, |\dfrac{\nabla \phi}{\sqrt{\phi}}| \le \dfrac{1}{\sigma_1 - \sigma_2}. \end{cases} \tag{2.1}$$

We will also use the following notations to denote our domains. Let $R > 0$ and $R \in (0,1)$. We write $B_R = B(0,R)$ and

$$P(R) = B_R \times (-R^2, 0], \quad P(R_1, R_2) = B_{R_1}/B_{R_2} \times (-R_1^2, 0] \text{ for } R_1 > R_2.$$

Consider the function $u = |\Gamma|^q$, $q > 1$ and the test function $\psi_R(y,s) = \phi_R(|y|)\eta_R(s) = \phi(\frac{y}{R})\eta(\frac{s}{R^2})$. Testing (1.3) by $q|\Gamma|^{2q-2}\Gamma\psi_R^2$ gives

$$\frac{1}{2}\int\int (\partial_s u^2 + b \cdot \nabla u^2 + \frac{2}{r}\partial_r u^2)\psi_R^2 = q\int\int \Delta\Gamma|\Gamma|^{2q-2}\Gamma\psi_R^2. \tag{2.2}$$

Using Cauchy–Schwartz's inequality and integration by parts, we compute

$$\begin{aligned}
& q\int\int \Delta\Gamma|\Gamma|^{2q-2}\Gamma\psi_R^2 \\
=\, & q\int\int \Delta|\Gamma||\Gamma|^{2q-1}\psi_R^2 \\
=\, & -q\int\int (2q-1)|\nabla\Gamma|^2\Gamma^{2q-2}\psi_R^2 + |\nabla\Gamma||\Gamma|^{2q-1}\nabla\psi_R^2 \\
=\, & -\int\int \frac{2q-1}{q}|\nabla\Gamma^q|^2\psi_R^2 + 2\psi_R|\Gamma^q|\nabla\psi_R\cdot|\nabla\Gamma^q| \\
=\, & -\int\int \frac{2q-1}{q}|\nabla u|^2\psi_R^2 + 2\psi_R u\nabla\psi_R\cdot\nabla u \\
=\, & -\int\int (2-\frac{1}{q})|\nabla(u\psi_R)|^2 - (2-\frac{2}{q})u\nabla\psi_R\cdot\nabla(u\psi_R) - \frac{1}{q}u^2|\nabla\psi_R|^2 \\
\lesssim\, & -\int\int |\nabla(u\psi_R)|^2 + \int\int u^2|\nabla\psi_R|^2
\end{aligned} \tag{2.3}$$

and

$$\frac{1}{2}\int\int \partial_s u^2\psi_R^2 = \frac{1}{2}\int_{B(\sigma_1 R)} u^2(\cdot,t)\psi_R^2 - \frac{1}{2}\int\int u^2\partial_s\psi_R^2. \tag{2.4}$$

Moreover, by the fact that $\Gamma = 0$ on the axis $r = 0$, we have

$$\begin{aligned}
& -\frac{1}{r}\int\int \partial_r u^2\psi_R^2 \\
=\, & -2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty} \partial_r u^2\psi_R^2 drdz \\
=\, & 2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty} u^2\partial_r\psi_R^2 drdz \\
=\, & -2\pi\int_{-\infty}^{+\infty}\int_0^{+\infty} \partial_r(u^2\partial_r\psi_R^2)rdrdz \\
\lesssim\, & \int\int u^2|\partial_r(\partial_r\psi_R^2)| + u\partial_r u\psi_R\partial_r\psi_R \\
\lesssim\, & \int\int u^2(|\nabla\psi_R|^2 + |\nabla^2\psi_R|) + u\partial_r\psi_R(\partial_r(u\psi_R) - u\partial_r\psi_R) \\
\lesssim\, & \int\int u^2(|\nabla\psi_R|^2 + |\nabla^2\psi_R|) + \frac{1}{4}\int\int |\nabla(u\psi_R)|^2.
\end{aligned} \tag{2.5}$$

Consequently, using (2.2) and combining (2.3), (2.4) and (2.5), we get

$$\begin{aligned}
&\int\limits_{B(\sigma_1 R)} u^2(\cdot,t)\psi_R^2+\int\int|\nabla(u\psi_R)|^2\\
&\lesssim\int\int u^2(|\nabla\psi_R|^2+|\nabla^2\psi_R|+|\partial_s\psi_R^2|)-\frac{1}{2}\int\int b\cdot\nabla u^2\psi_R^2\\
&\lesssim\frac{1}{(\sigma_1-\sigma_2)^2R^2}\int\int\limits_{P(\sigma_1 R)} u^2-\frac{1}{2}\int\int b\cdot\nabla u^2\psi_R^2.
\end{aligned}\tag{2.6}$$

By the divergence-free property of the drift term b and using (2.1), we have

$$\begin{aligned}
&-\frac{1}{2}\int\int b\cdot\nabla u^2\psi_R^2\\
&=\int\int u^2\psi_R b\cdot\nabla\psi_R\\
&=\int\int(\psi_R u)^{3/2}u^{\frac{1}{2}}b\cdot\frac{\nabla\psi_R}{\sqrt{\psi_R}}\\
&\lesssim\frac{1}{(\sigma_1-\sigma_2)R}\int\int\limits_{P(\sigma_1 R,\sigma_2 R)}|b|(\psi_R u)^{3/2}u^{\frac{1}{2}}\\
&\lesssim\frac{1}{(\sigma_1-\sigma_2)R}\int\left[\left(\int|b|^p dy\right)^{\frac{1}{p}}\left(\int(\psi_R u)^6 dy\right)^{\frac{1}{4}}\left(\int u^{\frac{2p}{3p-4}}dy\right)^{\frac{3p-4}{4p}}\right]ds\\
&\lesssim\frac{\|b\|_{L_t^\infty L_x^p(P(\sigma_1 R))}}{(\sigma_1-\sigma_2)R}\int\left[\left(\int|\nabla(\psi_R u)|^2 dy\right)^{\frac{3}{4}}\left(\int u^{\frac{2p}{3p-4}}dy\right)^{\frac{3p-4}{4p}}\right]ds\\
&\lesssim\frac{\|b\|_{L_t^\infty L_x^p(P(\sigma_1 R))}}{(\sigma_1-\sigma_2)R}\left(\int\int|\nabla(\psi_R u)|^2 dyds\right)^{\frac{3}{4}}\left(\int\int u^{\frac{2p}{3p-4}}dyds\right)^{\frac{3p-4}{4p}}\left(\int\limits_{-(\sigma_2 R)^2}^{-(\sigma_1 R)^2}ds\right)^{\frac{2-p}{2p}}\\
&\lesssim\left[\frac{((\sigma_1-\sigma_2)^2R^2)^{\frac{2-p}{2p}}}{(\sigma_1-\sigma_2)R}\|b\|_{L_t^\infty L_x^p(P(\sigma_1 R))}\right]^4\left(\int\int u^{\frac{2p}{3p-4}}\right)^{\frac{3p-4}{p}}+\frac{1}{2}\int\int|\nabla(\psi_R u)|^2\\
&\lesssim\left[\frac{1}{[(\sigma_1-\sigma_2)R]^{3-\frac{5}{p}}}E_R\right]^4\left(\int\int u^{\frac{2p}{3p-4}}dyds\right)^{\frac{3p-4}{p}}+\frac{1}{2}\int\int|\nabla(\psi_R u)|^2 dyds.
\end{aligned}\tag{2.7}$$

Combining (2.6) and (2.7), using the Cauchy–Schwartz inequality, we get

$$\int\limits_{B(\sigma_1 R)} u^2(\cdot,t)\psi_R^2+\int\int\limits_{P(\sigma_1 R)}|\nabla(u\psi_R)|^2$$

$$
\begin{aligned}
&\lesssim \frac{1}{(\sigma_1-\sigma_2)^2R^2}\int\int_{P(\sigma_1 R)} u^2 dyds + \frac{E_R^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}}\left(\int\int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} dyds\right)^{\frac{3p-4}{p}}\\
&\lesssim \frac{1}{(\sigma_1-\sigma_2)^2R^2}\left(\int\int_{P(\sigma_1 R)} u^{2\frac{p}{3p-4}} dyds\right)^{\frac{3p-4}{p}}\left(\int\int_{P(\sigma_1 R,\sigma_2 R)} dyds\right)^{\frac{4-2p}{p}}\\
&\quad + \frac{E_R^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}}\left(\int\int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} dyds\right)^{\frac{3p-4}{p}}\\
&\lesssim \frac{1+E_R^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}}\left(\int\int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} dyds\right)^{\frac{3p-4}{p}}.
\end{aligned}
$$

At last, we get the estimate

$$
\begin{aligned}
&\sup_{-\sigma_1^2R^2\le t\le 0}\int_{B(\sigma_1 R)} u^2(\cdot,t)\psi_R^2 + \int\int_{P(\sigma_1 R)} |\nabla(u\psi_R)|^2\\
&\lesssim \frac{(1+E_R)^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}}\left(\int\int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} dyds\right)^{\frac{3p-4}{p}}.
\end{aligned}
\tag{2.8}
$$

Our next step is to derive a mean value inequality based on (2.8) using Moser's iteration.

Lemma 2.1. *Suppose u satisfies (2.8) for $p \in (\frac{5}{3}, 2]$, then for $0 < R \le 1$, there is the estimate*

$$
\sup_{P(\frac{1}{2}R)} \Gamma \lesssim (1+E_R)^{\frac{5p}{2(3p-5)}}\left(\int\int_{P(R)} \frac{1}{R^5}\Gamma^2\right)^{\frac{1}{2}}. \tag{2.9}
$$

Proof. By Hölder inequality and Sobolev imbedding theorem, we have

$$
\begin{aligned}
&\int\int_{P(\sigma_1 R)} (u\psi_R)^{\frac{10}{3}}\\
&\lesssim \int\left(\|u\psi_R(\cdot,s)\|^{\frac{4}{3}}_{L^2_{(B(\sigma_1 R))}}\|\nabla(u\psi_R)\|^2_{L^2_{B(\sigma_1 R)}}\right)ds\\
&\lesssim \sup_{-(\sigma_1 R)^2\le t\le 0}\|u\psi_R(\cdot,t)\|^{\frac{4}{3}}_{L^2_{B(\sigma_1 R)}}\|\nabla(u\psi_R)\|^2_{L^2_{P(\sigma_1 R)}}.
\end{aligned}
$$

Using (2.1) and (2.8), we get

$$\int\int_{P(\sigma_2 R)} u^{\frac{10}{3}} \lesssim \left\{ \frac{(1+E_R)^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}} \right\}^{\frac{5}{3}} \left(\int\int_{P(\sigma_1 R)} u^{\frac{2p}{3p-4}} \right)^{\frac{5(3p-4)}{3p}}.$$

Remember $u = \Gamma^q$, then we obtain

$$\int\int_{P(\sigma_2 R)} \left(\Gamma^{\frac{2p}{3p-4}q} \right)^{\frac{3p-4}{2p}\times\frac{10}{3}} \lesssim \left\{ \frac{(1+E_R)^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}} \right\}^{\frac{5}{3}} \left(\int\int_{P(\sigma_1 R)} \Gamma^{\frac{2p}{3p-4}q} \right)^{\frac{5(3p-4)}{3p}}. \tag{2.10}$$

For convenience of computation, we denote $\kappa = \frac{p}{3p-4}$, then (2.10) is

$$\int\int_{P(\sigma_2 R)} \left(\Gamma^{2\kappa q} \right)^{\frac{5}{3}\kappa^{-1}} dyds \lesssim \left\{ \frac{(1+E_R)^4}{[(\sigma_1-\sigma_2)R]^{12-\frac{20}{p}}} \right\}^{\frac{5}{3}} \left(\int\int_{P(\sigma_1 R)} \Gamma^{2\kappa q} dyds \right)^{\frac{5}{3}\kappa^{-1}}. \tag{2.11}$$

For integer $j \geq 0$ and a constant $\sigma = \frac{1}{2}$, set $\sigma_2 = \frac{1}{2}(1+\sigma^{j+1})$ and $\sigma_1 = \frac{1}{2}(1+\sigma^j)$. Let $q = (\frac{5}{3}\kappa^{-1})^j$, then we get

$$\left(\int\int_{P(\frac{R}{2}(1+\sigma^{j+1}))} \Gamma^{2\kappa(\frac{5}{3}\kappa^{-1})^{j+1}} dyds \right)^{\frac{1}{2\kappa}(\frac{3}{5}\kappa)^{j+1}} \lesssim \frac{(1+E_R)^{\frac{20}{3}\frac{1}{2\kappa}(\frac{3}{5}\kappa)^{j+1}}}{[\sigma^{(j+1)}R]^{\frac{5}{3}(12-\frac{20}{p})\frac{1}{2\kappa}(\frac{3}{5}\kappa)^{j+1}}} \times \left(\int\int_{P(\frac{R}{2}(1+\sigma^j))} \Gamma^{2\kappa(\frac{5}{3}\kappa^{-1})^j} dyds \right)^{\frac{1}{2\kappa}(\frac{3}{5}\kappa)^j}.$$

By iterating j, the above inequality gives

$$\left(\int \int_{P(\frac{R}{2}(1+\sigma^{j+1}))} \Gamma^{2\kappa(\frac{5}{3}\kappa^{-1})^{j+1}} dyds\right)^{\frac{1}{2\kappa}(\frac{3}{5}\kappa)^{j+1}}$$

$$\lesssim \frac{(1+E_R)^{\frac{10}{3\kappa}\sum_{i=0}^{j}(\frac{3}{5}\kappa)^{i+1}}}{\left[\sigma^{\sum_{i=0}^{j}(i+1)(\frac{3}{5}\kappa)^{i+1}} R^{\sum_{i=0}^{j}(\frac{3}{5}\kappa)^{i+1}}\right]^{\frac{5}{3}(12-\frac{20}{p})\frac{1}{2\kappa}}} \times$$

$$\left(\int \int_{P(R)} \Gamma^{2\kappa} dyds\right)^{\frac{1}{2\kappa}}.$$

Note that $\frac{3}{5}\kappa \in [\frac{3}{5}, 1)$ when we assume $p \in (\frac{5}{3}, 2]$. So all the sums on the above are convergent, let $j \to \infty$ yield that

$$\begin{aligned}
\sup_{(x,t)\in P(\frac{R}{2})} |\Gamma| &\lesssim \frac{(1+E_R)^{\frac{15p-20}{6p-10}}}{R^{\frac{5}{2\kappa}}} \left(\int \int_{P(R)} \Gamma^{2\kappa} dyds\right)^{\frac{1}{2\kappa}} \\
&\lesssim (1+E_R)^{\frac{15p-20}{6p-10}} \left(\int \int_{P(R)} \frac{1}{R^5}\Gamma^{2\kappa} dyds\right)^{\frac{1}{2\kappa}} \\
&\lesssim (1+E_R)^{\frac{15p-20}{6p-10}} \left(\int \int_{P(R)} \frac{1}{R^5}\Gamma^{\frac{2p}{3p-4}} dyds\right)^{\frac{3p-4}{2p}}. \qquad (2.12)
\end{aligned}$$

Next we use (2.12) and an algebraic trick to improve our estimate (2.12). This is from Li–Schoen [20]. From the process of proving (2.12), we have for $\gamma \in (0, \frac{1}{2}]$, $\theta \in [\frac{1}{2}, 1-\gamma]$

$$\begin{aligned}
&\sup_{P(\theta R)} \Gamma^{\frac{2p}{3p-4}} \\
&\lesssim (1+E_R)^{\frac{5p}{3p-5}} \frac{1}{R^5} \int \int_{P((\theta+\gamma)R)} \Gamma^{\frac{2p}{3p-4}} \\
&\lesssim (1+E_R)^{\frac{5p}{3p-5}} \frac{1}{R^5} \sup_{P((\theta+\gamma)R)} \Gamma^{\frac{2p}{3p-4}-2} \int \int_{P((\theta+\gamma)R)} \Gamma^2.
\end{aligned}$$

Let $K \triangleq \frac{1}{R^5}\int\int_{P(R)} \Gamma^2$, then we have

$$\sup_{P(\theta R)} \Gamma^{\frac{2p}{3p-4}} \lesssim (1+E_R)^{\frac{5p}{3p-5}} K \left(\sup_{P((\theta+\gamma)R)} \Gamma^{\frac{2p}{3p-4}}\right)^{1-\frac{3p-4}{p}}.$$

Define $M(\theta) = \sup_{P(\theta R)} \Gamma^{\frac{2p}{3p-4}}$, then we yield that

$$M(\theta) \lesssim (1+E_R)^{\frac{5p}{3p-5}} K M(\theta+\gamma) K^{\lambda},$$

where $\lambda = 1 - \frac{3p-4}{p}$. Choosing $\theta_0 = \frac{1}{2}$, $\theta_i = \theta_{i-1} + \frac{1}{2^{i+1}}$ and $\gamma = \frac{1}{2^{i+1}}$, then we get

$$M(\theta_0) \lesssim \left[K(1+E_R)^{\frac{5p}{3p-5}} \right]^{\sum_{i=1}^{j} \lambda^{i-1}} M(\theta_j)^{\lambda^j}.$$

For $\lambda < 1$, letting $j \to \infty$, then we have

$$M(\theta_0) \lesssim \left[K(1+E_R)^{\frac{5p}{3p-5}} \right]^{\frac{p}{3p-4}}.$$

That is

$$\sup_{P(\frac{1}{2}R)} \Gamma^{\frac{2p}{3p-4}} \lesssim \left[K(1+E_R)^{\frac{5p}{3p-5}} \right]^{\frac{p}{3p-4}}.$$

So

$$\begin{aligned} &\sup_{P(\frac{1}{2}R)} |\Gamma| \\ &\lesssim (1+E_R)^{\frac{5p}{2(3p-5)}} K^{\frac{1}{2}} \\ &\lesssim (1+E_R)^{\frac{5p}{2(3p-5)}} \left(\int \int_{P(R)} \frac{1}{R^5} \Gamma^2 dyds \right)^{\frac{1}{2}}. \end{aligned}$$

This proves our Lemma 2.1. □

3. Proof of Theorem 1.2

In this section we study the continuity of Γ by using the local maximum estimate (2.9) and Nash type method for parabolic equations. First let us introduce some notations.

For $0 < R \leq 1$, we define

$$m_R = \inf_{P(R)} \Gamma, \quad M_R = \sup_{P(R)} \Gamma, \quad J_R = M_R - m_R.$$

Define

$$u = \begin{cases} \dfrac{2(M_R - \Gamma)}{J_R} & \text{if } M_R > -m_R, \\ \dfrac{2(\Gamma - m_R)}{J_R} & \text{else,} \end{cases} \tag{3.1}$$

hence

$$0 \le u \le 2, \quad a \triangleq u|_{r=0} \ge 1. \tag{3.2}$$

Lower bound on $\|u\|_{L^q}$

We give a lemma to state that there is a lower bound on $\|u\|_{L^q}$ where $q \in (0, 1)$. This bound depends on our $E_R(b)$ norm and will serve as an input for Nash's argument as we will describe it later on.

Lemma 3.1. *If u is a solution of (1.3) and satisfies (3.2). Then for $\forall q \in (0, 1)$, we have*

$$\frac{1}{R^{\frac{5}{q}}} \|u\|_{L^q(P(\frac{R}{2}))} \gtrsim a(1 + E_R)^{-\frac{8}{q}}. \tag{3.3}$$

Proof. Let $\psi(x,t) = \phi(x)\eta(t)$, where $\phi \in C_0^\infty$ s.t. $\phi = 1$ in $B_{\frac{1}{2}}$ and $\phi = 0$ in B_1^c. $\frac{\nabla\phi}{\sqrt{\phi}}$ and $\nabla(\frac{\nabla\phi}{\sqrt{\phi}})$ are bounded. $\eta \in C_0^\infty$ s.t. $\eta = 1$ in $[-\frac{7}{8}, -\frac{1}{8}]$ and η is supported in $(-1, 0)$. Define $\psi_R(y,s) = \phi(\frac{y}{R})\eta(\frac{s}{R^2})$ Let us test (1.3) by $qu^{q-1}\psi_R^2$, where $q \in (0, \frac{1}{2})$. Then we have

$$\int\int (\partial_s u^q + b \cdot \nabla u^q + \frac{2}{r}\partial_r u^q)\psi_R^2 dyds = q \int\int \Delta u u^{q-1}\psi_R^2 dyds. \tag{3.4}$$

Similarly as in [18], we have

$$\begin{aligned} -\int\int \frac{2}{r}\partial_r u^q \psi_R^2 dyds &= -\int\int 2\partial_r u^q \psi_R^2 drdzds \\ &= \int_{-R^2}^{0}\int 2u^q\psi_R^2|_{r=0} dzds + \int\int \frac{4}{r}u^q\psi_R\partial_r\psi_R dyds \\ &\ge -\frac{C}{R^2}\int\int u^q dyds + \frac{3}{2}R^3 a^q. \end{aligned} \tag{3.5}$$

Here we note that $\frac{\partial_r\psi_R}{r} = \frac{\partial_\rho\psi_R}{\rho}$ for ϕ is a radial function. Because $\phi_R = 1$ near $\rho = 0$, $\frac{\partial_r\psi_R}{r}$ has no singularity.

Moreover

$$\int\int (-\partial_s u^q + q\Delta u u^{q-1})\psi_R^2 dyds$$

$$
\begin{aligned}
&= \int\int 2u^q[\psi_R\partial_s\psi_R + |\nabla\psi_R|^2 - \frac{q-2}{q}\psi_R\Delta\psi_R]dyds - \frac{4(q-1)}{q}\int\int|\nabla(u^{\frac{q}{2}}\psi_R)|^2dyds \\
&\geq -\frac{C}{R^2}\int\int u^q dyds - \frac{4(q-1)}{q}\int\int|\nabla(u^{\frac{q}{2}}\psi_R)|^2dyds. \qquad (3.6)
\end{aligned}
$$

For the term involving b, we compute the same as (2.7)

$$
\begin{aligned}
&\int\int b\cdot\nabla u^q\psi_R^2dyds \\
&\leq \frac{C}{R^{12-20/p}}E_R^4\left(\int\int u^{\frac{q}{2}\frac{2p}{3p-4}}dyds\right)^{\frac{3p-4}{p}} - \frac{4(q-1)}{q}\int\int|\nabla(u^{\frac{q}{2}}\psi_R)|^2. \qquad (3.7)
\end{aligned}
$$

Combining (3.4), (3.5), (3.6) and (3.7), we derive

$$
R^{-2}\int\int u^q dyds + \frac{1}{R^{12-20/p}}E_R^4\left(\int\int u^{\frac{q}{2}\frac{2p}{3p-4}}dyds\right)^{\frac{3p-4}{p}} \gtrsim a^qR^3.
$$

Using Hölder inequality, we have

$$
\left(\int\int u^{2q}dyds\right)^{\frac{1}{2}} + E_R^4\left(\int\int u^{2q}dyds\right)^{\frac{1}{2}} \gtrsim a^qR^{5/2},
$$

then we have

$$
(1+E_R^4)^{\frac{1}{q}}\left(\int\int u^{2q}\right)^{\frac{1}{2q}} \gtrsim aR^{5/2q}.
$$

So

$$
\frac{1}{R^{\frac{5}{2q}}}\left(\int\int u^{2q}dyds\right)^{\frac{1}{2q}} \gtrsim a(1+E_R)^{-\frac{4}{q}}.
$$

This proves our lemma, since $q \in (0, \frac{1}{2})$ is arbitrary. □

Nash's lower bound

Before proving the Nash's lower bound estimate, we recall a Nash inequality, whose proof can be found in [6].

Lemma 3.2. *Let $M \geq 1$ be a constant and μ be a probability measure. Then for all $0 \leq f \leq M$, there holds*

$$
|\ln\int f d\mu - \int \ln f d\mu| \leq \frac{M\|g\|_{L^2}}{\int f d\mu}
$$

where $g = \ln f - \int \ln f d\mu$.

Now we come to prove Nash's lower bound estimate. We define a Lipschitz continuous cut-off function such that

$$\zeta = 1 \text{ in } B(\frac{1}{2}),\ \zeta = 0 \text{ in } B(1)^c,\ \int_{\mathbb{R}^3} \zeta^2 dx = 1.$$

In fact we take

$$\zeta = c \begin{cases} 1 & \text{in } B(\frac{1}{2}); \\ 2(1-|x|) & \text{in } B(1)/B(\frac{1}{2}), \end{cases} \tag{3.8}$$

where c is a constant to ensure $\int_{\mathbb{R}^3} \zeta^2 dx = 1$. Let $\zeta_R(x) = \frac{1}{R^{\frac{3}{2}}}\zeta(\frac{x}{R})$.

Lemma 3.3. *Let $0 \le u \le 2$ be a solution of (1.3) in $P(R)$ which is assumed to satisfy*

$$\|u\|_{L^1(P(\frac{R}{2}))} \ge c_1(1+E_R)^{-8}R^5. \tag{3.9}$$

Moreover, we assume that $u|_{r=0}$ is a constant bigger than 1, then there exists a $\tau > 0$ such that

$$-\int_{\mathbb{R}^3} \ln u \zeta_R^2 dx \lesssim (1+E_R)^{24}, \quad \textit{for } -\tau R^2 \le t < 0.$$

Proof. First, let us define $u_R(x,t) = u(Rx, R^2t)$, $b_R(x,t) = Rb(Rx, R^2t)$. It is clear that $u_R(x,t)$ solves the equation

$$\partial_t u_R + b_R \cdot \nabla u_R + \frac{2}{r}\partial_r u = \Delta u_R \quad \text{in P(1)}$$

and $0 \le u_R \le 2$, $\|u_R\|_{L^1(P(\frac{1}{2}))} \ge c_1(1+E_{1,p}(b_R))^{-8}$. The estimate we are going to get is

$$-\int_{\mathbb{R}^3} \ln u_R \zeta^2 dx \lesssim (1+E_{1,p}(b_R))^{24}, \quad \text{for } -\tau \le t < 0.$$

For convenience, we shall drop all R and the subscript from now on and set $R = 1$.

Also denote

$$\mathbf{E} = \mathbf{E_{1,p}}(\mathbf{b_R}).$$

Let $v = -\ln u$. It is easy to see that v solves the equation

$$\partial_s v + b \cdot v + \frac{2}{r}\partial_r v - \Delta v + |\nabla v|^2 = 0. \tag{3.10}$$

Testing (3.10) by ζ^2, we have

$$\int \partial_s v\zeta^2 dx + \int |\nabla v|^2\zeta^2 dx = \int [\Delta v - \frac{2}{r}\partial_r v - b\cdot v]\zeta^2 dx. \tag{3.11}$$

Using the Cauchy–Schwartz inequality and integration by parts, we have

$$\begin{aligned}\int \Delta v\zeta^2 dx &= -2\int \nabla v\cdot\nabla\zeta\,\zeta dx\\ &\le \frac{1}{4}\int |\nabla v|^2\zeta^2 + 4\int |\nabla\zeta|^2 dx\\ &\le \frac{1}{4}\int |\nabla v|^2\zeta^2 + C.\end{aligned}\tag{3.12}$$

Let $\overline{v}(t) = \int v(\cdot,t)\zeta^2 dx$, by recalling the assumption that $u|_{r=0}$ is a non-zero constant and the weighted Poincaré inequality

$$\int |v-\overline{v}|^2\zeta^2 dx \le C\int |\nabla v|^2\zeta^2 dx, \tag{3.13}$$

one can estimate

$$\begin{aligned}-\int \frac{2}{r}\partial_r v\zeta^2 &= -\int \frac{2}{r}\partial_r(v-\overline{v})\zeta^2 dx\\ &= -4\pi\int_{-\infty}^{+\infty}\int_0^{+\infty}\partial_r(v-\overline{v})\zeta^2 drdz\\ &= -4\pi\int_{-\infty}^{+\infty}\zeta^2(v-\overline{v})|_{r=0}^{\infty}dz + 4\pi\int_{-\infty}^{+\infty}(v-\overline{v})2\zeta\,\partial_r\zeta\,drdz\\ &= 4\pi\int_{-\infty}^{+\infty}v\zeta^2|_{r=0}dz - 4\pi\overline{v}\int_{-\infty}^{+\infty}\zeta^2 dz + 4\int (v-\overline{v})\zeta\frac{\partial_r\zeta}{r}rdrd\theta dz\\ &\le C - C\overline{v}(s) + \frac{1}{4}\int |\nabla v|^2\zeta^2 dx.\end{aligned}\tag{3.14}$$

Before estimating the term involving b, we need a more general weighted Poincaré inequality. Let B_R be a ball centered at 0 in $\mathbb{R}^n$. Let $1 \le r \le q < \infty$ satisfy $\frac{1}{q} \ge \frac{1}{r} - \frac{1}{n}$ and $1 - \frac{n+2}{r} + \frac{n+2}{q} \le 0$, then we have

$$\left(\frac{1}{|B_R|}\int_{B_R}(v-\overline{v})^q\zeta_R^2 dx\right)^{\frac{1}{q}} \le CR^{1-\frac{2}{r}+\frac{2}{q}}\left(\frac{1}{|B_R|}\int_{B_R}|\nabla v|^r\zeta_R^2 dx\right)^{\frac{1}{r}}, \tag{3.15}$$

here $|B_R|$ means the Lebesgue measure of the ball B_R and C depends only on q, r, n. One can see [10] for its proof. Hence, due to the divergence-free of b and Hölder inequality, we have

$$\begin{aligned}-\int (b\cdot\nabla)v\zeta^2dx &= \int 2\zeta(v-\overline{v})b\cdot\nabla\zeta dx\\ &\le \left(\int |b|^p|\nabla\zeta|^p dx\right)^{\frac{1}{p}}\left(\int (v-\overline{v})^q\zeta^q dx\right)^{\frac{1}{q}}\\ &\le \left(\int |b|^p|\nabla\zeta|^p dx\right)^{\frac{1}{p}}\left(\int (v-\overline{v})^q\zeta^2 dx\right)^{\frac{1}{q}},\end{aligned}$$

here $\frac{1}{p}+\frac{1}{q}=1$.

In (3.15), let $R=1$, $r=2$, $n=3$. When $p\in(\frac{5}{3},2]$, $q=\frac{p}{p-1}$ can satisfy $\frac{1}{q}\ge\frac{1}{r}-\frac{1}{n}$ and $1-\frac{n+2}{r}+\frac{n+2}{q}\le 0$. So

$$\left(\int (v-\overline{v})^q\zeta^2dx\right)^{\frac{1}{q}}\le C\left(\int|\nabla v|^2\zeta^2dx\right)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}-\int (b\cdot\nabla)v\zeta^2dx &\le C\left(\int |b|^p|\nabla\zeta|^p dx\right)^{\frac{1}{p}}\left(\int |\nabla v|^2\zeta^2 dx\right)^{\frac{1}{2}}\\ &\le C\left(\int_{B(1)} |b|^p dx\right)^{\frac{2}{p}}+\frac{1}{4}\int|\nabla v|^2\zeta^2dx\\ &\le CE^2+\frac{1}{4}\int|\nabla v|^2\zeta^2dx.\end{aligned}\tag{3.16}$$

Combining (3.11), (3.12), (3.14) and (3.16), we have

$$\partial_s\int v\zeta^2dx+C\overline{v}(s)\le -\frac{1}{4}\int|\nabla v|^2\zeta^2dx+(1+E)^2.$$

Now we apply the Nash inequality, taking $f=u$, $d\mu=\zeta^2dx$ in Lemma 3.2, one has

$$|\ln\int u\zeta^2dx+\int v\zeta^2dx|^2\left(\int u\zeta^2dx\right)^2\le M^2\int|-v+\int v\zeta^2dy|^2\zeta^2dx,$$

here $M=2$ is the upper bound of u. Using the weighted Poincaré inequality (3.13) once again, we have

$$|\ln\int u\zeta^2dx+\int v\zeta^2dx|^2\left(\int u\zeta^2dx\right)^2\le C\int|\nabla v|^2\zeta^2dx.$$

Then finally we obtain

$$\partial_s \overline{v}(s) + C_0 \overline{v}(s) \le (1+E)^2 - \frac{1}{4C} |\ln \int u\zeta^2 dx + \overline{v}(s)|^2 \left(\int u\zeta^2 dx \right)^2 .$$

Recalling (3.9)

$$\|u\|_{L^1(P(\frac{1}{2}))} \ge c_1 (1+E)^{-8}.$$

Let χ be the characteristic function of the non-empty set

$$W = \left\{ s \in [-\frac{1}{4}, 0] : \|u\|_{L^1(B(\frac{1}{2}))} \ge \frac{c_1(1+E)^{-8}}{10} \right\}.$$

We assert that $|W| \ge \frac{c_1(1+E)^{-8}}{20}$. In fact, if $|W| \le \frac{c_1(1+E)^{-8}}{20}$, then

$$\begin{aligned} \|u\|_{L^1(P(\frac{1}{2}))} &< \int\limits_W 2|B(\frac{1}{2})| ds + \int\limits_{W^c} \frac{c_1(1+E)^{-8}}{10} ds \\ &< \frac{\pi}{3}|W| + \frac{c_1(1+E)^{-8}}{40} \\ &< (\frac{\pi}{60} + \frac{1}{40}) c_1 (1+E)^{-8} \\ &< c_1(1+E)^{-8}, \end{aligned}$$

this is a contradiction with (3.9). Thus, one has

$$\begin{aligned} \partial_s \overline{v}(s) + C_0 \overline{v}(t) &\le (1+E)^2 - \frac{1}{4C} \chi(s) |\ln \int u\zeta^2 dx + \overline{v}(s)|^2 \left(\int u\zeta^2 dx \right)^2 \\ &\le (1+E)^2 - \frac{1}{4C} \chi(s) |\ln \int u\zeta^2 dx + \overline{v}(s)|^2 \frac{c_1^2 (1+E)^{-16}}{100}. \qquad (3.17) \end{aligned}$$

The last inequality is due to

$$\left(\int u\zeta^2 dx \right)^2 \ge \left(\int\limits_{B(\frac{1}{2})} u dx \right)^2 \ge \frac{c_1^2(1+E)^{-16}}{100},$$

when $s \in W$.

From (3.17), we first have $\partial_s\overline{v}(s)+C_0\overline{v}(s)\le(1+E)^2$. This gives, for $-\frac{1}{4}\le s_1\le s_2\le 0$,

$$\begin{aligned}\overline{v}(s_2)&\le e^{C_0(s_1-s_2)}\overline{v}(s_1)+e^{-s_2}\int\limits_{s_1}^{s_2}(1+E)^2ds\\&\le e^{C_0|s_1-s_2|}\overline{v}(s_1)+C(1+E)^2.\end{aligned}\tag{3.18}$$

Now we consider two cases.

Case one: if there exists some $s_0\in[-\frac{1}{4},-\frac{c_1}{40}]$, such that

$$\overline{v}(s_0)\le\frac{2}{C_0}(1+E)^2+4|\ln\frac{10}{c_1(1+E)^{-8}}|.$$

Then for $s\in(s_0,0)$, from (3.18), we have

$$\begin{aligned}\overline{v}(s)&\lesssim\overline{v}(s_0)+(1+E)^2\\&\lesssim(1+E)^2+\ln\frac{10}{c_1}+\ln(1+E)\\&\lesssim(1+E)^2.\end{aligned}$$

Choosing $\tau=s_0$, this completes the proof of the lemma.

Case two: if for any $s\in[-\frac{1}{4},-\frac{c_1}{40}]$,

$$\overline{v}(s)\ge\frac{2}{C_0}(1+E)^2+4|\ln\frac{10}{c_1(1+E)^{-8}}|.$$

Then when $s\in W\cap[-\frac{1}{4},-\frac{c_1}{40}]$,

$$\ln\int u\zeta^2dx\ge\ln\int\limits_{B(\frac{1}{2})}u\zeta^2\ge\ln\frac{c_1(1+E)^{-8}}{10}.$$

So we have

$$\overline{v}(s)+\ln\int u\zeta^2dx\ge C\overline{v}(s).$$

From (3.17), we have

$$\partial_s\overline{v}(s)+C_0\overline{v}(s)\lesssim-C\chi(s)\frac{c_1^2(1+E)^{-16}}{100}\overline{v}^2(s).$$

Then integrating the above inequality from $[-\frac{1}{4}, -\frac{c_1}{40}]$, one gets

$$\begin{aligned}\overline{v}(-\frac{1}{4})^{-1} - \overline{v}(-\frac{c_1}{40})^{-1} &\lesssim -c_1^2(1+E)^{-16}\int \chi ds \\ &\lesssim -c_1^2(1+E)^{-16}|W| \\ &\lesssim -c_1^3(1+E)^{-24}.\end{aligned}$$

Since $\overline{v}(-\frac{1}{4}) \geq 0$ in this case, we have

$$\overline{v}(-\frac{c_1}{40}) \lesssim \frac{(1+E)^{24}}{c_1^3}.$$

Then we use (3.18), for $s \in [-\frac{c_1}{40}, 0]$,

$$\overline{v}(s) \lesssim (1+E)^2 + (1+E)^{24} \lesssim (1+E)^{24}.$$

So we can take $\tau = \frac{c_1}{40}$, this proves the lemma. □

As a corollary of Lemma 3.3, we derive a lower bound of positive solution of (1.3).

Corollary 3.1. *Let u, τ be given in Lemma 3.3 and E_R satisfies the assumption (1.8). Then there exists a $\delta \in (0, 1)$, depending only on R, such that*

$$\inf_{P(\frac{\sqrt{\tau}}{2}R)} u \geq \frac{1}{2}\delta(R). \tag{3.19}$$

In fact, we take $\delta = |\ln \frac{R}{3}|^{-1}$.

Proof. Using Lemma 3.3, one has

$$\begin{aligned}(1+E_R)^{24} &\gtrsim -\int \zeta_R^2(x)\ln u(x,t)dx \\ &= -\int_{\delta<u\leq 1} \zeta_R^2(x)\ln u dx - \int_{u\leq\delta} \zeta_R^2(x)\ln u dx \\ &\quad \int_{1<u\leq 2} \zeta_R^2(x)\ln u dx \\ &\geq -\int_{u\leq\delta} \zeta_R^2(x)\ln u dx - \ln 2\int_{1<u\leq 2} \zeta_R^2(x)dx \\ &\geq -\int_{u\leq\delta} \zeta_R^2(x)\ln u dx - \ln 2.\end{aligned}$$

This implies that

$$-\int_{u\leq\delta}\zeta_R^2(x)\ln u dx \lesssim (1+E_R)^{24}.$$

For $t\in[-\tau R^2,0]$, consequently, one has

$$|\left\{x\in B(\frac{R}{2})|u\leq\delta\right\}|\lesssim \frac{R^3}{-\ln\delta}(1+E_R)^{24}.$$

Using the mean value inequality (2.9), one has

$$\begin{aligned}\sup_{P(\frac{\sqrt{\tau}}{2}R)}(\delta-u)_+ &\lesssim (1+E_R)^{\frac{5p}{2(3p-5)}}\left(\int\int_{P(\sqrt{\tau}R)}\frac{1}{(\sqrt{\tau}R)^5}(\delta-u)_+^2 dyds\right)^{\frac{1}{2}}\\ &\lesssim (1+E_R)^{\frac{5p}{2(3p-5)}}\delta\left[\frac{(\sqrt{\tau}R)^2}{(\sqrt{\tau}R)^5}\frac{R^3}{-\ln\delta}(1+E_R)^{24}\right]^{\frac{1}{2}}\\ &\lesssim (1+E_R)^{\frac{5p}{2(3p-5)}+12}\frac{\delta}{\sqrt{-\ln\delta}}.\end{aligned}$$

This gives

$$\inf_{P(\frac{\sqrt{\tau}}{2}R)} u\geq\delta\left[1-C\frac{(1+E_R)^{\frac{5p}{2(3p-5)}+12}}{\sqrt{-\ln\delta}}\right].$$

Under the assumption (1.8), one has

$$\begin{aligned}\inf_{P(\frac{\sqrt{\tau}}{2}R)} u &\geq\delta\left\{1-C\frac{\left[\left(\ln\ln\frac{3}{R}\right)^{\frac{3p-5}{77p-120}-}\right]^{\frac{77p-120}{6p-10}}}{\sqrt{-\ln\delta}}\right\}\\ &\geq\delta\left[1-C\frac{\left(\ln\ln\frac{3}{R}\right)^{\frac{1}{2}-}}{\sqrt{-\ln\delta}}\right],\end{aligned}$$

when $R\in(0,1)$. We can take $\delta(R)=(\ln\frac{3}{R})^{-1}$ to ensure $\inf_{P(\frac{\sqrt{\tau}}{2}R)} u\geq\frac{1}{2}\delta(R)$. This proves the corollary. □

Proof of Theorem 1.2. We define

$$m_\tau = \inf_{P(\frac{\sqrt{\tau}}{2}R)} \Gamma, \quad M_\tau = \sup_{P(\frac{\sqrt{\tau}}{2}R)} \Gamma.$$

Then from (3.1) and (3.19), one has

$$\frac{1}{2}\delta(R) \leq \inf_{P(\frac{\sqrt{\tau}}{2}R)} u = \begin{cases} 2(M_R - M_\tau)/J_R & \text{if } M_R > -m_R; \\ 2(m_\tau - m_R)/J_R & \text{else.} \end{cases}$$

We add the two cases together to get that

$$\delta(R) \leq \frac{4}{J_R}\left\{J_R - \mathrm{osc}(\Gamma, \frac{\sqrt{\tau}}{2}R)\right\},$$

here $\mathrm{osc}(\Gamma, \frac{\sqrt{\tau}}{2}R) = M_\tau - m_\tau$ and $\mathrm{osc}(\Gamma, R) = M_R - m_R = J_R$. So

$$\mathrm{osc}(\Gamma, \frac{\sqrt{\tau}}{2}R) \leq \left(1 - \frac{\delta(R)}{4}\right)\mathrm{osc}\,(\Gamma, R)\,.$$

We write it also as

$$J_{\frac{\sqrt{\tau}}{2}R} \leq \left(1 - \frac{\delta(R)}{4}\right) J_R.$$

For any small R, there exists some integer $j \geq 0$ such that $(\frac{\sqrt{\tau}}{2})^{j+1} < R/3 \leq (\frac{\sqrt{\tau}}{2})^j$. Using the above inequality, an iteration argument gives

$$J_{R/3} \leq J_{(\frac{\sqrt{\tau}}{2})^j} \leq \prod_{k=0}^{j-1}\left[1 - \frac{\delta((\frac{\sqrt{\tau}}{2})^k)}{4}\right] J_1.$$

Noting that $\ln(1-x) \leq -x$ for sufficiently small positive x, one has

$$\begin{aligned} J_{R/3} &\leq \exp\ln J_{R/3} \\ &\leq J_1 \exp\sum_0^{j-1}\ln\left[1 - \frac{\delta\left((\frac{\sqrt{\tau}}{2})^k\right)}{4}\right] \\ &\leq J_1 \exp\left\{-\frac{1}{4}\sum_0^{j-1}\delta((\frac{\sqrt{\tau}}{2})^k)\right\} \\ &\leq J_1 \exp\left\{-\frac{1}{4}\sum_0^{j-1}\left[\ln\frac{3}{(\frac{\sqrt{\tau}}{2})^k}\right]^{-1}\right\} \end{aligned}$$

$$
\begin{aligned}
&\le J_1 \exp\left\{-\frac{1}{4}\sum_0^{j-1}\left[\ln 3 + k|\ln(\frac{\sqrt{\tau}}{2})|\right]^{-1}\right\}\\
&\le J_1 \exp\left\{-\frac{1}{4}\sum_0^{j-1}[C(k+1)]^{-1}\right\}\\
&\lesssim J_1 \exp\{-c_0 \ln j\}\\
&\lesssim J_1 j^{-c_0}\\
&\lesssim J_1 |\ln\frac{R}{3}|^{-c_0}.
\end{aligned}
$$

Since, $\Gamma|_{r=0}=0$, the above estimate proves our Theorem 1.2. □

4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1 and get the regularity of the solution under the assumption (1.2). The idea comes from [Chen–Strain–Tsai–Yau]'s proof where they assume $|v| \le Cr^{-1}$.

We divide the proof into 3 steps.

Step one: scaling of the solution and set up of a equation

Let M be the maximum of $|v|$ up to a fixed time t_0 and we may assume $M > 1$ is large. Define the scaled solution

$$
v^M(X,T) = M^{-1} v(\frac{X}{M}, \frac{T}{M^2}), \quad X = (X_1, X_2, Z).
$$

Denote $x = (x_1, x_2, z)$ and $X = (X_1, X_2, Z)$, $r = \sqrt{x_1^2 + x_2^2}$ and $R = \sqrt{X_1^2 + X_1^2}$. We have the following estimate for r and R for time $t < t_0$ and $T < M^2 t_0$:

$$
|\nabla^k v^M| \le C_k. \tag{4.1}
$$

This inequality follows from $\|v^M\|_{L^\infty} \le 1$ for $t < t_0$ and the standard regularity theorem of Navier–Stokes equations. Its angular component (we omit the time dependence below) $v_\theta^M(R,Z)$ satisfies $v_\theta^M(0,Z) = 0 = \partial_Z v_\theta^M(0,Z)$ for all Z. By mean value theorem and (4.1),

$$
|v_\theta^M(R,Z)| \lesssim R, \ |\partial_Z v_\theta^M(R,Z)| \lesssim R \quad \text{for } R \le 1.
$$

Together with (4.1) for $R \ge 1$, we get

$$
|v_\theta^M| \lesssim \min\{R,1\}, \ |\partial_z v_\theta^M| \lesssim \min\{R,1\}. \tag{4.2}
$$

Due to Theorem 1.2,

$$
|\Gamma| = |r v_\theta(r,z)| \lesssim \begin{cases} (\ln\frac{3}{r})^{-c_0} & \text{for } r \le 1, \\ 1 & \text{for } r > 1. \end{cases}
$$

That is

$$|v_\theta(r,z)| \lesssim \begin{cases} \dfrac{(\ln\frac{3}{r})^{-c_0}}{r} & \text{for } r \le 1, \\ \dfrac{1}{r} & \text{for } r > 1. \end{cases} \tag{4.3}$$

Then $v_\theta^M(R,Z)$ satisfies the estimate

$$|v_\theta^M(R,Z)| = M^{-1}\left|v_\theta(\frac{X}{M},\frac{T}{M^2})\right| \lesssim \begin{cases} \dfrac{(\ln\frac{3M}{R})^{-c_0}}{R} & \text{for } R \le M, \\ \dfrac{1}{R} & \text{for } R > M. \end{cases}$$

Combining this with (4.2), one has

$$|v_\theta^M(R,Z)| \lesssim \begin{cases} \min\left\{R, \dfrac{(\ln\frac{3M}{R})^{-c_0}}{R}\right\} & \text{for } R \le 1, \\ \min\left\{1, \dfrac{(\ln\frac{3M}{R})^{-c_0}}{R}\right\} & \text{for } 1 < R < M, \\ \dfrac{1}{R} & \text{for } R \ge M. \end{cases} \tag{4.4}$$

Now consider the angular component of the rescaled vorticity. Recall $\Omega = \frac{w_\theta}{r}$. Let

$$f = \Omega^M(X,T) = M^{-3}\Omega(\frac{X}{M},\frac{T}{M^2}) = M^{-3}w_\theta(\frac{X}{M},\frac{T}{M^2})\frac{M}{R}$$
$$= \frac{w_\theta^M(X,T)}{R},$$

where

$$w_\theta^M(X,T) = w_\theta(\frac{X}{M},\frac{T}{M^2})M^{-2}.$$

Note that w_θ^M and ∇w_θ^M are bounded by (4.1) and also $w_\theta^M|_{R=0} = 0$, so one has

$$|f| \lesssim \frac{1}{1+R}.$$

From the equation (1.6), f satisfies

$$(\partial_T - L)f = g, \quad L = \Delta + \frac{2}{R}\partial_R - b^M \cdot \nabla,$$

where $g = R^{-2}\partial_Z(v_\theta^M)^2$ and $b^M = v_R^M e_R + v_Z^M e_Z$, $|b^M| \le 1$.

Combining the estimates (4.2) and (4.4), one has

$$g = \frac{2}{R^2} v_\theta^M \partial_Z (v_\theta^M) \lesssim \begin{cases} \min\left\{1, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^2}\right\} & \text{for } R \le 1, \\ \min\left\{\frac{1}{R^2}, \frac{(\ln \frac{3M}{R})^{-c_0}}{R^3}\right\} & \text{for } 1 < R < M, \\ \frac{1}{R^3} & \text{for } R \ge M. \end{cases} \tag{4.5}$$

Let $P(X, T; Y, S)$ be the kernel of $\partial_T - L$. By Duhamel's formula

$$\begin{aligned} f(X,T) &= \int P(X,T;Y,S) f(Y,S) dY + \int_S^T \int P(X,T;Y,\tau) g(Y,\tau) dY d\tau \\ &:= I_1 + I_2. \end{aligned} \tag{4.6}$$

Step two: bounding of f

In the following, we will estimate (4.6) and give a bound for $f(X, T)$.

The kernel $P(X, T; Y, S)$ satisfies $P \ge 0$, $\int P(X, T; Y, S) dY \le 1$ and

$$P(X,T;Y,S) \le C(T-S)^{-3/2} \exp\left\{-C \frac{|X-Y|^2}{T-S} \left(1 - \frac{T-S}{|X-Y|}\right)_+^2\right\}. \tag{4.7}$$

The proof of estimate (4.7) is based on [7], but due to the singularity of the term $\frac{2}{r}\partial_r$, the proof is more involved. For completeness of our paper, we will prove it in Section 5 as Theorem 1.3.

Now we give estimates of P in two cases.

Case one: when $1 - \frac{T-S}{|X-Y|} > \frac{1}{2}$, that is $|X - Y| > 2(T - S)$,

$$\begin{aligned} \exp\left\{-\frac{|X-Y|^2}{T-S}\left(1 - \frac{T-S}{|X-Y|}\right)_+^2\right\} &\le \exp\left\{-\frac{1}{4}\frac{|X-Y|^2}{T-S}\right\} \\ &\le \begin{cases} \exp\left\{-\frac{1}{4}\frac{|X-Y|}{T-S}\right\} & \text{for } |X-Y| \ge 1, \\ \exp\left\{-\frac{1}{4}\frac{|X-Y|^2}{T-S}\right\} & \text{for } |X-Y| < 1. \end{cases} \end{aligned}$$

Case two: when $1 - \frac{T-S}{|X-Y|} \le \frac{1}{2}$, that is $|X - Y| \le 2(T - S)$,

$$\exp\left\{-\frac{|X-Y|^2}{T-S}\left(1 - \frac{T-S}{|X-Y|}\right)_+^2\right\} \le 1 \le e^2 \exp\left\{-\frac{|X-Y|}{T-S}\right\}.$$

With these estimates and Hölder inequality, one gets, for I_1 in (4.6),

$$
\begin{aligned}
|I_1| &\le \left[\int P(X,T;Y,S)|f(Y,S)|^3 dY\right]^{\frac{1}{3}} \left[\int P(X,T;Y,S) dY\right]^{\frac{2}{3}} \\
&\le \left\{\int_{-\infty}^{+\infty}\int_0^{+\infty} (T-S)^{-\frac{3}{2}} \left[e^{-\frac{|X_3-Y_3|}{T-S}} + e^{-\frac{|X_3-Y_3|^2}{T-S}}\right] \frac{R}{(R+1)^3} dR dY_3\right\}^{1/3} \\
&\lesssim (T-S)^{-\frac{1}{2}} \left\{\int_{-\infty}^{+\infty} \left[e^{-\frac{|X_3-Y_3|}{T-S}} + e^{-\frac{|X_3-Y_3|^2}{T-S}}\right] dY_3\right\}^{1/3} \\
&\lesssim (T-S)^{-\frac{1}{2}} \left\{(T-S) + (T-S)^{\frac{1}{2}}\right\}^{1/3} \\
&\lesssim (T-S)^{-\frac{1}{6}}
\end{aligned} \tag{4.8}
$$

for $T - S \ge 1$, next

$$
\begin{aligned}
|I_2| \le& \int_S^T (T-\tau)^{-\frac{3}{2}} \left\{ \int_{|X-Y|\le 2(T-\tau)} e^{-\frac{|X-Y|}{T-\tau}} |g| dY \right. \\
&\left. + \int_{|X-Y|\ge 2(T-\tau), |X-Y|>1} e^{-\frac{1}{4}\frac{|X-Y|}{T-\tau}} |g| dY + \int_{|X-Y|\ge 2(T-\tau), |X-Y|<1} e^{-\frac{1}{4}\frac{|X-Y|^2}{T-\tau}} |g| dY \right\} d\tau \\
:=& \int_S^T (T-\tau)^{-\frac{3}{2}} \left\{I_{2,1} + I_{2,2} + I_{2,3}\right\} d\tau.
\end{aligned} \tag{4.9}
$$

We deal with $I_{2,1}$, $I_{2,2}$, $I_{2,3}$ in (4.9) as follows,

$$
\begin{aligned}
I_{2,1} + I_{2,2} &\lesssim \int_{-\infty}^{+\infty}\int_0^{+\infty} \left(e^{-\frac{1}{4}\frac{|X_3-Y_3|}{T-\tau}} + e^{-\frac{|X_3-Y_3|}{T-\tau}}\right) |g| R dR dY_3 \\
&\lesssim (T-\tau) \int_0^{+\infty} |g| R dR \\
&\lesssim (T-\tau) \left\{ \int_0^1 \min\left\{R, \frac{(\ln\frac{3M}{R})^{-c_0}}{R}\right\} dR \right. \\
&\qquad \left. + \int_1^M \min\left\{\frac{1}{R}, \frac{(\ln\frac{3M}{R})^{-c_0}}{R^2}\right\} dR + \int_M^{+\infty} \frac{1}{R^2} dR \right\} \\
&:= (T-\tau)(I_{2,4} + I_{2,5} + I_{2,6})
\end{aligned} \tag{4.10}
$$

For $I_{2,4}$, when $R \in (0, 1]$, the function R is increasing while $\frac{(\ln \frac{3M}{R})^{-c_0}}{R}$ is decreasing. Let R_0 be such that

$$R_0 = \frac{(\ln \frac{3M}{R_0})^{-c_0}}{R_0}.$$

This makes

$$(\frac{1}{R_0})^{\frac{2}{c_0}} = \ln 3M + \ln \frac{1}{R_0} \le \ln 3M + \frac{c_0}{2}(\frac{1}{R_0})^{\frac{2}{c_0}}.$$

That is

$$(1 - \frac{c_0}{2})(\frac{1}{R_0})^{\frac{2}{c_0}} \le \ln 3M \le (\frac{1}{R_0})^{\frac{2}{c_0}}.$$

So, there exists a $C > 1$, such that

$$C^{-1}(\ln 3M)^{-\frac{c_0}{2}} \le R_0 \le C(\ln 3M)^{-\frac{c_0}{2}}. \tag{4.11}$$

Then

$$\min\{R, \frac{(\ln \frac{3M}{R})^{-c_0}}{R}\} = \begin{cases} R & \text{for } 0 \le R \le R_0; \\ \frac{(\ln \frac{3M}{R})^{-c_0}}{R} & \text{for } R_0 < R < 1. \end{cases}$$

By (4.11), the control of $I_{2,4}$ in (4.10) is

$$\begin{aligned} I_{2,4} &\le \int_0^{R_0} R\,dR + \int_{R_0}^{1} \frac{(\ln \frac{3M}{R})^{-c_0}}{R} dR \\ &\le \frac{1}{2}R_0^2 + (\ln 3M)^{-c_0} \int_{R_0}^{1} \frac{1}{R} dR \\ &\lesssim (\ln 3M)^{-c_0}(1 + \ln \frac{1}{R_0}) \\ &\lesssim (\ln 3M)^{-c_0}(1 + \frac{1}{R_0}) \\ &\lesssim (\ln 3M)^{-c_0}(1 + (\ln 3M)^{\frac{c_0}{2}}) \\ &\lesssim (\ln 3M)^{-\frac{c_0}{2}}. \end{aligned} \tag{4.12}$$

For $I_{2,5}$ in (4.10), one has

$$
\begin{aligned}
I_{2,5} &\le \int_1^M \frac{(\ln\frac{3M}{R})^{-c_0}}{R^2} dR \\
&= M^{-1} \int_{M^{-1}}^1 \frac{(\ln\frac{3}{R})^{-c_0}}{R^2} dR \\
&= M^{-1} \left(\int_{M^{-1}}^{M^{-\frac{1}{2}}} + \int_{M^{-\frac{1}{2}}}^1 \right) \frac{(\ln\frac{3}{R})^{-c_0}}{R^2} dR \\
&\le M^{-1} \left\{ (\ln \frac{3}{M^{-\frac{1}{2}}})^{-c_0} \int_{M^{-1}}^{M^{-\frac{1}{2}}} \frac{1}{R^2} dR + \int_{M^{-\frac{1}{2}}}^1 \frac{(R\ln\frac{3}{R})^{-c_0}}{R^{2-c_0}} dR \right\}.
\end{aligned}
$$

Since $R\ln\frac{3}{R}$ is increasing when $R \in (0, 1)$, one has,

$$
\begin{aligned}
I_{2,5} &\lesssim M^{-1} \left\{ (\ln 3M)^{-c_0} M + (M^{-\frac{1}{2}} \ln 3M)^{-c_0} M^{-\frac{c_0-1}{2}} \right\} \\
&\lesssim (\ln 3M)^{-c_0} + M^{-\frac{1}{2}} (\ln 3M)^{-c_0} \\
&\lesssim (\ln 3M)^{-c_0}. \qquad (4.13)
\end{aligned}
$$

For $I_{2,6}$ in (4.10), obviously

$$I_{2,6} \lesssim M^{-1}. \qquad (4.14)$$

Hence, combining (4.12), (4.13) and (4.14), from (4.10), one has

$$I_{2,1} + I_{2,2} \lesssim (T-\tau)(\ln 3M)^{-\frac{c_0}{2}}. \qquad (4.15)$$

For $I_{2,3}$ in (4.9), using the Cauchy–Schwartz inequality, one has

$$
\begin{aligned}
I_{2,3} &= \int_{|X-Y|\ge 2(T-\tau),|X-Y|<1} e^{-\frac{1}{4}\frac{|X-Y|^2}{T-\tau}} |g| dY \\
&= \int e^{-\frac{1}{4}\frac{|X'-Y'|^2}{T-\tau} - \frac{1}{8}\frac{|X_3-Y_3|^2}{T-\tau}} e^{-\frac{1}{8}\frac{|X_3-Y_3|^2}{T-\tau}} |g| dY \\
&\le \left(\int e^{-\frac{1}{2}\frac{|X'-Y'|^2}{T-\tau} - \frac{1}{4}\frac{|X_3-Y_3|^2}{T-\tau}} dY \right)^{\frac{1}{2}} \left(\int e^{-\frac{1}{4}\frac{|X_3-Y_3|^2}{T-\tau}} g^2 dY \right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \left(\int e^{-\frac{1}{2}\frac{|X'-Y'|^2}{T-\tau}-\frac{1}{4}\frac{|X_3-Y_3|^2}{T-\tau}}dY\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}\int_0^{+\infty} e^{-\frac{1}{4}\frac{|X_3-Y_3|^2}{T-\tau}} g^2 R dR dY_3\right)^{\frac{1}{2}}$$

$$\leq (T-\tau)^{\frac{3}{4}}\cdot(T-\tau)^{\frac{1}{4}}\left(\int_0^{+\infty}\sup_{Y_3} g^2 R dR\right)^{\frac{1}{2}}$$

$$\leq (T-\tau)\left(\int_0^{+\infty}\sup_{Y_3} g^2 R dR\right)^{\frac{1}{2}}.$$

Following (4.5) one has

$$g^2 \leq \begin{cases} \min\left\{1, \frac{(\ln\frac{3M}{R})^{-c_0}}{R^2}, \frac{(\ln\frac{3M}{R})^{-2c_0}}{R^4}\right\} & \text{for } R\leq 1; \\ \min\left\{\frac{1}{R^4}, \frac{(\ln\frac{3M}{R})^{-c_0}}{R^5}, \frac{(\ln\frac{3M}{R})^{-c_0}}{R^6}\right\} & \text{for } 1<R<M; \\ \frac{1}{R^6} & \text{for } R\geq M. \end{cases}$$

As the previous proof for (4.15), one can get

$$I_{2,3} \lesssim (T-\tau)(\ln 3M)^{-\frac{c_0}{2}}. \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.9), one has

$$|I_2| \lesssim \int_S^T (T-\tau)^{-\frac{1}{2}}(\ln 3M)^{-\frac{c_0}{2}}d\tau \lesssim (T-S)^{\frac{1}{2}}(\ln 3M)^{-\frac{c_0}{2}}. \tag{4.17}$$

So, combining (4.8) and (4.17), from (4.6), one has

$$|f(X,T)| \lesssim (T-S)^{-\frac{1}{6}} + (T-S)^{\frac{1}{2}}(\ln 3M)^{-\frac{c_0}{2}}.$$

Let $S = T - (\ln 3M)^{-\frac{3}{4}c_0} > -M^2$ (hence f is defined), so

$$|f(X,T)| \lesssim (\ln 3M)^{-\frac{1}{8}c_0}.$$

Step three: bounding the solution v from f

First

$$|w_\theta(x,t)| \leq M^2|w_\theta^M(rM, zM, tM^2)| \leq |\Omega^M(rM, zM, tM^2)|M^2 rM \leq CM^3 r(\ln 3M)^{-\frac{1}{8}c_0}.$$

Therefore

$$|w_\theta(x,t)| \le CM^2(\ln 3M)^{-\frac{1}{16}c_0}, \quad \text{for } r \le M^{-1}(\ln 3M)^{\frac{1}{16}c_0}. \tag{4.18}$$

In the following, we bound $b = v_r e_r + v_z e_z$.

Denote $B_\rho(x_0) = \{x : |x - x_0| < \rho\}$, where $\rho > 0$ to be determined later. By Biot–Savart law, b satisfies

$$-\Delta b = \mathrm{curl}(w_\theta e_\theta).$$

From the estimates of elliptic equation [12], for $q > 1$,

$$\sup_{B_\rho(x_0)} |b| \le C\left(\rho^{-\frac{3}{q}}\|b\|_{L^q(B_{2\rho}(x_0))} + \rho \sup_{B_{2\rho}(x_0)} |w_\theta|\right). \tag{4.19}$$

Take

$$\rho = M^{-1}(\ln 3M)^{\frac{1}{32}c_0},\ \ x_0 \in \{(r,\theta,z) : r < \rho\} \text{ and } 1 < q < 2.$$

By the assumption (1.2) on b,

$$\begin{aligned}
\rho^{-\frac{3}{q}}\|b\|_{L^q(B_{2\rho}(x_0))} &\le \rho^{-\frac{3}{q}}\Big\|\frac{(\ln|\ln\frac{r}{3}|)^\alpha}{r}\Big\|_{L^q(B_{2\rho}(x_0))} \\
&\le C\rho^{-\frac{3}{q}}\left[\int_{z_0-2\rho}^{z_0+2\rho} dz \int_0^{3\rho} \frac{(\ln|\ln\frac{r}{3}|)^{\alpha q}}{r^q} r\,dr\right]^{\frac{1}{q}} \\
&\le C\rho^{-\frac{2}{q}}\left[\int_0^{3\rho} \frac{(\ln|\ln\frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr\right]^{\frac{1}{q}}.
\end{aligned} \tag{4.20}$$

We compute $\int_0^{3\rho} \frac{(\ln|\ln\frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr$ as follows,

$$\begin{aligned}
\int_0^{3\rho} \frac{(\ln|\ln\frac{r}{3}|)^{\alpha q}}{r^{q-1}} dr &= \int_{\frac{1}{3\rho}}^{+\infty} (\ln\ln 3r)^{\alpha q} r^{q-3} dr \quad r \text{ replaced by } \frac{1}{r} \\
&= \left(\int_{\frac{1}{3\rho}}^{\frac{1}{3}e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}}} + \int_{\frac{1}{3}e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}}}^{+\infty}\right)(\ln\ln 3r)^{\alpha q} r^{q-3} dr
\end{aligned}$$

$$\leq \left(\ln\ln\frac{3}{\rho}\right)^{2\alpha q}\int_{\frac{1}{3\rho}}^{+\infty} r^{q-3}dr + \int_{\frac{1}{3}e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}}}^{+\infty}\left(\frac{\ln\ln 3r}{r}\right)^{\alpha q} r^{q-3+\alpha q}dr.$$

Here $\frac{\ln\ln 3r}{r}$ is a decreasing function in the integral domain. Also we can pick a $q \in (1,2)$ such that $q-3+\alpha q < -1$. So

$$\begin{aligned}\int_0^{3\rho}\frac{(\ln|\ln\frac{r}{3}|)^{\alpha q}}{r^{q-1}}dr &\leq C\left(\ln\ln\frac{3}{\rho}\right)^{2\alpha q}\left(\frac{1}{\rho}\right)^{q-2} + \frac{(\ln\ln\frac{3}{\rho})^{2\alpha q}}{\left(\frac{1}{3}e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}}\right)^{\alpha q}}\left(\frac{1}{3}e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}}\right)^{q-2+\alpha q}\\ &\leq C\left(\ln\ln\frac{3}{\rho}\right)^{2\alpha q}\left(\frac{1}{\rho}\right)^{q-2}.\end{aligned}\tag{4.21}$$

The last inequality holds because $e^{e^{\left(\ln\ln\frac{3}{\rho}\right)^2}} > \frac{3}{\rho}$.

Since $\rho = M^{-1}(\ln 3M)^{\frac{1}{32}c_0}$ and $\rho^{-1} \leq M$, so, from (4.20) and (4.21),

$$\begin{aligned}\rho^{-\frac{3}{q}}\|b\|_{L^q(B_{2\rho}(x_0))} &\leq C\rho^{-\frac{2}{q}}\left(\ln\ln\frac{3}{\rho}\right)^{2\alpha}\left(\frac{1}{\rho}\right)^{1-\frac{2}{q}}\\ &\leq C\left(\ln\ln\frac{3}{\rho}\right)^{2\alpha}\left(\frac{1}{\rho}\right)\\ &\leq CM(\ln 3M)^{-\frac{c_0}{32}}\cdot(\ln\ln 3M)^{2\alpha}\\ &\leq CM(\ln 3M)^{-\frac{c_0}{64}}.\end{aligned}\tag{4.22}$$

While, due to (4.18), when $x_0 \in B_\rho(x_0)$,

$$\begin{aligned}\rho\sup_{B_\rho(x_0)}|w_\theta| &\leq M^{-1}(\ln 3M)^{\frac{c_0}{32}}M^2(\ln 3M)^{-\frac{c_0}{16}}\\ &\leq CM(\ln 3M)^{-\frac{c_0}{32}}.\end{aligned}\tag{4.23}$$

Combining (4.19), (4.22) and (4.23), we have

$$|b| \leq CM(\ln 3M)^{-\frac{c_0}{64}} \quad \text{for } r < M^{-1}(\ln 3M)^{\frac{1}{32}c_0};\tag{4.24}$$

next, when $M^{-1}(\ln 3M)^{\frac{c_0}{32}} \leq r < 1$,

$$|b| \leq C\frac{(\ln\ln\frac{3}{r})^\alpha}{r} \leq CM(\ln 3M)^{-\frac{c_0}{32}}(\ln\ln 3M)^\alpha \leq CM(\ln 3M)^{-\frac{c_0}{64}};\tag{4.25}$$

thirdly, when $1 \le r$,

$$|b| \le \frac{C}{r} \le C. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we get, for any $r > 0$,

$$|b| \le CM(\ln 3M)^{-\frac{c_0}{64}}. \tag{4.27}$$

In the following, we bound v_θ.

Recall that v_θ satisfies (4.3), then

$$v_\theta(r,z) = M|v_\theta^M(rM, zM)| \le M \begin{cases} \min\left\{\dfrac{(\ln\frac{3}{r})^{-c_0}}{rM}, rM\right\} & \text{for } r < \dfrac{1}{M}; \\ \min\left\{\dfrac{(\ln\frac{3}{r})^{-c_0}}{rM}, 1\right\} & \text{for } \dfrac{1}{M} < r < 1. \end{cases}$$

Firstly, when $r < \frac{1}{M}$, $\frac{(\ln\frac{3}{r})^{-c_0}}{rM}$ is a decreasing function while rM is an increasing function with respect to r. Let r_0 be such that

$$\frac{(\ln\frac{3}{r_0})^{-c_0}}{r_0 M} = r_0 M.$$

This gives

$$\left(\frac{1}{r_0 M}\right)^{\frac{2}{c_0}} = \ln\frac{3}{r_0} > \ln 3M.$$

So

$$r_0 < (\ln 3M)^{-\frac{c_0}{2}} M^{-1}.$$

Then

$$|v_\theta| \lesssim r_0 M^2 \lesssim (\ln 3M)^{-\frac{c_0}{2}} M. \tag{4.28}$$

Next, when $\frac{1}{M} \le r < 1$,

$$|v_\theta| \lesssim \frac{(\ln\frac{3}{r})^{-c_0}}{r} \lesssim (\ln 3M)^{-c_0} M. \tag{4.29}$$

Thirdly, when $1 \le r$,

$$|v_\theta| \le \frac{C}{r} \le C. \tag{4.30}$$

Combining (4.28), (4.29) and (4.30), we have, for any $r > 0$,

$$|v_\theta| \le CM(\ln 3M)^{-\frac{c_0}{64}}. \tag{4.31}$$

Since M is the maximum of $|v|$, $M = \max\{\sup|b|, \sup|v_\theta|\}$. Due to the estimates (4.27) and (4.31), we get

$$M \le CM(\ln 3M)^{-\frac{c_0}{64}}.$$

This gives an upper bound for M which completes the proof of Theorem 1.1. □

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 and give the estimate (4.7) of the fundamental solution.

Following Davies [8] and Carlen–Loss [7], for a fixed constant vector $\alpha \in \mathbb{R}^3$, let $\psi(x) = \alpha \cdot x$. For any $f \in C_0^\infty(\mathbb{R}^3; (0, +\infty))$, define

$$\begin{aligned} P_{t,s}^{\psi} f(x) &= e^{-\psi(x)} \int f(y)p(x,t;y,s)e^{\psi(y)}dy \\ &= e^{-\alpha x} \int f(y)p(x,t;y,s)e^{\alpha y}dy \\ &\triangleq f_t(x). \end{aligned}$$

In fact, let $Q_R = B_R(0) \times (s, +\infty)$ and define the Dirichlet fundamental solution $p^R(x,t;y,s)$ in Q_R the same as Definition 1.1 which satisfies the boundary condition

$$p^R(x,t;y,s)|_{(x,t)\in\partial B_R\times(s,+\infty)} = 0.$$

Due to the maximum principle, we have

$$p^{R_1}(x,t;y,s) \le p^{R_2}(x,t;y,s) \le p(x,t;y,s), \quad \text{when } R_1 < R_2.$$

Also

$$\lim_{R\to+\infty} p^R(x,t;y,s) = p(x,t;y,s). \quad a.e.$$

In a rigorous computation, all the integrals in the following should be done in $B_R(0)$ with the function $f_t(x)$ replaced by $f_t^R(x) \triangleq e^{-\alpha x}\int_{\mathbb{R}^3} f(y)p^R(x,t;y,s)e^{\alpha y}dy$ which satisfies $f_t^R|_{x\in\partial B_R} = 0$ for all $t \ge s$. Then let $R \to +\infty$ to reach the estimate of $p(x,t;y,s)$. But for simplicity, we just carry out this process on $f_t(x)$ and assume that $f_t(x)$ vanishes on the boundary which means, $f_t(x)|_{|x|=+\infty} = 0$.

We divide the proof into 3 parts.

Part one: $L^2 \to L^\infty$ estimate of $P_{t,s}^{\psi}$.

Let $k(t) : [s, T] \to [2, \infty]$ be a continuously differentiable increasing function to be determined later. By direct computation, we have

$$
\begin{aligned}
& k(t)^2 \| f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt} \| f_t \|_{k(t)} \\
= & k'(t) \int f_t^{k(t)} \ln \left(\frac{f_t^{k(t)}}{\| f_t\|_{k(t)}^{k(t)}} \right) dx + k(t)^2 \int f_t^{k(t)-1} \frac{d}{dt} f_t dx \\
= & k'(t) \int f_t^{k(t)} \ln \left(\frac{f_t^{k(t)}}{\| f_t\|_{k(t)}^{k(t)}} \right) dx + k(t)^2 \int (f_t^{k(t)-1} e^{-\alpha x})(\Delta + \frac{2}{r}\partial_r - b \cdot \nabla)(e^{\alpha x} f_t) dx \\
= & k'(t) \int f_t^{k(t)} \ln \left(\frac{f_t^{k(t)}}{\| f_t\|_{k(t)}^{k(t)}} \right) dx + k(t)^2 \Big\{ - \int \nabla (f_t^{k(t)-1} e^{-\alpha x}) \cdot \nabla (e^{\alpha x} f_t) \\
& + \int \frac{2}{r} f_t^{k(t)-1} \left[\partial_r f_t + \partial_r (\alpha x) f_t \right] - \int f_t^{k(t)-1} (b \cdot \alpha f_t + b \cdot \nabla f_t) \Big\} \\
\triangleq & k'(t) \int f_t^{k(t)} \ln \left(\frac{f_t^{k(t)}}{\| f_t\|_{k(t)}^{k(t)}} \right) dx + k(t)^2 \{ I_1 + I_2 + I_3 \}.
\end{aligned} \tag{5.1}
$$

Using Cauchy–Schwartz inequality, we have

$$
\begin{aligned}
I_1 = & - \int [(k(t) - 1) f_t^{k(t)-2} \nabla f_t - f_t^{k(t)-1} \alpha] \cdot \left[\nabla f_t + \alpha f_t \right] \\
= & - \int \frac{4(k(t) - 1)}{k(t)^2} |\nabla f_t^{\frac{k(t)}{2}}|^2 - \int (k(t) - 2) f_t^{k(t)-1} \alpha \cdot \nabla f_t + \int \alpha^2 f_t^{k(t)} \\
= & - \int \frac{4(k(t) - 1)}{k(t)^2} |\nabla f_t^{\frac{k(t)}{2}}|^2 - \int \frac{2(k(t) - 2)}{k(t)} f_t^{\frac{k(t)}{2}} \alpha \cdot \nabla f_t^{\frac{k(t)}{2}} + \int \alpha^2 f_t^{k(t)} \\
\leq & - \int \frac{4(k(t) - 1)}{k(t)^2} |\nabla f_t^{\frac{k(t)}{2}}|^2 + \int \frac{(k(t) - 2)^2 \varepsilon}{k(t)^2} |\nabla f_t^{\frac{k(t)}{2}}|^2 + \int (1 + \frac{1}{\varepsilon}) \alpha^2 f_t^{k(t)},
\end{aligned} \tag{5.2}
$$

where $\varepsilon > 0$ is to be determined later on.

Also,

$$
\begin{aligned}
I_2 = & \int \frac{2}{r} f_t^{k(t)-1} [\partial_r f_t + \partial_r (\alpha x) f_t] \\
\leq & \int \frac{2}{k(t)} \partial_r f_t^{k(t)} dr d\theta dz + \int |\alpha| \frac{2}{r} f_t^{k(t)} \\
= & - \frac{2}{k(t)} \int dz d\theta f_t^{k(t)} |_{r=0} + \int |\alpha| \frac{2}{r} f_t^{k(t)} \\
\leq & \int |\alpha| \frac{2}{r} f_t^{k(t)}.
\end{aligned} \tag{5.3}
$$

The estimate (5.3) is due to our choice of $f \geq 0$. So $f_t|_{r=0} \geq 0$.

Moreover, due to the divergence-free property of b,

$$
I_3 = - \int f_t^{k(t)-1} (b \cdot \alpha f_t + b \cdot f_t)\}
$$

$$
\begin{aligned}
&= -\int f_t^{k(t)} b \cdot \alpha \\
&\le \int (C_0 + \frac{1}{r})|\alpha| f_t^{k(t)}, \qquad (5.4)
\end{aligned}
$$

while using integration by parts,

$$
\begin{aligned}
\int \frac{|\alpha|}{r} f_t^{k(t)} &= |\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{+\infty} |\alpha| f_t^{k(t)} dr d\theta dz| \\
&= |-\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{+\infty}\int 2|\alpha| r f_t^{\frac{k(t)}{2}} \partial_r f_t^{\frac{k(t)}{2}} dr d\theta dz| \\
&\le \int 2|\alpha| f_t^{\frac{k(t)}{2}} \partial_r f_t^{\frac{k(t)}{2}} dx \\
&\le \varepsilon \int |\nabla f_t^{\frac{k(t)}{2}}|^2 + \frac{\alpha^2}{\varepsilon}\int f_t^{k(t)}. \qquad (5.5)
\end{aligned}
$$

Thus combining (5.3), (5.4) and (5.5), we have

$$
I_2 + I_3 \le 3\varepsilon \int |\nabla f_t^{\frac{k(t)}{2}}|^2 + 3\frac{\alpha^2}{\varepsilon}\int f_t^{k(t)} + C_0|\alpha| \int f_t^{k(t)}. \qquad (5.6)
$$

Now we recall the 3-d *log − Sobolev* inequality.

For all functions u on $\mathbb{R}^3$*, together with their distributional gradients* ∇u *are square integrable, then*

$$
\int u^2 \ln\left(\frac{u^2}{\|u\|_2^2}\right) + \left(3 + \frac{3}{2}\ln a\right)\int u^2 \le \frac{a}{\pi}\int |\nabla u|^2 \qquad (5.7)
$$

for all $a > 0$.

Inserting (5.2), (5.6) and (5.7) into (5.1), one has

$$
\begin{aligned}
k(t)^2 \|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} \le\ & k'(t)\left[\frac{a}{\pi}\int |\nabla f_t^{\frac{k(t)}{2}}|^2 - \left(3 + \frac{3}{2}\ln a\right)\int f_t^{k(t)}\right] \\
&+ k(t)^2 \left\{\left(\frac{-4(k(t)-1) + \varepsilon(k(t)-2)^2}{k(t)^2} + 3\varepsilon\right)\int |\nabla f_t^{\frac{k(t)}{2}}|^2 \right. \\
&\left. + \left((1 + \frac{4}{\varepsilon})\alpha^2 + C_0|\alpha|\right)\int f_t^{k(t)}\right\},
\end{aligned}
$$

then

$$
\|f_t\|_{k(t)}^{k(t)-1} \frac{d}{dt}\|f_t\|_{k(t)} \le \left[\frac{k'(t)}{k(t)^2}\frac{a}{\pi} - \frac{4(k(t)-1)}{k(t)^2} + \frac{(k(t)-2)^2}{k(t)^2}\varepsilon + 3\varepsilon\right]\int |\nabla f_t^{\frac{k(t)}{2}}|^2
$$

$$+\left[(1+\frac{4}{\varepsilon})\alpha^2+C_0|\alpha|-\frac{k'(t)}{k(t)^2}\left(3+\frac{3}{2}\ln a\right)\right]\int f_t^{k(t)}.$$

Here we can not choose $k(t):[s,T]\to[1,+\infty]$ as Carlen–Loss in [7] do. Because when $k(s)=1$, the coefficient of $\int|\nabla f_t^{\frac{k(t)}{2}}|^2$ is $k'(s)\frac{a}{\pi}+4\varepsilon$ which is obviously positive when $k(t)$ is a continuously differentiable increasing function and $a>0$. It can not reach zero as Carlen–Loss in [7] do. So we choose $k(t):[s,T]\to[2,+\infty]$ to ensure the coefficient of $\int|\nabla f_t^{\frac{k(t)}{2}}|^2$ is zero.

When $k(t)\in[2,\infty)$,

$$\frac{k(t)-1}{k(t)}\geq\frac{1}{2},\qquad\frac{(k(t)-2)^2}{k(t)^2}<1.$$

So

$$\|f_t\|_{k(t)}^{k(t)-1}\frac{d}{dt}\|f_t\|_{k(t)}\leq\left[\frac{k'(t)}{k(t)^2}\frac{a}{\pi}-\frac{2}{k(t)}+4\varepsilon\right]\int|\nabla f_t^{\frac{k(t)}{2}}|^2$$
$$+\left[(1+\frac{4}{\varepsilon})\alpha^2+C_0|\alpha|-\frac{k'(t)}{k(t)^2}\left(3+\frac{3}{2}\ln a\right)\right]\int f_t^{k(t)}.$$

Let $k(t)=2\sqrt{\frac{T}{T+s-t}}$, $4\varepsilon=\frac{k'(t)}{k(t)^2}\frac{a}{\pi}=\frac{1}{k(t)}$, then

$$a=\frac{\pi k(t)}{k'(t)}=2\pi(T+s-t),$$
$$-\frac{k'(t)}{k(t)^2}=-\frac{1}{4\sqrt{T(T+s-t)}}.$$

Then

$$\|f_t\|_{k(t)}^{-1}\frac{d}{dt}\|f_t\|_{k(t)}\leq\left[\left(1+32\sqrt{\frac{T}{T+s-t}}\right)\alpha^2+C_0|\alpha|\right.$$
$$\left.-\frac{3}{4\sqrt{T(T+s-t)}}-\frac{3\ln(2\pi(T+s-t))}{8\sqrt{T(T+s-t)}}\right].$$

Integrating the above inequality in $[s,T]$, we get

$$\ln\|f_T\|_\infty-\ln\|f_s\|_2\leq\int_s^T\left[\left(1+32\sqrt{\frac{T}{T+s-t}}\right)\alpha^2+C_0|\alpha|\right.$$
$$\left.-\frac{3}{4\sqrt{T(T+s-t)}}-\frac{3\ln(2\pi(T+s-t))}{8\sqrt{T(T+s-t)}}\right]dt$$
$$\leq(\alpha^2+C_0|\alpha|+64\alpha^2)(T-s)-\frac{3\ln 2\pi}{4}-\frac{3\ln(T-s)}{4}$$
$$\leq(C_0|\alpha|+65\alpha^2)(T-s)+\ln(2\pi(T-s))^{-\frac{3}{4}}.$$

So

$$\|f_T\|_\infty \le (2\pi(T-s))^{-\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)(T-s)\}\|f\|_{L^2}.$$

That is

$$\|P^{\psi}_{t,s} f\|_\infty \le (2\pi(t-s))^{-\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)(t-s)\}\|f\|_{L^2}. \tag{5.8}$$

Part two: $L^2 \to L^\infty$ estimate of the adjoint $(P^{\psi}_{t,s})^*$ of $P^{\psi}_{t,s}$.

Now we come to investigate the adjoint $(P^{\psi}_{t,s})^*$ of $P^{\psi}_{t,s}$, for any $f, g \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}(P^{\psi}_{t,s} f(x), g(x)) &= \int g(x)e^{-\psi(x)} \int f(y)e^{\psi(y)} p(x,t;y,s)dydx \\ &= \int f(y)e^{\psi(y)}dy \int g(x)e^{-\psi(x)} p(x,t;y,s)dx \\ &\triangleq ((P^{\psi}_{t,s})^* g(y), f(y))\end{aligned}$$

So

$$(P^{\psi}_{t,s})^* g(y) = e^{\psi(y)} \int g(x)e^{-\psi(x)} p(x,t;y,s)dx.$$

Here, note that we do not require $t \ge s$. We denote $y = (y_1, y_2, y_3)$ and $y' = (y_1, y_2, 0)$.

Let $p(x,t;y,s)$ be the fundamental solution of (1.10), that is

$$\partial_t p(x,t;y,s) = \Delta_x p(x,t;y,s) - b \cdot \nabla_x p(x,t;y,s) + \frac{2}{r_x}\partial_{r_x} p(x,t;y,s),$$

when $t > s$. Here $r_x = \sqrt{x_1^2 + x_2^2}$. Then $p(x,t;y,s)$, with respect to (y,s), satisfies

$$-\partial_s p(x,t;y,s) = \Delta_y p(x,t;y,s) + b \cdot \nabla_y p(x,t;y,s) - \frac{2}{r_y}\partial_{r_y} p(x,t;y,s).$$

Let $\rho = -t$, $\tau = -s$. $p(x,\rho;y,\tau)$, with respect to (y,τ), satisfies

$$\partial_\tau p(x,\rho;y,\tau) = \Delta_y p(x,\rho;y,\tau) + b \cdot \nabla_y p(x,\rho;y,\tau) - \frac{2}{r_y}\partial_{r_y} p(x,\rho;y,\tau).$$

Let $p^*(y,\tau;x,\rho) = p(x,\rho;y,\tau)$, then

$$(P^{\psi}_{\rho,\tau})^* g(y) = e^{\psi(y)} \int g(x)e^{-\psi(x)} p^*(y,\tau;x,\rho)dx.$$

When $\tau > \rho$, $p^*(y,\tau;x,\rho)$ satisfies

$$\partial_\tau p^*(y,\tau;x,\rho) = \Delta_y p^*(y,\tau;x,\rho) + b \cdot \nabla_y p^*(y,\tau;x,\rho) - \frac{2}{r_y}\partial_{r_y} p^*(y,\tau;x,\rho).$$

Then $p^*(y,\tau;x,\rho)$ is a fundamental solution of

$$\partial_\tau v = \Delta v + b\cdot\nabla v - \frac{2}{r}\partial_r v, \tag{5.9}$$

with respect to variables (y,τ) and $e^{-\psi(y)}(P^{\psi}_{\rho,\tau})^* g(y)$ is a solution of (5.9).

We now restrict the solution v of (5.9) such that $v(y,\tau)|_{|y'|=0} = 0$. The reason is the following: let $v = rh$, then by direct computation, h satisfies

$$\partial_s h = \Delta h - \frac{1}{r^2}h + b\cdot\nabla h + \frac{b_r}{r}h, \tag{5.10}$$

where $b = b_r e_r + b_\theta e_\theta + b_z e_z$.

If $|b| \le C_0 + \frac{1}{r}$, using Nash–Moser iteration argument as in the section 2 and noting that the integral of $\frac{b_r}{r}h$ can be absorbed by that of $-\frac{1}{r^2}h$ which is a good term in the energy estimate due to its minus sign. We can derive that the weak solution of (5.10) is bounded. So we can assume

$$v|_{|y'|=0} = rh|_{|y'|=0} = 0.$$

Then we have $(P^{\psi}_{t,s})^* g(y)|_{|y'|=0} = 0$ when $s \ge t$.

Now we can follow the proof of $L^2 \to L^\infty$ estimate for $P^{\psi}_{t,s}$ to derive the $L^2 \to L^\infty$ estimate for $(P^{\psi}_{t,s})^*$. $(P^{\psi}_{t,s})^*$ has nearly the same form as $P^{\psi}_{t,s}$, but the signs on the terms $\frac{2}{r}\partial_r$ and $b\cdot\nabla$ are reversed. This makes the estimate a little different when we deal with the term I_2. If we denote

$$(P^{\psi}_{t,s})^* g(y) = g_s(y)$$

then

$$\begin{aligned}
I_2 &= -\int \frac{2}{r} g_s^{k(t)-1}[\partial_r g_s + \partial_r(\alpha x)] \\
&\le -\int \frac{2}{k(t)}\partial_r g_t^{k(t)} dr d\theta dz + \int |\alpha|\frac{2}{r} g_s^{k(t)} \\
&= \frac{2}{k(t)}\int dz d\theta g_s^{k(t)}|_{r=0} + \int |\alpha|\frac{2}{r} g_s^{k(t)} \\
&= \int |\alpha|\frac{2}{r} g_s^{k(t)}.
\end{aligned}$$

Due to the vanishing property of $(P^{\psi}_{t,s})^* g(y)$ at $|y'| = 0$, we can also get the estimate (5.3) for $(P^{\psi}_{t,s})^* g(y)$, so the $L^2 \to L^\infty$ estimate (5.8) is also right to $(P^{\psi}_{t,s})^* g(y)$.

$$\|(P^{\psi}_{t,s})^* g\|_\infty \le (2\pi|t-s|)^{-\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)|t-s|\}\|g\|_{L^2}.$$

Part three: $L^1 \to L^\infty$ estimate of $P^{\psi}_{t,s}$.

Using the duality, we have the $L^1 \to L^2$ estimate of $P^{\psi}_{t,s}$.

$$\|(P^{\psi}_{t,s}) f\|_{L^2} \le (2\pi(t-s))^{-\frac{3}{4}} \exp\{(65\alpha^2 + C_0|\alpha|)(t-s)\}\|f\|_{L^1}.$$

So

$$\begin{aligned}\|(P^{\psi}_{t,s})f\|_{L^\infty} = \|(P^{\psi}_{t,\frac{t+s}{2}}P^{\psi}_{\frac{t+s}{2},s})f)\| &\leq (2\pi(t-\frac{t+s}{2}))^{-\frac{3}{4}}\exp\{(65\alpha^2 \\ &\quad + C_0|\alpha|)(t-\frac{t+s}{2})\}\|P^{\psi}_{\frac{t+s}{2},s}f\|_{L^2} \\ &\leq (\pi(t-s))^{-\frac{3}{4}}\exp\{(65\alpha^2+C_0|\alpha|)\frac{t-s}{2}\}\|P^{\psi}_{\frac{t+s}{2},s}f\|_{L^2}. \\ &\leq (\pi(t-s))^{-\frac{3}{2}}\exp\{(65\alpha^2+C_0|\alpha|)(t-s)\}\|f\|_{L^1}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}p(x,t;y,s) &\leq (\pi(t-s))^{-\frac{3}{2}}\exp\{(65\alpha^2+C_0|\alpha|)(t-s)\}\exp\{\alpha(x-y)\} \\ &\leq (\pi(t-s))^{-\frac{3}{2}}\exp\{65\alpha^2(t-s)+C_0|\alpha|(t-s)+\alpha(x-y)\}.\end{aligned}$$

Let $\alpha = -\frac{1}{65(t-s)}\frac{x-y}{|x-y|}\left[|x-y|-C_0(t-s)\right]_+$. With this choice of α, we have

$$\alpha\cdot(x-y)+C_0|\alpha|(t-s)+65\alpha^2(t-s) = -\frac{1}{65t}\left[|x-y|-C_0(t-s)\right]_+^2,$$

then

$$p(x,t;y,s) \leq (\pi(t-s))^{-\frac{3}{2}}\exp\left\{-\frac{1}{65(t-s)}\left[|x-y|-C_0(t-s)\right]_+^2\right\}.$$

This gives the estimate (1.11) of $p(x,t;y,s)$.

Moreover, when $t \geq s$,

$$\begin{aligned}\partial_t\int_{\mathbb{R}^3} p(x,t;y,s)dx &= \int_{\mathbb{R}^3}\partial_t p(x,t;y,s)dx \\ &= \int_{\mathbb{R}^3}(\Delta_x+\frac{2}{r_x}\partial_{r_x}-b\cdot\nabla_x)p(x,t;y,s)dx \\ &= \int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{+\infty}\frac{2}{r_x}\partial_{r_x}p(x,t;y,s)r_x dr_x d\theta dz \\ &= \int_{-\infty}^{+\infty}\int_0^{2\pi} -2p(x,t;y,s)|_{r_x=0}d\theta dz \\ &\leq 0,\end{aligned}$$

so

$$\int_{\mathbb{R}^3} p(x,t;y,s)dx \le \int_{\mathbb{R}^3} p(x,s;y,s)dx = 1.$$

Also, when $t \le s$,

$$\begin{aligned}
\partial_s \int_{\mathbb{R}^3} p(x,t;y,s)dy &= \int_{\mathbb{R}^3} \partial_s p(x,t;y,s)dy \\
&= \int_{\mathbb{R}^3} (\Delta_y - \frac{2}{r_y}\partial_{r_y} + b\cdot\nabla_y)p(x,t;y,s)dy \\
&= \int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^{+\infty} -\frac{2}{r_y}\partial_{r_y} p(x,t;y,s)r_y dr_y d\theta dz \\
&= \int_{-\infty}^{+\infty}\int_0^{2\pi} 2p(x,t;y,s)|_{r_y=0} d\theta dz \\
&= 0,
\end{aligned}$$

so

$$\int_{\mathbb{R}^3} p(x,t;y,s)dy = \int_{\mathbb{R}^3} p(x,t;y,t)dy = 1.$$

Thus we complete the proof of Theorem 1.3. □

Acknowledgments

I want to express my heartfelt gratitude to my advisor, Professor Qi S. Zhang in University of California, Riverside, for his helpful discussion and guidance about this subject and Professor Zhen Lei in Fudan University for his inspiring idea for this paper. I also want to thank my co-advisor, Professor Huicheng Yin in Nanjing Normal University, for his constant encouragement.

Appendix A

Claim. *If b satisfies (1.2), then we can get the estimate (1.8).*

Proof. We compute $E^p_{R,p}(b)$ as follows.

$$E^p_{R,p}(b) \le \frac{1}{R^{3-p}} \int_{B_R} \left[\frac{(\ln|\ln\frac{r}{3}|)^\alpha}{r}\right]^p dx$$

$$\le \frac{2\pi}{R^{3-p}} \int_0^R \int_0^R \left[\frac{(\ln|\ln \frac{r}{3}|)^{\alpha}}{r} \right]^p r dr dz$$

$$\le \frac{C}{R^{2-p}} \int_0^R (\ln|\ln \frac{r}{3}|)^{\alpha p} r^{1-p} dr$$

$$\le \frac{C}{R^{2-p}} \int_{\frac{1}{R}}^{\infty} (\ln\ln 3s)^{\alpha p} s^{p-3} ds \qquad (\text{let } s = 1/r)$$

$$\le \frac{C}{R^{2-p}} \left(\int_{\frac{1}{R}}^{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}} + \int_{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}}^{\infty} \right) (\ln\ln 3s)^{\alpha p} s^{p-3} ds \quad \text{for any } \varepsilon > 0$$

$$\le \frac{C}{R^{2-p}} \left\{ \int_{\frac{1}{R}}^{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}} (\ln\ln 3s)^{\alpha p} s^{p-3} ds \right.$$

$$\left. + \int_{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}}^{\infty} \left(\frac{\ln\ln 3s}{3s} \right)^{\alpha p} (3s)^{p-3+\alpha p} ds \right\}.$$

Here $\ln\ln 3s$ is a monotone-increasing function while $\frac{\ln\ln 3s}{3s}$ a monotone-decreasing function, so

$$E^p_{R,p}(b) \le \frac{C}{R^{2-p}} \left\{ \left(\ln\ln \frac{3}{R} \right)^{\alpha p + \varepsilon p} \int_{\frac{1}{R}}^{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}} s^{p-3} ds \right.$$

$$\left. + \frac{\left(\ln\ln \frac{3}{R}\right)^{\alpha p + \varepsilon p}}{\left(e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}} \right)^{\alpha p}} \int_{\frac{1}{3}e^{e^{\left(\ln\ln \frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}}^{\infty} (3s)^{p-3+\alpha p} ds \right\},$$

we need to choose a $p \in (\frac{5}{3}, 2]$ such that $p - 3 + \alpha p < -1$, that is $\alpha < \frac{2}{p} - 1$.

For such a p

$$E_{R,p}^{p}(b) \le \frac{C}{R^{2-p}} \left\{ \left(\ln\ln\frac{3}{R}\right)^{\alpha p+\varepsilon p} \frac{1}{2-p}\left(\frac{1}{R}\right)^{p-2} + \frac{1}{2-\alpha p-p} \frac{\left(\ln\ln\frac{3}{R}\right)^{\alpha p+\varepsilon p}}{\left(e^{e^{\left(\ln\ln\frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}\right)^{\alpha p}} \left(e^{e^{\left(\ln\ln\frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}}\right)^{p-2+\alpha p} \right\},$$

here $e^{e^{\left(\ln\ln\frac{3}{R}\right)^{1+\frac{\varepsilon}{\alpha}}}} \ge \frac{3}{R}$. So

$$E_{R,p}^{p}(b) \le \frac{C}{R^{2-p}} \left(\ln\ln\frac{3}{R}\right)^{\alpha p+\varepsilon p} \left(\frac{1}{R}\right)^{p-2} \le C\left(\ln\ln\frac{3}{R}\right)^{\alpha p+\varepsilon p}.$$

Here ε can be chosen sufficiently small. For satisfying (1.8), we need $\alpha < \frac{3p-5}{77p-120}$. So

$$\alpha < \min\left\{\frac{3p-5}{77p-120}, \frac{2}{p}-1\right\}.$$

Let $f(p) = \min\left\{\frac{3p-5}{77p-120}, \frac{2}{p}-1\right\}$, $f(p_0) = \max\limits_{\frac{5}{3}<p\le 2} f(p)$. We compute that

$$p_0 = \frac{279+\sqrt{1041}}{160} \approx 1.945.$$

We choose p_0 to ensure $\alpha < \min\left\{\frac{3p_0-5}{77p_0-120}, \frac{2}{p_0}-1\right\} \approx 0.028$. This is nearly the maximum value we can choose for α. Then for $\alpha \in [0, 0.028]$,

$$E_{R,p_0}(b) \le C\left(\ln|\ln\frac{R}{3}|\right)^{\frac{3p_0-5}{77p_0-120}-} \quad \forall R \in (0,1]$$

This finishes the proof of Claim. □

References

[1] Hammadi Abidi, Ping Zhang, Global smooth axisymmetric solutions of 3-D inhomogenenous incompressible Navier–Stokes system, arXiv:1409.2953.

[2] L.J. Burke, Qi S. Zhang, A priori bounds for the vorticity of axially symmetric solutions to the Navier–Stokes equations, Adv. Differential Equations 15 (5–6) (2010) 531–560.

[3] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, Comm. Pure Appl. Math. 35 (6) (1982) 771–831.

[4] Dongho Chae, Jihoon Lee, On the regularity of the axisymmetric solutions of the Navier–Stokes equations, Math. Z. 239 (4) (2002) 645–671.

[5] Chiun-Chuan Chen, Robert M. Strain, Horng-Tzer Yau, Tai-Peng Tsai, Lower bound on the blow-up rate of the axisymmetric Navier–Stokes equations, Int. Math. Res. Not. IMRN 9 (2008) rnn016.

[6] Chiun-Chuan Chen, Robert M. Strain, Tai-Peng Tsai, Horng-Tzer Yau, Lower bounds on the blow-up rate of the axisymmetric Navier–Stokes equations. II, Comm. Partial Differential Equations 34 (1–3) (2009) 203–232.

[7] Eric A. Carlen, Michael Loss, Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier–Stokes equation, Duke Math. J. 81 (1) (1995) 135–157 (1996).

[8] E.B. Davies, Explicit constants for Gaussian upper bounds on heat kernels, Amer. J. Math. 109 (2) (1987) 319–333.

[9] Ennio De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957) 25–43 (in Italian).

[10] Seng-Kee Chua, Richard L. Wheeden, Weighted Poincaré inequalities on convex domains, Math. Res. Lett. 17 (5) (2010) 993–1011.

[11] E.B. Fabes, D.W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, Arch. Ration. Mech. Anal. 96 (4) (1986) 327–338.

[12] David Gilbarg, Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Springer-Verlag, Berlin, 1983.

[13] Thomas Y. Hou, Zhen Lei, Congming Li, Global regularity of the 3D axi-symmetric Navier–Stokes equations with anisotropic data, Comm. Partial Differential Equations 33 (7–9) (2008) 1622–1637.

[14] Thomas Y. Hou, Congming Li, Dynamic stability of the three-dimensional axisymmetric Navier–Stokes equations with swirl, Comm. Pure Appl. Math. 61 (5) (2008) 661–697.

[15] Quansen Jiu, Zhouping Xin, Some regularity criteria on suitable weak solutions of the 3-D incompressible axisymmetric Navier–Stokes equations, in: Lectures on Partial Differential Equations, in: New Stud. Adv. Math., vol. 2, Int. Press, Somerville, MA, 2003, pp. 119–139.

[16] Gabriel Koch, Nikolai Nadirashvili, Gregory A. Seregin, Vladimir Sverak, Liouville theorems for the Navier–Stokes equations and applications, Acta Math. 203 (1) (2009) 83–105.

[17] O.A. Ladyzenskaja, Unique global solvability of the three-dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry, Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 7 (1968) 155–177 (in Russian).

[18] Zhen Lei, Qi S. Zhang, A Liouville theorem for the axially-symmetric Navier–Stokes equations, J. Funct. Anal. 261 (8) (2011) 2323–2345.

[19] Zhen Lei, Qi S. Zhang, Structure of solutions of 3D axisymmetric Navier–Stokes equations near maximal points, Pacific J. Math. 254 (2) (2011) 335–344.

[20] Peter Li, Richard Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds, Acta Math. 153 (3–4) (1984) 279–301.

[21] Vitali Liskevich, Qi S. Zhang, Extra regularity for parabolic equations with drift terms, Manuscripta Math. 113 (2) (2004) 191–209.

[22] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math. 13 (1960) 457–468.

[23] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.

[24] J. Neustupa, M. Pokorny, An interior regularity criterion for an axially symmetric suitable weak solution to the Navier–Stokes equations, J. Math. Fluid Mech. 2 (4) (2000) 381–399.

[25] Hirofumi Osada, Diffusion processes with generators of generalized divergence form, J. Math. Kyoto Univ. 27 (4) (1987) 597–619.

[26] Gregory Seregin, Luis Silvestre, V. Sverak, A. Zlatos, On divergence-free drifts, J. Differential Equations 252 (1) (2012) 505–540.

[27] Gang Tian, Zhouping Xin, One-point singular solutions to the Navier–Stokes equations, Topol. Methods Nonlinear Anal. 11 (1) (1998) 135–145.

[28] M.R. Ukhovskii, V.I. Iudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, Prikl. Mat. Mekh. 32 (1968) 59–69 (in Russian), translated as J. Appl. Math. Mech. 32 (1968) 52–61.

[29] Chi Hin Chan, Alexis Vasseur, Log improvement of the Prodi–Serrin criteria for Navier–Stokes equations, Methods Appl. Anal. 14 (2) (2007) 197–212.

[30] Yi Zhou, Zhen Lei, Logarithmically improved criteria for Euler and Navier–Stokes equations, Commun. Pure Appl. Anal. 12 (6) (2013) 2715–2719.
[31] Qi S. Zhang, A strong regularity result for parabolic equations, Comm. Math. Phys. 244 (2) (2004) 245–260.